PHY 293F – WAVES AND PARTICLES
DEPARTMENT OF PHYSICS, UNIVERSITY OF TORONTO

PROBLEM SET #8 – SOLUTIONS

Marked Q2 (parts (a) and (b); out of 7 points) and Q4 (out of 3 points) for a total of 10.

1. Calculate the Helmholtz free energy of a photon gas, such as the one we used to investigate blackbody radiation by doing the following:

   a. Calculate the Helmholtz free energy directly from $F = U - TS$. Simplify your expression to get the most compact expression possible.

We know U and S in terms of T, V and α (related to a used in class by $a=\alpha V$):

$$U = \frac{8\pi^5 (kT)^4 V}{15(hc)^3} = \alpha V T^4$$

$$S = \frac{32\pi^5}{45} V \left(\frac{kT}{hc}\right)^3 k = \frac{4}{3}\alpha V T^3,$$

$$\text{where} \qquad \alpha = \frac{8\pi^5 k^4}{15(hc)^3}.$$

Then we calculate the Helmholtz free energy and get:

$$\begin{aligned} F &= U - TS \\ &= \alpha V T^4 - T \cdot \frac{4}{3}\alpha V T^3 \\ &= T^4\left(\alpha V - \frac{4}{3}\alpha V\right) \\ &= -\frac{1}{3}\alpha V T^4 = -\frac{1}{3}U. \end{aligned}$$

   b. Verify your answer in part (a) by calculating the entropy, S and comparing it to the result we got in class. To get a compact answer in terms of V and T, you can use $\alpha = 8\pi^5 k^4 / 15(hc)^3$.

The entropy is equal to:

$$S = -\left(\frac{\partial F}{\partial T}\right)_V.$$

So, using the result from part (a), we calculate the entropy as:

$$\begin{aligned} -\left(\frac{\partial F}{\partial T}\right)_V &= -\frac{\partial}{\partial T}\left(-\frac{1}{3}\alpha V T^4\right) \\ &= \frac{4}{3}\alpha V T^3 = S. \end{aligned}$$

The expression is the same as the one we derived in class, except this one is in terms of α, V and T instead of a ($\alpha\cdot V$) and T.

c. Differentiate F with respect to V to get the pressure of a photon gas. Again, present your answer as the most compact expression possible.

The expression for pressure is and, using the result from part (a), we calculate pressure for the photon gas to be:

$$\begin{aligned} P &= -\left(\frac{\partial F}{\partial V}\right)_{T,N} \\ &= -\frac{\partial}{\partial V}\left(-\frac{1}{3}\alpha V T^4\right) = \frac{1}{3}\alpha T^4 \\ &= \frac{1}{3}\frac{U}{V}. \end{aligned}$$

2. Problem 7.51 on page 303-304 of Schroeder (only parts (a) – (d)).

a) *The power radiated by the filament is given by the Stefan-Boltzmann Law,* $\dfrac{power}{area} = \varepsilon\sigma T^4$.

Therefore, the area of the filament is:

$$A = \frac{power}{\varepsilon\sigma T^4} = \frac{100\text{ W}}{\frac{1}{3}\cdot 5.67\text{x}10^{-8}\cdot(3000\text{ K})^4} = 6.53\text{x}10^{-5}\text{ m}^2 = 65\text{ mm}^2$$

b) *Using Wien's Law, we can calculate the peak in the photon energy spectrum (energy density versus photon energy):*

$$\varepsilon = 2.82kT = 2.82\cdot 8.62\text{x}10^{-5}\text{ eV/K}\cdot 3000\text{ K} = 0.73\text{ eV}.$$

Here the most appropriate units are eV. The answer is $1.2\text{x}10^{-19}$ J.

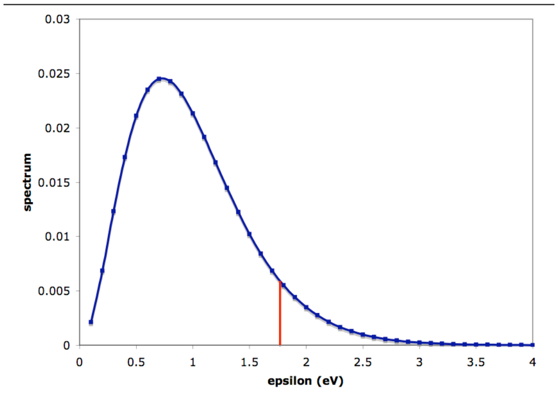
The wavelength of the photon can be calculated from:

$$\lambda = \frac{c}{f} = \frac{hc}{\varepsilon} = \frac{4.136\text{x}10^{-15}\text{ eVs}\cdot 2.998\text{x}10^{8}\text{ m/s}}{0.73\text{ eV}} = 1.7\text{x}10^{-6}\text{ m}.$$

This is a photon wavelength of 1.7µm or 1700 nm, which is in the near-infrared region of the spectrum.

c) *The spectrum can be plotted using a computer or sketched by hand. To identify the visible region, we need to convert the red end (700 nm = 1.7 eV) and violet end (400 nm = 3.1 eV) values to eV.*

The red line indicates the red end of the visible range and the violet line for the violet end of the range is essentially invisible under the spectrum. This is graph is calculated using only the ε-dependent part of the Planck spectrum, e.g. $\varepsilon^3/(\exp(\varepsilon/kT)-1)$.



d) *By numerically integrating over part of energy range of the Planck spectrum, we can calculate the fraction of the bulb's light that is in the visible range. Also, it is useful to convert from ε to the unitless variable (or reduced energy) $x = \varepsilon/kT$. For the visible range, x extends from 6.8 (red end) to 11.9 (violet end) by using $x = hc/\lambda kT$.*

$$fraction = \int_{6.8}^{11.9} \frac{x^3}{e^x - 1} dx \Big/ \int_0^{\infty} \frac{x^3}{e^x - 1} dx = \frac{15}{\pi^4} \int_{6.8}^{11.9} \frac{x^3}{e^x - 1} dx.$$

The value obtained from the numerical integration is about 0.082. Therefore, about 8% of the light emitted by the bulb is in the visible range.

3. Problem 7.56 on page 307 of Schroeder.

a) *The brightness of a source decreases as the square of the distance so the solar constant at Venus is equal to:*

$$\text{solar constant at Venus} = \frac{1}{r^2} \cdot 1370 \text{ W/m}^2 = \frac{1}{(0.7)^2} \cdot 1370 \text{ W/m}^2 = 2796 \text{ W/m}^2.$$

Therefore, the average temperature of the surface of Venus (without an atmosphere and reflecting no sunlight) is:

$$\text{solar constant at Venus} \cdot \pi R^2 = 4\pi R^2 \cdot \sigma T^4$$

$$T = \left(\frac{2796 \text{ W/m}^2}{4 \cdot 5.67\text{x}10^{-8}} \right)^{\frac{1}{4}} = 333 \text{ K}.$$

b) *The reflectivity of the clouds on Venus is 77% (or albedo of 0.77). So, our calculation becomes:*

$$\text{solar constant at Venus} \cdot (1 - 0.77) \cdot \pi R^2 = 4\pi R^2 \cdot \sigma T^4$$

$$T = \left(\frac{2796 \text{ W/m}^2 \cdot 0.23}{4 \cdot 5.67\text{x}10^{-8}} \right)^{\frac{1}{4}} = 231 \text{ K}.$$

This reduces the average surface temperature significantly. So, the atmosphere must play a significant role on Venus to have the very high temperatures that exist there.

c) *To consider the effect of the atmosphere on Venus, you consider a series of blankets as described by Schroeder in the hint. Here is his explanation of how to solve this one:*

(c) Consider a time period in which one unit of sunlight is absorbed. The planet as a whole (including the atmosphere) must also emit one unit of energy, and it must be emitted by the uppermost atmospheric blanket. This implies that the uppermost blanket is at 231 K, as calculated in part (b). But the blanket also sends a unit of energy down to the next blanket. Since it emits two units total, it must also absorb two units, and these must come from the next blanket down. The second blanket therefore sends twice as much energy upward as the first, so it must be hotter by a factor of $2^{1/4}$. Meanwhile, the second blanket also sends two units of energy downward to the third. Since the second blanket is emitting four units total, and receiving one unit from the first blanket, it must receive three more from the third blanket. The third blanket must therefore send three units of energy upward, which implies that it is hotter than the first blanket by a factor of $3^{1/4}$. Continuing downward, we might guess that the fourth blanket is hotter than the first by a factor of $4^{1/4}$. To check this, note that the third blanket sends three units of energy downward, so it emits six units total, but it receives two units from the second blanket, so it must receive four more units from the fourth blanket. We could prove by induction that the nth blanket must emit n units of energy upward and therefore must be hotter than the first by a factor of $n^{1/4}$; I'll skip the proof and take the theorem as established by the first four cases. Thus, the 70th blanket is hotter than the first by a factor of $(70)^{1/4}$, and the ground (finally) is hotter by a factor of $(71)^{1/4} = 2.90$. Multiplying by 231 K, we therefore predict that the ground temperature should be about 670 K. Ouch!

4. In equation 7.116 (p.311), there are two contributions to the heat capacity of a solid. At what temperature, T_{eq}, are these contributions equal for a sample of solid copper. Use data from Figure 7.28 (plot of C/T versus T^2), where the slope is $5\text{x}10^{-5}$ J/K^4 and the intercept is 0.7 mJ/K^2 for solid copper, for your calculation. Name which contribution to the heat capacity is greater above T_{eq}?

From class, we have the expression for the heat capacity of a solid of the form: $C = \gamma T + bT^3$. *In a graph of C/T versus T^2, slope gives us b ($5\text{x}10^{-5}$ J/K^4) and the intercept gives us γ (0.7 mJ/K^2).*

These two contributions are equal at T_{eq}, which is found by:

$$\gamma T_{eq} = bT_{eq}^3$$

$$T_{eq} = \sqrt{\frac{\gamma}{b}} = \sqrt{\frac{0.7\text{x}10^{-3}\ \text{J/K}^2}{5\text{x}10^{-5}\ \text{J/K}^4}} = 3.7\ \text{K}.$$

The contribution from lattice vibrations (bT^3) is higher than the contribution from conduction electrons (γT) above T_{eq}.