# PHY293 Oscillations Lecture #2

September 13, 2010

1. Tutorials start this week
2. First problem set will be posted Tuesday, during the day

Start of material

5. Other oscillating systems
   - a) Pendulum: $ml\ddot{\theta} + mg\theta = 0$ and $\omega_0 = \sqrt{g/l}$
   - b) Torsion Oscillator: $I\ddot{\theta} + c\theta = 0$ and $\omega_0 = \sqrt{c/I}$
   - e) Water U-tube: $\rho l\ddot{x} + 2g\rho x = 0$ and $\omega_0 = \sqrt{2g/l}$
   - g) Bobber in fluid Torsion: $m\ddot{x} + Ag\rho x = 0$ and $\omega_0 = \sqrt{AG\rho/m}$
   - d) Mass on a tensioned string: $m\ddot{x} + 2Tx/l = 0$ and $\omega_0 = \sqrt{2T/lm}$

1. Damped Simple Harmonic Motion
   - Perpetual motion machines do **not** exist because of damping
   - Energy juggling cannot go on forever
   - Resistance, friction, dissipation is **always** present
     - We only know about oscillating systems because of dissipative phenomena
     - Hearing a tuning fork ⇒ vibrations transmitted to air
     - Observe LC oscillation ⇒ voltage dropped across a resistor
     - Feel waves ⇒ because of the viscosity of the medium they are moving in
   - Dissipation proportional to $\dot{x}$ (velocity)
     - Microscopically there is no dissipation
     - Dissipation comes from increased randomness, averaged over a system gives the overall damping
     - eg. Heat is generated by increased randomness, averaged over whole system that gets hot)
   - What form should damping take? $x$, $\dot{x}$ or $\ddot{x}$?
     (a) If we reverse time, should reverse the effect (things always slow down as time goes forward)
       - $F \propto d^2x/dt^2$ reversed gives $d^2x/d(-t)^2$ ⇒ same effect
       - $F \propto x$ doesn't depend on time
       - Only $F \propto dx/dt$ gives the opposite in $dx/d(-t)$
       - Tempted to conclude dissipation must be proportional to $\dot{x}$

     (b) What about higher odd derivatives?
       - $x = \frac{1}{2}gt^2; \ddot{x} = g \Rightarrow \frac{d\ddot{x}}{dt} = 0$
       - Physical effects will vanish for third and all higher derivatives
       - Could have powers of $\dot{x}$ or combinations but $\dot{x}$ is simplest non-trivial effect
   - As with linear restoring force, the damping force acts opposite the velocity: $F_{\text{damp}} = -b\dot{x}$

2. Equation of Motion for Damped SHO

$$F = -kx - b\dot{x} = m\ddot{x}$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

   - This is a second order, ordinary differential equation
     - Has two linearly independent solutions
     - Requires two initial conditions to be fully determined
   - Look for solutions like $x \propto e^{\alpha t}$ subbing this into equation of motion get:

$$(\alpha^2 + \gamma\alpha + \omega_0^2)e^{\alpha t} = 0$$

- Which means:

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0$$

- Find roots: $\alpha_{1,2} = \frac{-\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$
- General solution is of the form: $x = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}$
  - Where $C_{1,2}$ are determined from initial conditions (eg. $x(0)$ and $\dot{x}(0)$)
  - Note: If $\alpha_1 = \alpha_2$ (ie $\gamma^2 = 4\omega_0^2$) then the solution is of the form: $x = (C_1 + C_2 t)e^{-\gamma t/2}$
  - This is a special case of critical damping (come back next time)

3. Underdamped Solution

- This is the only solution that exhibits oscillations
- Roots $\alpha_{1,2}$ are complex since for $\omega_0^2 > \gamma^2/4$ the argument of the square-root is negative

$$\alpha + 1, 2 = \frac{-\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = \frac{-\gamma}{2} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = \frac{-\gamma}{2} \pm i\omega'$$

- We'll see that $\omega'$ is the damped oscillation frequency $\omega'^2 = \omega_0^2 - \frac{\gamma^2}{4}$
- The general solution can be written in several ways

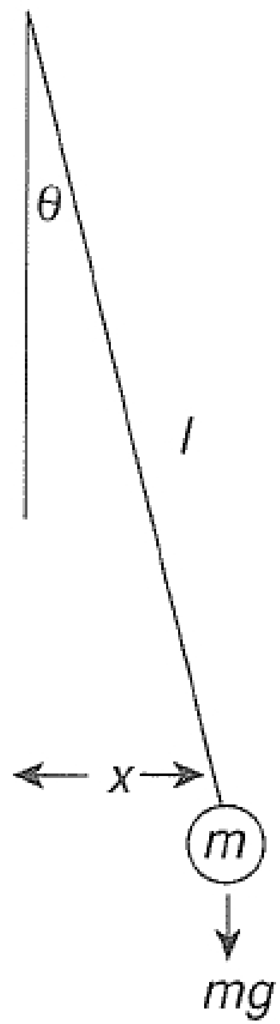
$$x = C_1 e^{-\gamma t/2} e^{i\omega' t} + C_2 e^{-\gamma t/2} e^{-i\omega' t}$$

$$x = e^{-\gamma t/2}[D_1 \sin(\omega' t) + D_2 \cos(\omega' t)]$$

$$x = A e^{-\gamma t/2} \cos(\omega' t + \phi)$$

- In each case there are two constants to be determined from initial conditions ($C_{1,2}; D_{1,2}, A; \phi$)
- The first practice problems suggest you find the relationship between these different forms of the solution for practice
  - ie. the relationship between the different constants
- The last form of the solution makes manifest how damping modifies our original (un-damped) SHO solution (see figures)
  - The oscillations are shifted to lower frequencies: $\omega' = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} < \omega_0$
  - Damping increases the period of a full oscillation: $T_d = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}} \to \infty$ as $\gamma \to 2\omega_0$
  - But in this case oscillatory motion stops and the nature of the solution changes (see next time)

(a)




$$m\ddot{x} + mg\,\frac{x}{l} = 0$$

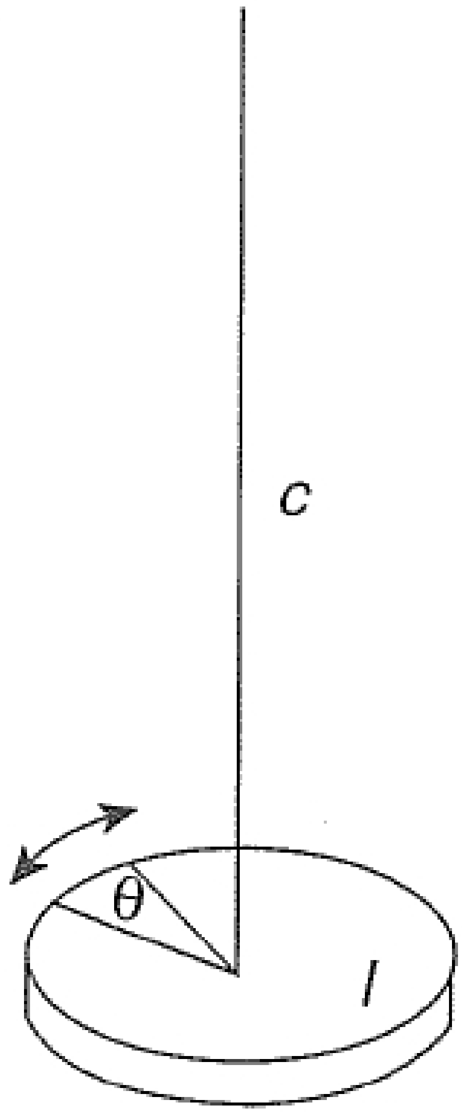
$$ml\,\ddot{\theta} + mg\,\theta = 0$$

$$\omega^2 = g/l$$

$$mg\sin\theta \approx mg\,\theta$$

$$\approx mg\,\frac{x}{l}$$

(b)

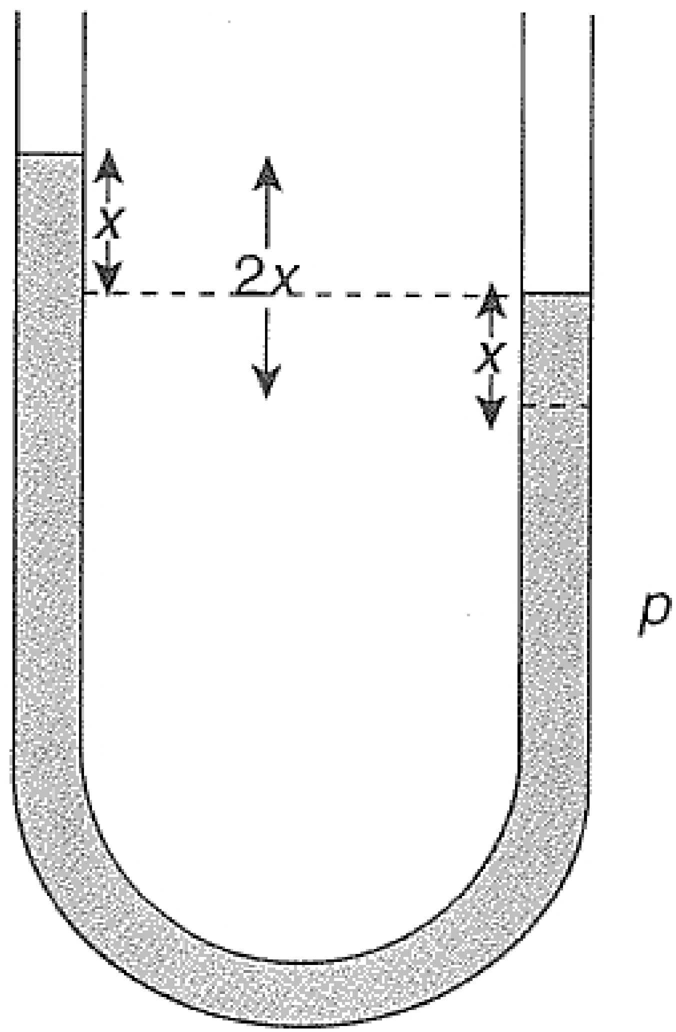



$$I\ddot{\theta} + c\,\theta = 0$$

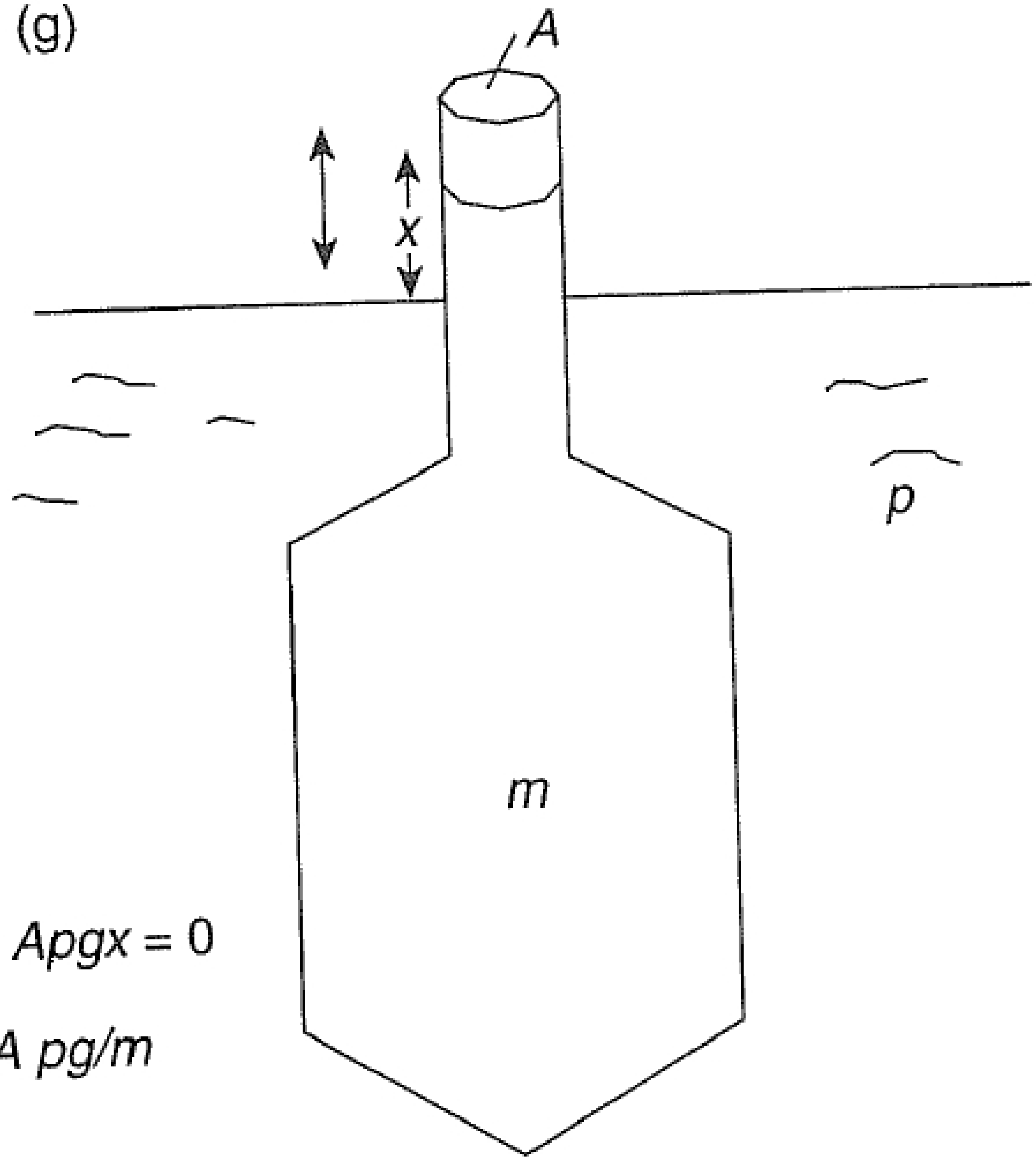
$$\omega^2 = \frac{c}{I}$$

(e) $\rho l\ddot{x} + 2\rho g x = 0$

$\omega^2 = 2g/l$



(g)



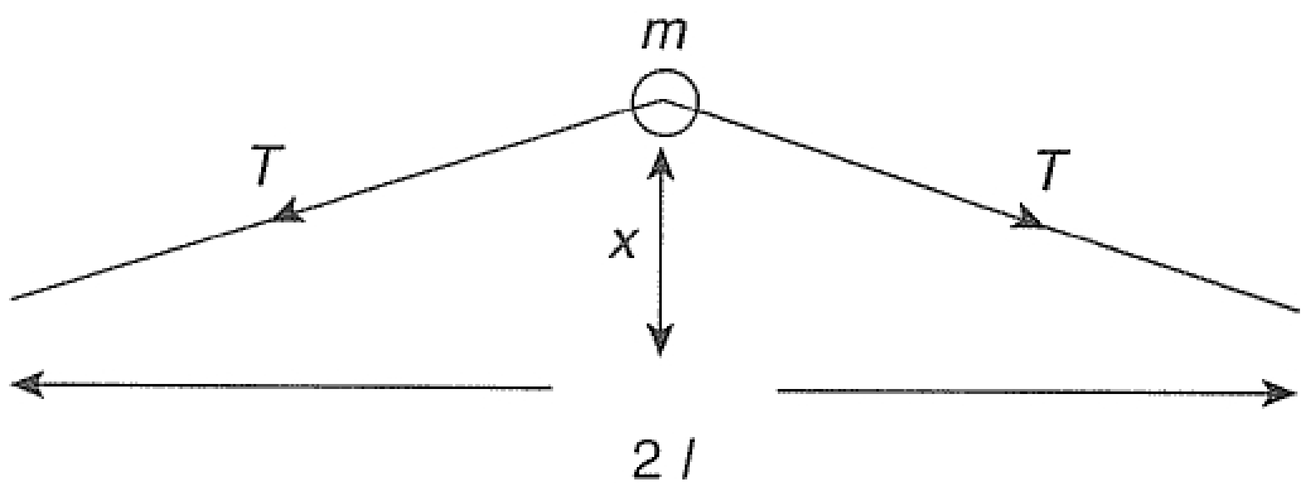

$$m\ddot{x} + A\rho g x = 0$$

$$\omega^2 = A\rho g/m$$

(d)

$$m\ddot{x} + 2T\frac{x}{l} = 0$$

$$\omega^2 = \frac{2T}{l\,m}$$



# Underdamped solution

$\gamma = 0.1\omega_{\mathrm{d}}$