PHY293 Oscillations Lecture #4

September 17, 2010

1. First problem set (http://www.physics.utoronto.ca/ phy293h1f/waves/phy293_ps1.pdf) Due Monday at 5:00pm
2. Flash Demo (http://www.upscale.utoronto.ca/PVB/Harrison/Flash/ClassMechanics/DrivenSHM/DrivenSHM.html)

Start of material

1. Driven Simple Harmonic Motion: Equation of Motion
   - Take mass on spring and move platform that spring is attached to
   - This gives driven oscillations: $X_F(t) = a_0 \cos(\omega t)$ (Note: Dimensions of a_0 are length.)
   - Now the position of the spring's fixed point is time-dependent

$$F_{\text{spring}} = -k(x - x_F(t))$$

   - So the equation of motion for the mass becomes:

$$F = -k(x - x_F(t)) - r\dot{x} = m\ddot{x}$$

   - Or dividing through by m as usual gives the canonical form:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{ka_0}{m}\cos(\omega t)$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = A_0 \cos(\omega t)$$

   - Where we've simplified the coefficient in front of the driving term to $A_0 = \omega_0^2 a_0 = \frac{ka_0}{m}$ (Note: Dimensions of A_0 are length/time^2)
   - This is the general equation of motion for a forced simple harmonic oscillator
   - This is a complicated as we'll make it in this course. All other cases we considered up to now were simplifications ($A_0 = 0$ Free SHO, $\gamma = 0$ undamped SHO, etc.)
   - We now have a second-order, linear, non-homogeneous, ordinary, differential equation
   - From the general theory of O.D.E. solutions (eg. Boyce & Diprima secs. 3.6 and 3.9)
     - Look for a solution of the form: $X(t) = C_1 x_1(t) + C_2 x_2(t) + \mathcal{X}(t)$
     - The first two terms are the complementary solutions of the homogeneous equation (ie. un-driven SHO studied up to now)
     - These solutions are transient (oscillations will die away as energy is dissipated)
   - The third term is more interesting. It describes the forced response of the system
   - After the transient solutions have died away (like $e^{-\gamma t/2}$) this will be the only steady-state part of the solution
   - Since our forcing function has $\cos(\omega t)$ time dependence "try" a solution like:

$$\mathcal{X}(t) = G\cos(\omega t) + H\sin(\omega t)$$

   - To solve we'll need the time derivatives:

$$\dot{\mathcal{X}}(t) = -G\omega\sin(\omega t) + H\omega\cos(\omega t)$$

$$\ddot{\mathcal{X}}(t) = -G\omega^2\cos(\omega t) - H\omega^2\sin(\omega t)$$

   - Plugging these back into the equation of motion get:

$$[-G\omega^2 + H\omega\gamma + G\omega_0^2]\cos(\omega t) + [-H\omega^2 - G\omega\gamma + H\omega_0^2]\sin(\omega t) = A_0\cos(\omega t)$$

   - But $\sin(\omega t)$ and $\cos(\omega t)$ are linearly independent function forms so this leads to separate constraints:

$$G(\omega_0^2 - \omega^2) + H\omega\gamma = A_0$$

$$H(\omega_0^2 - \omega^2) - G\omega\gamma = 0$$

- We can solve these two equations for G and H giving:

$$G = A_0 \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2} \qquad \mathrm{In-Phase}$$

$$H = A_0 \frac{\omega\gamma}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2} \qquad \mathrm{Out-of-Phase}$$

- Now use the identity $a\cos\phi + b\sin\phi = c\cos(\phi - \delta)$ where $a^2 + b^2 = c^2$ and $\delta = tan^{-1}(b/a)$ to get:

$$\mathcal{X}(t) = a(\omega)\cos(\omega t - \delta)$$

- With $a(\omega) = \frac{A_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$
- Sometimes see this written as: $a(\omega) = \frac{a_0\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$
  - Where a_0 is the amplitude (in length) of the driving force, while A_0 is the driving acceleration (in units of length/time^2).
- And $\tan\delta = \frac{\omega\gamma}{(\omega_0^2 - \omega^2)}$
- This is the forced response of a damped simple harmonic oscillator (see plots)
- Note that $\tan^{-1}$ only has a result between $-\pi/2$ and $\pi/2$ so we can also specify:

$$\delta = \cos^{-1}[\frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}]$$

- This form gives a phase δ for all ω_0, γ, ω etc.

2. Resonance in Forced Oscillations

- Looking at $a(\omega)$ we find a frequency where the amplitude is maximal
- As γ is reduced the peak gets higher and narrower (see plots)
- The peak frequency can be found by differentiation:

$$a(\omega) = \frac{A_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \quad \text{stationary at} \quad \frac{da(\omega)}{d\omega} = 0 \Rightarrow \frac{d}{d\omega}[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2] = 0$$

- This is the condition for the minimum or maximum and gives:

$$2(\omega_0^2 - \omega^2)(-2\omega) + 2\gamma^2\omega = 0$$

$$2\omega(2\omega^2 - 2\omega_0^2 + \gamma^2) =$$

- Has two real solutions: $\omega = [0, \sqrt{\omega_0^2 - \gamma^2/2}]$
- The first solution is the local minimum at $\omega = 0$ (a_0)
- This second solution is a maximum at:

$$\omega_m = \sqrt{\omega_0^2 - \gamma^2/2}$$

- Note that for $\gamma^2 > 2\omega_0^2$ there is **no** resonance (this is **not (quite!)** the condition for critical damping and overdamped solutions $\Rightarrow$ no oscillations)
- Still resonance only happens for **low** damping
- At this resonance frequency the amplitude is

$$a(\omega_m) = \frac{A_0}{\sqrt{(\gamma^2/2)^2 + (\omega_0^2 - \gamma^2/2)\gamma^2}} = \frac{A_0}{\sqrt{\omega_0^2\gamma^2 - \gamma^4/4}} = a_0\frac{\omega_0}{\gamma}\frac{1}{\sqrt{1 - \frac{\gamma^2}{4\omega_0^2}}}$$

3. Phase Lag

- Up to now we've focused on the amplitude of the driven oscillations: $a(\omega)$

- There is also physics in the phase lag:

$$\mathcal{X}(t) = a(\omega)\cos(\omega t - \delta); \qquad \tan\delta = \frac{\omega\gamma}{(\omega_0^2 - \omega^2)}$$

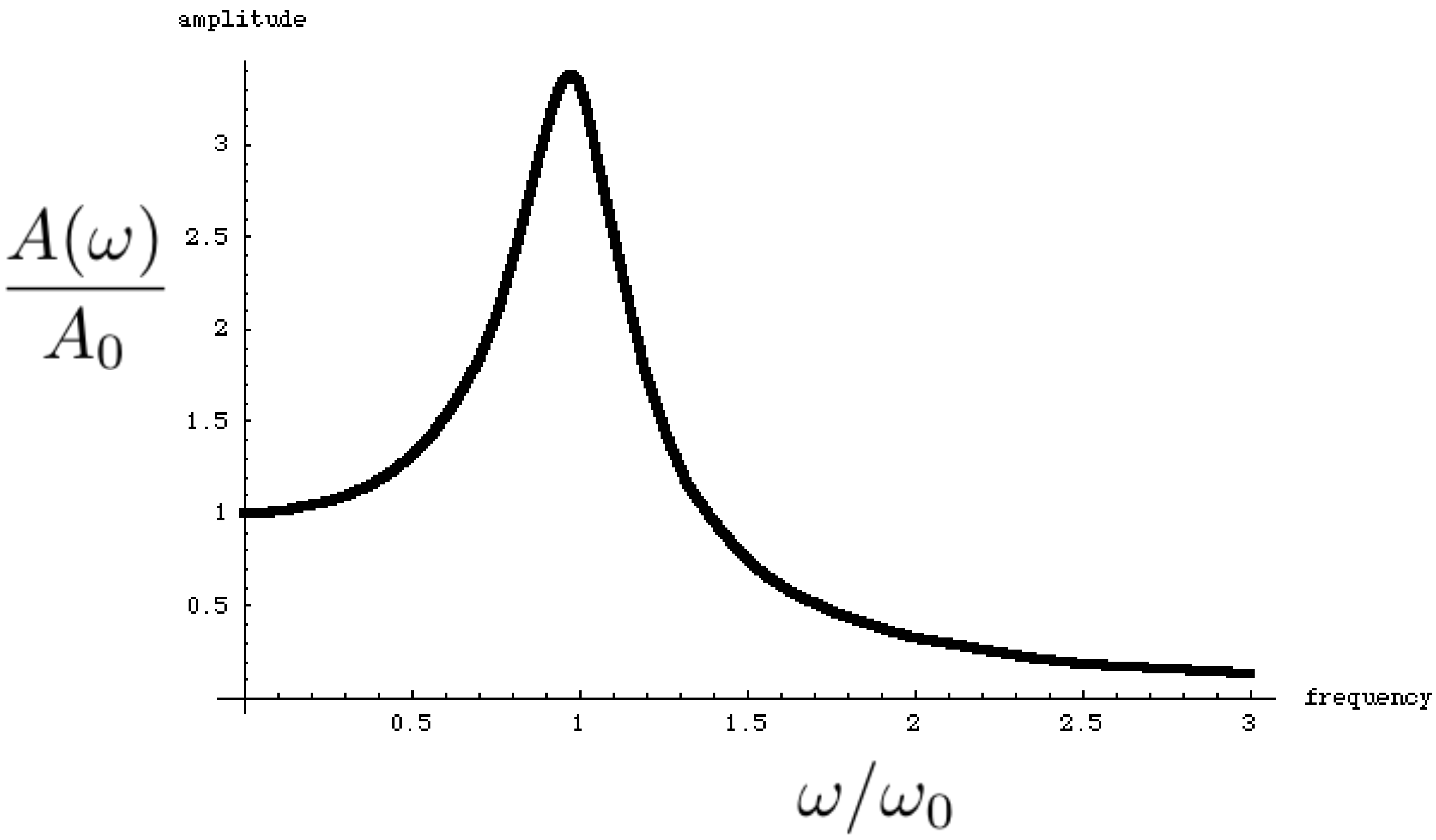
- The phase is a smooth connection between two regimes (see plots):
  (a) At low driving frequencies $\delta \to 0$
    - The mass and platform move in unison (rigid regime): $\mathcal{X}(t) = a_0\cos(\omega t)$ for $\omega \ll \omega_0$
    - In this regime $a(\omega) \to a_0 \frac{\omega_0^2}{\omega_0^2} = a_0$
    - The response of the system is determined by the stiffness of the spring

  (b) At high frequency

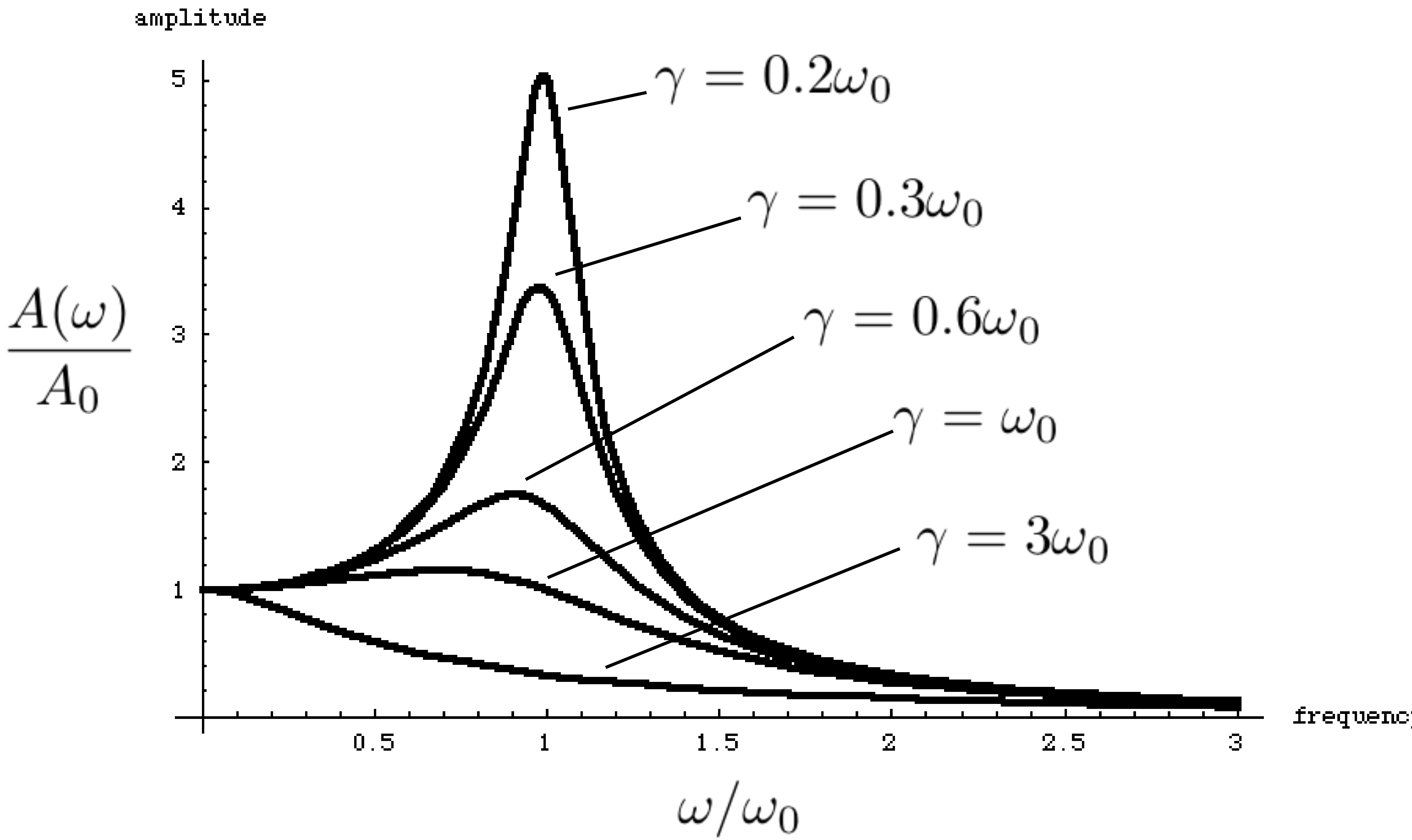
$$a(\omega_m) = \frac{A_0}{\sqrt{(\gamma^2/2)^2 + (\omega_0^2 - \gamma^2/2)\gamma^2}} \quad \xrightarrow[\omega \gg \omega_0]{} \quad a_0\frac{\omega_0^2}{\omega^2}$$

    - And the phase: $\tan\delta = \frac{\omega\gamma}{(\omega_0^2 - \omega^2)} \quad \xrightarrow[\omega \gg \omega_0]{} \quad \frac{-\gamma}{\omega}$
    - This gives $\delta \to 180°$
    - The mass lags $180°$ out of phase, ie. it moves in the opposite direction to the driving force
    - This system is inertia dominated (ie. dominated by the inertia of the mass)
    - The amplitude still falls off at high ω like ω^{-2} but it is not due to the damping

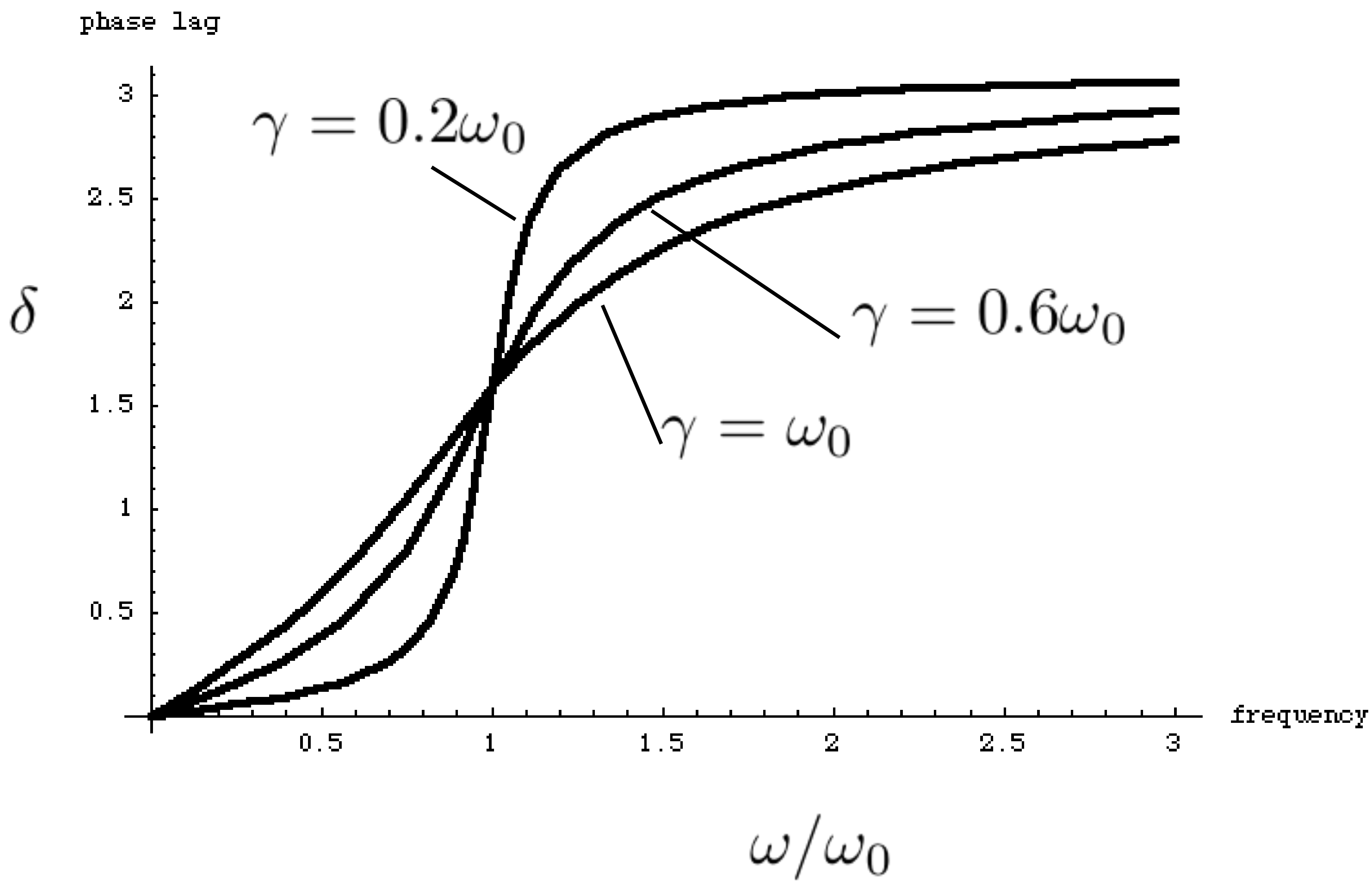
Amplitude response



Amplitude response

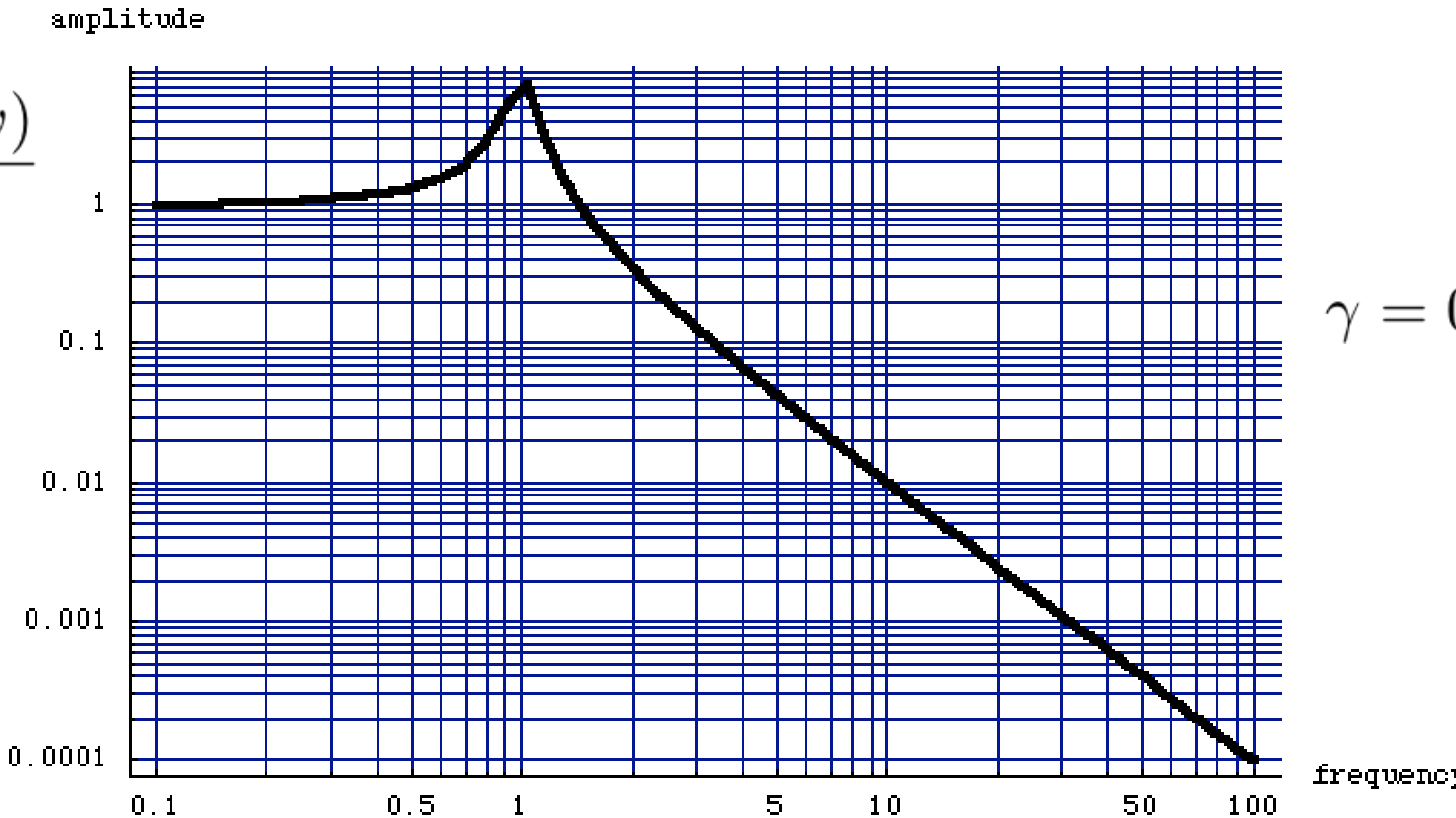


Phase lag



phase lag
3
2.5
2
1.5
1
0.5
δ
$\gamma = 0.2\omega_0$
$\gamma = 0.6\omega_0$
$\gamma = \omega_0$
0.5
1
1.5
2
2.5
3
frequency
ω/ω_0

Log-log plot of A



$$\frac{A(\omega)}{A_0}$$

$$\gamma = 0.1\omega_0$$

$$\omega/\omega_0$$