PHY293 Oscillations Lecture #5

September 20, 2010

4. Velocity Response

- There is also a resonance in velocity – mass moves faster than forced-end of spring
- This actually occurs at the natural frequency of the system

$$\dot{\mathcal{X}}(t) = -\omega a(\omega)\sin(\omega t - \delta)$$

- The velocity amplitude is:

$$V(\omega) = \frac{a_0\omega\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}} = \frac{a_0\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2/\omega^2+\gamma^2}}$$

- This is maximised when the denominator is minimised:

$$\frac{d}{d\omega}(\frac{(\omega_0^2-\omega^2)^2}{\omega^2}+\gamma^2) = 0$$

- But the γ^2 term just vanishes from the derivative
- Taking derivative of a ratio: $\frac{d}{dx}(\frac{A}{B}) = \frac{A'B-B'A}{B^2}$ gives:

$$2(\omega_0^2-\omega^2)2\omega(\omega^2) - (2\omega)(\omega_0^2-\omega^2)^2 = 0$$

- Gathering terms together this gives:

$$(\omega_0^2-\omega^2)[2\omega](2\omega^2-\omega_0^2-\omega^2) = 0$$

- There are three different families of roots here:

$$\omega = [0, \pm\omega_0, \pm i\omega_0]$$

- Of these only $\omega = \omega_0$ is positive and real
- We don't have a physical interpretation of negative frequencies, nor for imaginary ones (though these lead back to decaying exponentials rather than oscillations)
- $\omega = 0$ is, again, a local minimum at the velocity of the forcing point
- The velocity amplitude at this frequency is:

$$|\omega_0 a(\omega_0)| = \frac{a_0\omega_0\omega_0^2}{\sqrt{(\omega_0^2-\omega_0^2)^2+(\gamma\omega_0)^2}} = a_0\omega_0\frac{\omega_0}{\gamma}$$

- This simple result ($V_{\max}$ occurs at $\omega = \omega_0$ and the simple form of $V_{\max}$) often leads to the statement: "Resonance is a velocity phenomenon"
- Another way of thinking of this is that $\dot{\mathcal{X}}$ is the most natural variable to study resonant phenomena
- In terms of $V_{\max}$ the general velocity amplitude can be written as:

$$\dot{\mathcal{X}} = \frac{V_{\max}\gamma}{\sqrt{(\omega_0^2-\omega_0^2)^2/\omega^2+(\gamma)^2}}$$

- Where $V_{\max} = \frac{a_0\omega_0^2}{\gamma}$ [See plots of $V(\omega)$ discussed in class – appended below]

5. Power Absorbed by a Driven SHO

- In the steady-state (after complementary solutions have died out) the solution has a constant amplitude, $a(\omega)$.
- As with the free and free-damped oscillator the total energy will look something like: $E = \frac{1}{2}ka^2(\omega) =$ constant.
- Energy is still juggled between kinetic and potential energy inside, but the total stays constant
- So instead of looking at the energy, look at the energy flow (power) into/out-of the oscillator.

- The driving force delivers energy to the oscillator and then the damping takes energy back out of the oscillator
- The rate of energy flow is called **power** given by: $P = \vec{F} \cdot \vec{v}$
- Looking back at the SHO equation of motion we have:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = a_0\omega_0^2 \cos(\omega t)$$

- Power enters on RHS and leaves through the damping term on the LHS
- No power leaves/enters the oscillator through the $\ddot{x}$ or x terms.
- One way of looking at almost every oscillator is that it is just a mechanism for converting power from some driver to another dissipater (eg. gravitational energy from falling water drives turbines and this energy is, in turn, dissipated by the transformers generating current on the electrical grid).

6. Out-of-phase Oscillations

- Recall that we have the particular solution

$$\mathcal{X}(t) = a(\omega)\cos(\omega t - \delta)$$

- The solution actually came in two pieces:

$$G\cos(\omega t) + H\sin(\omega t) = a(\omega)\cos\delta\cos(\omega t) + a(\omega)\sin\delta\sin(\omega t)$$

- We called the "G" piece the **in-phase** part of the solution – since it is in-phase with the driving force.
- The "H" piece is the **out-of-phase** part of the solution
  - Compared to the driving force $\sin(\omega t + \pi/2)$ must be advanced by $\pi + 2$ to line up in phase with $\cos(\omega t)$.
- Now consider the instantaneous power the driver is delivering to the circuit

$$\text{Power} = (\text{Force of Driver}) \cdot (\text{Velocity of Mass})$$

$$\begin{aligned}
P_{\text{Drive}} &= F_{\text{Drive}} \cdot v_{\text{mass}} \\
&= F_0\cos(\omega t)[-a(\omega)\omega\sin(\omega t - \delta)] \\
&= -F_0 a(\omega)\omega\cos(\omega t)[\sin(\omega t)\cos\delta - \cos(\omega t)\sin\delta] \\
&= -F_0 a(\omega)\omega[\cos(\omega t)\sin(\omega t)\cos\delta - \cos^2(\omega t)\sin\delta]
\end{aligned}$$

- This is as far as we can get for general expression: understand more by averaging over a complete cycle
  - To do this recall $\int \cos^2\phi d\phi = 1/2$ while $\int \cos\phi\sin\phi d\phi = 0$
  - Note these are not integrals over time, but over ωt or complete cycles of the oscillator – does not change the dimensions
- This gives

$$< P_{\text{Drive}} >= \frac{1}{2}F_0 a(\omega)\omega[\sin\delta]$$

- We can simplify by looking back in our notes to find: $\sin\delta = \frac{\gamma\omega a(\omega)}{a_0\omega_0^2}$ to get

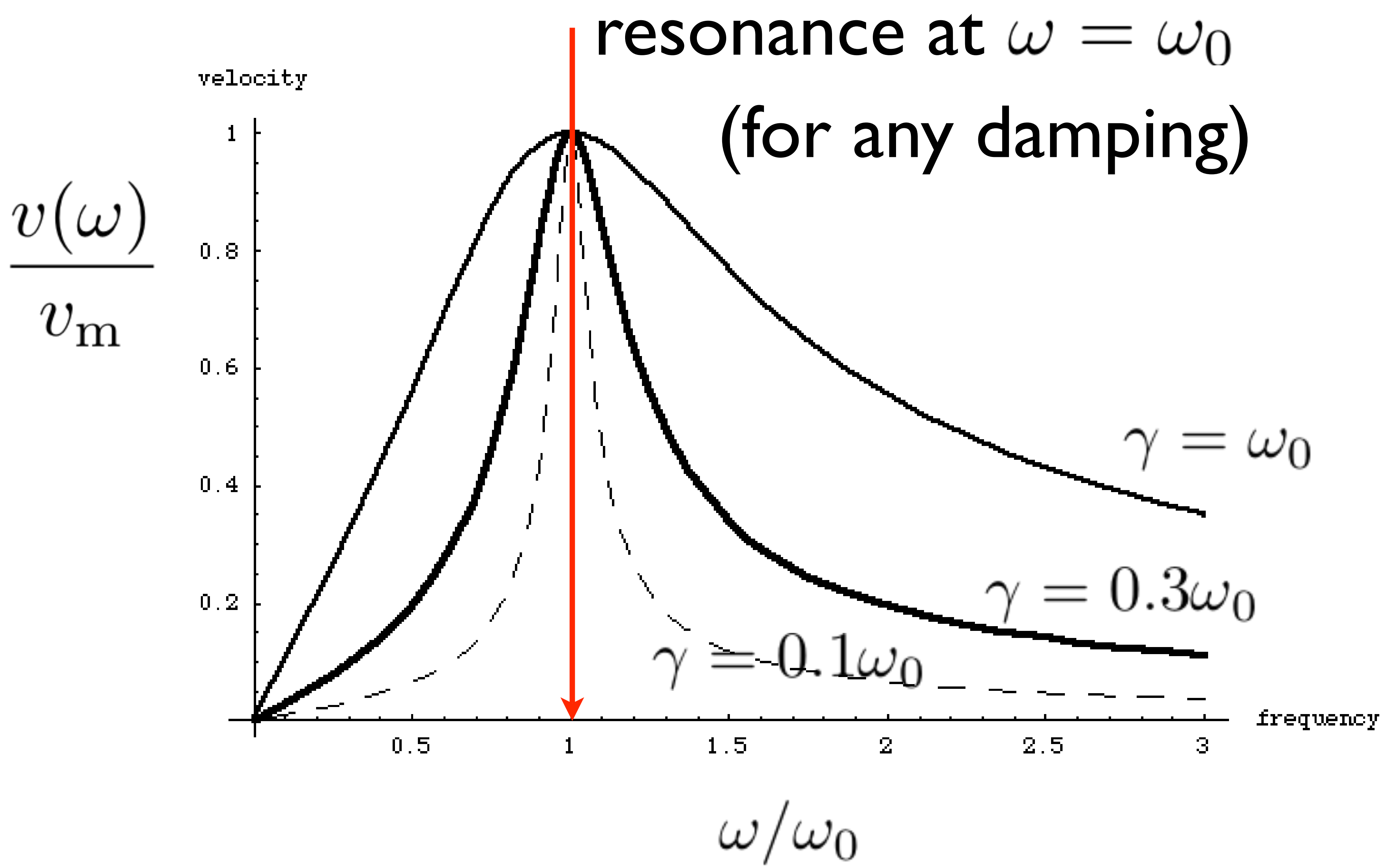
$$< P_{\text{Drive}} >= \frac{1}{2}F_0 a(\omega)\omega[\frac{\gamma\omega a(\omega)}{a_0\omega_0^2}] = \frac{1}{2}M\gamma[a(\omega)\omega]^2$$

- Where we used $F_0/M = a_0\omega_0^2$ from our coefficients on the RHS of the original driven SHO equation (two lectures ago)
- Summarising

$$< P_{\text{Drive}} >= - < P_{\text{Damp}} >= \frac{1}{2}M\gamma[a(\omega)\omega]^2$$

- Prove the relationship with damped power in next lecture
- Notice that the Power resonance occurs at ω_0
- So the maximum power is dissipated at $\omega = \omega_0$
  (a) $H(\omega)$ peaks at ω_0, so it is the out-of-phase component of our solution that is responsible for the power absorption in the oscillator.
  (b) $G(\omega)$ crosses through 0 [see plots attached] at $\omega = \omega_0$. It is out-of-phase with the driver at that point and thus can absorb no power.
- Since oscillators main roll is the transfer of energy or, in the case of suspension bridges avoiding the transfer of energy this is an important concept to understand.

Velocity response



Quadratures