PHY293 Oscillations Lecture #7

September 24, 2010

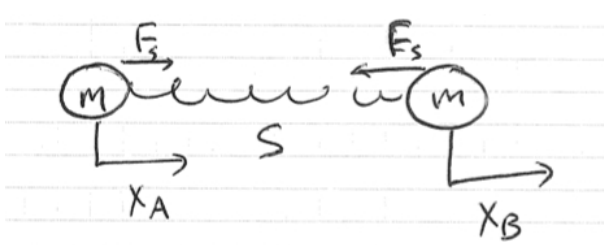
1. To better understand coupled oscillators I encourage you to have a look at:
   - http://faraday.physics.utoronto.ca/GeneralInterest/Harrison/Flash/ClassMechanics/CoupledSHM/CoupledSHM.html
   - http://www.walter-fendt.de/ph14e/cpendula.htm

Begin Lecture material

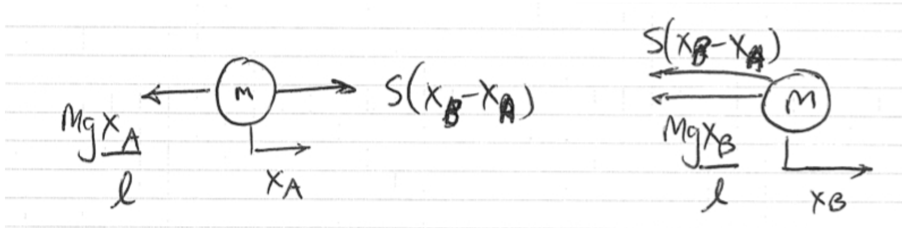
1. Coupled Oscillations
   - So far have considered systems that can only oscillate in **one way** – have a single natural frequency
   - Simple (only one variable and two degrees of freedom) Harmonic (linear restoring force $\Rightarrow$ linear differential equation) Oscillator (periodic motion)
   - In this next part of the course we'll consider more general harmonic oscillators $\Rightarrow$ more degrees of freedom
   - Real physical systems can vibrate in **many** ways, resonate at many frequencies
   - Each resonance or natural frequency corresponds to a normal mode
2. Equations of Motion for Coupled Oscillators
   - Consider a pair of pendula, coupled by a spring
   - Review equation of motion for a pendulum
     - Pendulum swings because of its gravitational potential energy
     - Mass moves on a circle of radius: l (where l is the length of the string)
     - Trajectory given by $y = \pm\sqrt{l^2 - x^2}$ where $+$ corresponds to top of the circle and $-$ corresponds to the bottom
     - Pendulum bob performs small oscillations near bottom of the circle

$$\begin{aligned}
U &= mgy \\
&= -mg\sqrt{l^2 - x^2} \\
&= -mgl\sqrt{1 - x^2/l^2} \\
&\approx -mgl(1 - \frac{x^2}{2l^2}) \qquad \text{for } x \ll l \\
m\ddot{x} &= \frac{-d}{dx}U(x) = \frac{-d}{dx}(1 - \frac{x^2}{2l^2})(-mgl) \\
&= \frac{x}{l^2}(-mgl) = -mgx/l \\
&\Rightarrow \ddot{x} + \frac{g}{l}x = 0
\end{aligned}$$

     - For a pendulum undergoing small oscillations about its minimum height
   - Now consider two pendula, side-by-side, joined by a spring



   - F_s pulls inward (or outward) on both masses: $F_x = k(x_B - x_A)$
   - Draw a free-body diagram for both masses, including both the spring force and the pendulum (gravity) force



- Apply Newton's law in each case:

$$m\ddot{x_A} = -mgx_A/l + k(x_B - x_A)$$
$$m\ddot{x_B} = -mgx_B/l - k(x_B - x_A)$$

- Divide through by m and bring terms to LHS

$$\ddot{x_A} + gx_A/l - k/m(x_B - x_A) = 0$$
$$\ddot{x_B} + gx_B/l + k/m(x_B - x_A) = 0$$

- Define $\omega_s^2 = k/m$ and $\omega_0^2 = g/l$ and collect like terms together

$$\ddot{x_A} + (\omega_0^2 + \omega_s^2)x_A - \omega_s^2 x_B = 0 \qquad [1]$$
$$\ddot{x_B} + (\omega_0^2 + \omega_s^2)x_B - \omega_s^2 x_A = 0 \qquad [2]$$

- This is a coupled system of linear, second order differential equations
- Come back to the general solution next time, but for now

3. Magic Guess Solution

- Try a substitution of the form $q_1 = x_A + x_B$ and $q_2 = x_A - x_B$
- Adding and subtracting [1] from [2] gives two new equations:

$$[1] + [2]: \quad \ddot{x_A} + \ddot{x_B} + (\omega_0^2 + \omega_s^2)(x_A + x_B) - \omega_s^2(x_B + x_A) = 0 \Rightarrow \ddot{q_1} + \omega_0^2 q_1 = 0$$
$$[1] - [2]: \quad \ddot{x_A} - \ddot{x_B} + (\omega_0^2 + \omega_s^2)(x_A - x_B) + \omega_s^2(x_A - x_B) = 0 \Rightarrow \ddot{q_2} + (\omega_0^2 + 2\omega_s^2)q_2 = 0$$

- But we already know the solutions to these equations:

$$q_1 = C\cos(\omega_0 t + \alpha) \qquad q_2 = D\cos(\sqrt{\omega_0^2 + 2\omega_s^2}t + \beta)$$

- Where C, D, α, β are constants to be determined by initial conditions
- Note: we are ignoring damping and there is no forcing, so these are the complementary solutions that don't die away in time
- Original formulation of the problem was in terms of x_A and x_B, the positions of the two pendulum bobs
- Have now solved the problem for q_1 and q_2, what do x_A and x_B look like?

$$q_1 = x_A + x_B; q_2 = x_A - x_B \quad \Rightarrow \quad x_A = \frac{1}{2}(q_1 + q_2); x_B = \frac{1}{2}(q_1 - q_2)$$

- So we get solutions that look like:

$$x_{A,B} = \frac{1}{2}(q_1 \pm q_2) = \frac{1}{2}[C\cos(\omega_0 t + \alpha) \pm D\cos(\sqrt{\omega_0^2 + 2\omega_s^2}t + \beta)]$$

- Consider the special case $\alpha = \beta = 0$ and $C = D$ in the first lecture I said this wrong, I said $D = 0$, but I MEAN'T $C = D$, sorry.

- This corresponds to $x_B(0) = 0$ and $X_A(0) = C$, solution looks like

$$x_A = \frac{1}{2}C[\cos(\omega_0 t) + \cos(\sqrt{\omega_0^2 + 2\omega_s^2}t)]$$
$$= C\cos(\frac{\omega_0 + \sqrt{\omega_0^2 + 2\omega_s^2}}{2}t)\cos(\frac{\omega_0^2 - \sqrt{\omega_0^2 + 2\omega_s^2}}{2}t)]$$

- This last step uses a not-so-obvious trig identity:

$$\cos(\theta+\phi)\cos(\theta-\phi) = [\cos\theta\cos\phi + \sin\theta\sin\phi][\cos\theta\cos\phi - \sin\theta\sin\phi]$$
$$= \cos^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi + (\text{cross terms cancel})$$
$$= \frac{1}{4}(1+\cos(2\theta))(1+\cos(2\phi)) - \frac{1}{4}(1-\cos(2\theta))(1-\cos(2\phi))$$

- Expand but find 1 and $\cos^2$ terms all cancel leaving only

$$\cos(\theta+\phi)\cos(\theta-\phi) = \frac{1}{2}(\cos 2\theta + \cos 2\phi)$$

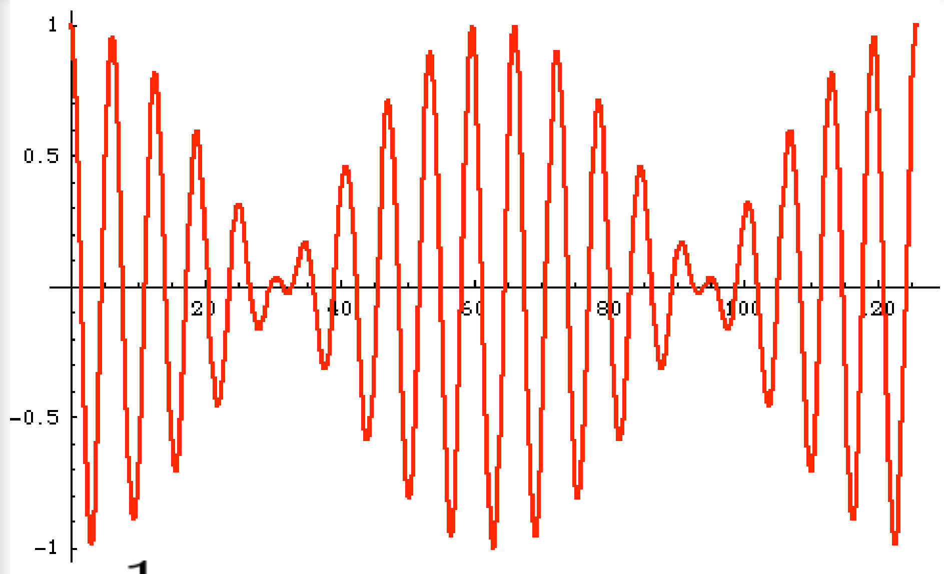
- So the sum of two $\cos$ can be transformed into a product of two related $\cos$
- Have a look at the product of the two $\cos$ terms we have
  - One has a higher frequency – oscillates fast – like the pendula swinging
  - The other has a lower frequency – the rate at which the system changes from one-pendulum swinging to the other
  - See plots shown in class
- This is classic "beat" behaviour; two nearby frequencies combine to give distinctive oscillation pattern
- Only gives distinctive "beat patterns" when the two frequencies are "close" together (5-10%)

Beats

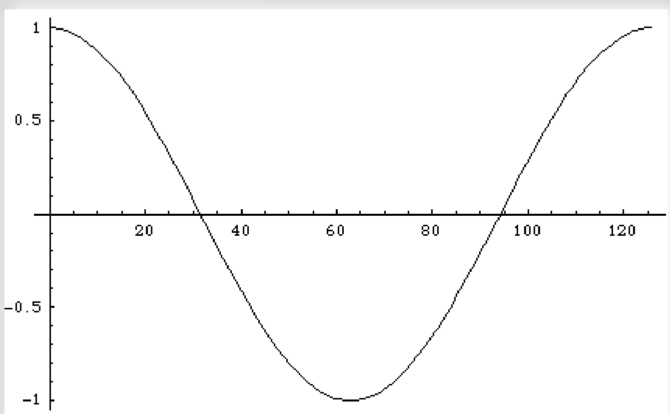
Example for

$$\omega_1 = 1.05$$

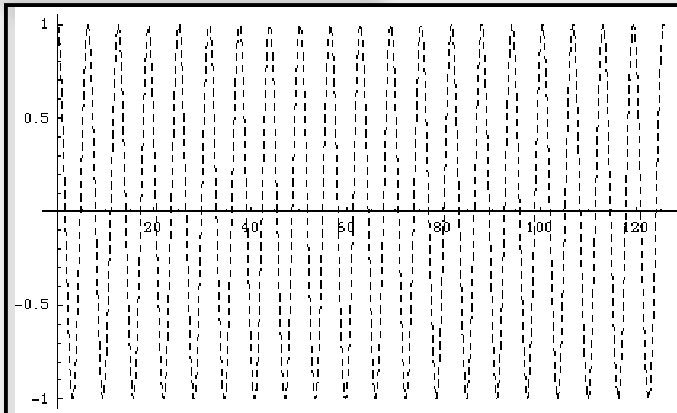
$$\omega_2 = 0.95$$



$$\frac{1}{2}\left[\cos\omega_1 t + \cos\omega_2 t\right]$$



$$\cos\frac{\omega_1 - \omega_2}{2}t$$



$$\cos\frac{\omega_1 + \omega_2}{2}t$$

Beats: envelope and phase