
Family name, Given name (Please print)

Student Number

Tutorial Leader's Name

**PHY293F – STATISTICAL MECHANICS
DEPARTMENT OF PHYSICS, UNIVERSITY OF TORONTO**

**EXAMPLE FOR MIDTERM TEST #2
10 November 2009**

PLEASE read carefully the following instructions.

AIDS ALLOWED: Equation sheet provided with test paper and a non-programmable calculator without text storage (Type 2 calculator).

Before starting, please **print** your name, tutorial group and student number **at the top of this page and on the cover of your answer booklet.**

This test has one cover sheet, two question pages and one equation sheet.

There are three questions on this midterm test. The value of each question is indicated next to the question part. The total number of points for the midterm is 35 points.

The test questions can be answered in any order. It is your responsibility to clearly indicate the question number (and part, where appropriate) for each of your answers.

Partial credit will be given for partially correct answers, so show any intermediate calculations that you do and write down, **in a clear fashion**, any relevant assumptions you are making along the way.

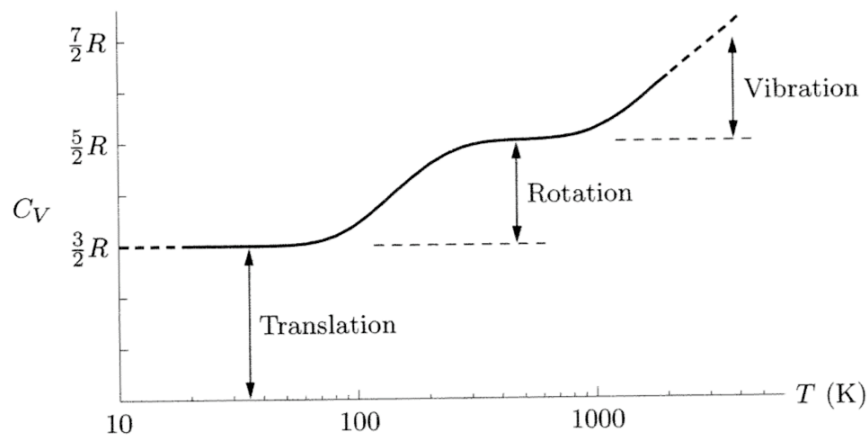
Do not separate the two stapled sheets of the question paper. Hand in the question sheet with your exam booklet at the end of the test.

- Please answer the following questions showing your reasoning as well as your final answer. To receive full marks for your answer, both your reasoning and your answer must be correct.

Here you need to provide enough details to convince the marker that you understand the concept or calculation. Usually about one sentence or point per mark is sufficient. Remember to answer what was asked in the question. There are often several points required.

- The heat capacity of a diatomic ideal gas, such as H_2 , decreases as the temperature of the gas decreases. Briefly explain how and why this happens using words and a diagram. Explicitly describe how this relates to the spacing between energy levels in different degrees of freedom. [4 points]

As the temperature of the gas decreases, different degrees of freedom “freeze out” at different temperatures. This reduces the heat capacity of the diatomic ideal gas as shown in the diagram below. [Must include properly labelled diagram to receive full marks]



The larger the spacing between the energy levels the higher the temperature needed for this degree of freedom to be active. The spacing between energy levels decreases from vibrational (largest energy spacing) to rotational to translational (smallest energy spacing).

- Consider two macrostates of a system of two Einstein solids. The system is 10^6 times more likely to be in the first macrostate than in the second macrostate. Based on this information, what is the difference in entropy between these two macrostates? Express your answer in units of Boltzmann's constant, k . [4 points]

We know that macrostate 1 is 10^6 times more likely than macrostate 2, so: $\Omega_1/\Omega_2 = 10^6$, because the ratio of the probabilities is the ratio of the multiplicities.

Since $S = k \ln \Omega$, can rearrange to get relation for Ω in terms of S , $\Omega = e^{S/k}$. Now we can combine these to get:

$$\frac{\Omega_1}{\Omega_2} = e^{+(S_1 - S_2)/k}$$

$$10^6 = e^{+(S_1 - S_2)/k}$$

$$\ln(10^6) = (S_1 - S_2)/k$$

$$S_1 - S_2 = 13.8k$$

The difference in entropy between the states, in units of Boltzmann's constant, is 13.8 k.

- c. Define the terms microstate and macrostate as they have been used in class and explain how they are related. [3 points]

A macrostate is a general description of the amount of energy in the system (for example: 3 units of energy in Einstein solid A with 5 oscillators).

A microstate is specific description of one of the possible ways that the energy can be distributed between the particles in a system (for example: 1 unit of energy in oscillator #1, 2 units of energy in oscillator #2 and zero units of energy in oscillators #3, 4 and 5).

One or more microstate corresponds to each macrostate of a given energy. The multiplicity of a macrostate is the number of microstates (or specific ways of distributing the energy) for that particular energy value.

- d. In the "low temperature" limit, the multiplicity of the Einstein solid and two-state paramagnet models are given by:

$$\Omega = \left(\frac{eN}{q} \right)^q \text{ and } \Omega = \left(\frac{Ne}{N_{\downarrow}} \right)^{N_{\downarrow}}, \text{ respectively.}$$

For each model system, define what is meant by "low temperature" in terms of system parameters (e.g. number of oscillators). Explain why, in the low temperature limit, the multiplicities of the Einstein solid and the two-state paramagnet systems are essentially the same. [4 points]

For the Einstein solid model system, the low temperature limit is when q (the number of units of energy) is much smaller than N (the number of oscillators). In this case, it is very unlikely to find an oscillator with more than one unit of energy.

For the two-state paramagnet model system, the low temperature limit is when the number of dipoles in the "spin-down" position (N_{\downarrow}) is much smaller than the total number of dipoles (N). In this case most of the dipoles are in the lower energy "spin-up" state.

In the low temperature limit, the multiplicities of the Einstein solid and the two-state paramagnet are essentially the same because there are so few units of energy compared to the number of particles. In this case, the Einstein solid has "essentially" only two energy levels: zero units of energy and one unit of energy. This makes it very similar to the two-state paramagnet that can only hold a maximum of one unit of energy per particle.

2. Consider an ideal monatomic gas that lives in a two-dimensional universe, occupying an area A instead of volume V . There are N molecules in this isolated two-dimensional gas with total energy U . The multiplicity of this system is

$$\Omega(N, U, A) = \frac{1}{N!} \frac{A^N}{h^{2N}} \frac{\pi^N}{N!} (2mU)^N$$

where h is Planck's constant and m is the molecular mass.

- a. Determine the entropy of this two-dimensional system. Explicitly indicate the terms that depend on area, A , and total energy, U . [3 points]

Calculate the entropy from the multiplicity:

$$S = k \ln \Omega$$

$$= k \ln \left[\frac{1}{N!} \frac{A^N}{h^{2N}} \frac{\pi^N}{N!} (2mU)^N \right]$$

$$S = Nk \ln A + Nk \ln U + k \ln \left[\left(\frac{2m\pi}{h^2} \right)^N \frac{1}{N!^2} \right]$$

In this equation, the first term depends on A and the second depends on U .

- b. Find an expression for the total energy, U , as a function of temperature, T . [3 points]

To determine the total energy, use:

$$\frac{1}{T} = \frac{\partial S}{\partial U}$$

$$= \frac{\partial}{\partial U} \left(Nk \ln A + Nk \ln U + k \ln \left[\left(\frac{2m\pi}{h^2} \right)^N \frac{1}{N!^2} \right] \right)$$

$$\frac{1}{T} = \frac{Nk}{U}$$

Rearrange to get:

$$U(T) = NkT.$$

- c. What is the heat capacity C_V at constant volume for this system? Comment on how your expression compares with the equipartition theorem. [4 points]

Heat capacity at constant volume is:

$$C_V = \left(\frac{\partial U}{\partial T} \right)_{N,V}$$

$$= \frac{\partial}{\partial T} (NkT)$$

$$C_V = Nk$$

From the equipartition theorem, we can calculate the heat capacity C_V . There are two translational degrees of freedom for this monatomic ideal gas in a 2-D world.

$C_V = N \cdot f \frac{1}{2} k = Nk$. The answer above agrees with the equipartition value.

3. As an experiment in statistical physics, you decide undertake a one-dimensional random walk. Each step you take is of equal size and each is chosen randomly to be either in the forward or backward direction. Using concepts we have discussed in PHY293, answer the following questions.

- a. Where are you most likely to find yourself at the end of a long random walk of N steps? [1 point]

*You are **most** likely to find yourself at the starting position.*

- b. Find an expression for the net distance you have traveled, L , in terms of l , the length of your step, and x , the excess number of forward steps over $N/2$. [3 points]

To find the net distance traveled, you need the difference between the number of forward and backward steps):

$$\begin{aligned} L &= l \cdot (\text{forward steps} - \text{backward steps}) \\ &= l \cdot \left(\left(\frac{N}{2} + x \right) - \left(\frac{N}{2} - x \right) \right) \\ L &= 2 \cdot l \cdot x \end{aligned}$$

- c. Using the relation for multiplicity of a large number of coin tosses ($\Omega \propto e^{-2x^2/N}$, where x is the excess number of heads) and your result from part (b), express the dependence of the distribution of ending positions for your random walk in terms of l , the length of your step, N , the number of steps you have taken, and L , the net distance you have traveled. [2 points]

In the expression given above, x , the excess number of heads, is similar to x , the excess number of forward steps. So substituting in for x you get:

$$\begin{aligned} \text{Since } x &= \frac{L}{2 \cdot l}, \\ \Omega &\propto e^{-2(L^2 / 4l^2) / N} . \\ \Omega &\propto e^{-L^2 / 2l^2 N} . \end{aligned}$$

- d. Suppose you take a random walk of 10,000 steps (each step, l , is 1 m long). About how far from your starting position could you expect to be at the end? Give an answer in terms of order of magnitude and provide justification for your answer using what you derived in part (c). [4 points]

You expect to find yourself within $1/e$ of the most probable position. This is within the width of the Gaussian distribution you found in part (c). The net distance traveled at the width is found by the following.

$$\begin{aligned} \text{Since } e^{-1} &= e^{-L^2 / 2l^2 N}, \\ L^2 &= 2l^2 N \\ L &= \sqrt{2l^2 N}. \end{aligned}$$

For $l = 1\text{ m}$ and $N = 10,000$ steps, you expect to find yourself within:

$$L = \sqrt{2 \cdot l^2 \cdot 10,000}$$

$$L = 141\text{ steps}$$

$$\text{magnitude} \approx 100\text{ steps}$$

You expect to find yourself within ~ 100 steps of your starting position.