PHY293 Dot Product in State Vector Spaces

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- I've been asked a number of times about how we define the dot product between two state vectors in a coupled oscillator system.
- It is a little bit more complicated than the usual dot product of Cartesian vectors, but no more complicated than taking the dot-product of two vectors in polar coordinates [1].
- When we assemble a state-vector (x_A, x_b for the two masses in coupled pendula, or y_A, y_B for the two bobs on the double sprung pendulum in the first problem of the third problem set) we are talking about the positions of two different objects.
- When we write down the equations of motion we are coupling equations that relate forces, or accelerations to each other for these two (or more) objects.
- When the objects have the same mass then we can take a short-cut
  - The state vector space ends up being Euclidean → accelerations in x_A and x_B are the same
  - The mass matrix ends up being proportional to the unit matrix
  - And we can take dot products in the usual way for Euclidean/Cartesian space
- If the objects have different masses
  - Then the mass matrix is not a simple multiple of the unit matrix
  - The coupled system must *know* about the greater inertia that is present for the more massive object
  - The mass matrix is the way this is encoded into the problem
  - The state-vector space ends up being non-Euclidean
- When we solve a coupled system, with different masses, the solution (the eigenfrequencies and eigenvectors) take the non-Euclidean nature of the state-vector space into account.
- They find the solutions that minimise the action of the coupled system.
- The normal modes/eigenvectors are a *basis* for the non-Euclidean space
- Thus the eigenvectors are only a basis in the distorted state-vector space that *includes* the differences between the masses etc.
- The way our arithmetic keeps track of this is to **define** the dot-product of any two vectors in the state-vector space as:

$$\vec{a} \odot \vec{b} = (\vec{a}[M]\vec{b})$$

- Strictly speaking this holds for any vector product in the state-vector space of a particular coupled oscillator problem.
- In particular this holds for the norm of a vector:

$$||\vec{\xi}_a||^2 = (\vec{\xi}_a[M]\vec{\xi}_a)$$

- So the normal of a basis vector in these coupled oscillator problems should really have units of $[1/\sqrt{\text{mass}}]$
- In this course we have only computed ratios of dot-products so the units of mass cancel out, viz:

$$B_n = \vec{\xi}_n \odot \vec{x}_0/||\vec{\xi}_n||^2 = (\vec{\xi}_n[M]\vec{x}_0)/||\vec{\xi}_n||^2 = (\vec{\xi}_n[M]\vec{x}_0)/(\vec{\xi}_n[M]\vec{\xi}_n)$$

- So there is one power of the mass matrix on the top and bottom and the units (of kg, say) will cancel out in the coefficients.
- But if you choose to take the short-cut and suppress the common factors of m in the numerator and denominator, you still need to keep the numerical factors from the mass matrix (say, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$) when computing the dot-product with eigenvectors (or indeed between any state-vectors in such a system) because it is only in doing so are we preserving the information that the x_B direction has twice as much inertia as the x_A direction (given the mass matrix two lines above), ie. that twice as much force is necessary to achieve a given acceleration in the x_B direction relative to the x_A direction.
- It is only with *this metric* that the different normal modes are orthogonal to one-another (ie. $\vec{\xi}_i \odot \vec{\xi}_j = \vec{\xi}_i[M]\vec{\xi}_j = 0 \quad \text{if} \quad i \neq j$)

[1] If you were given two vectors in polar coordinates $(r, \theta, \phi)_1 \cdot (r, \theta, \phi)_2 = (1, 30°, 180°) \cdot (2, 60°, 0°)$ you wouldn't just multiply each component ... you'd have to work out where the vectors pointed and, probably figure out that they were back-to-back in ϕ and so their dot-product would be 0. The math for figuring this out would be to know the metric for this system. It would tell you that in additiona to taking the product of the third components of this vector you have to also multiply them by the $r\sin(\phi_1 - \phi_2)$ (and $\sin(180°) = 0$). Similarly in taking the product of the second components you'd have to multiply them by $r\cos(\theta_1 - \theta_2)$ (I think...)