PHY293 Oscillations Lecture #8

September 27, 2010

1. Went back several times throughout this lecture to:
   - http://faraday.physics.utoronto.ca/GeneralInterest/Harrison/Flash/ClassMechanics/CoupledSHM/CoupledSHM.html
   - http://www.walter-fendt.de/ph14e/cpendula.htm

Begin Lecture material

1. General solution of Coupled Oscillator Problem
   - Need a general method to solve problems where guessing isn't so obvious
     - Cases where the pendulum bobs have different masses,
     - Where the pendulum lengths are different
     - Or where we have more degrees of freedom
   - Instead of *knowing* the answer, use the tools of linear algebra to find it
2. Equations of motion:

$$m_A \ddot{x_A} + (m_A g/l + k)x_A - kx_B = 0$$
$$m_B \ddot{x_B} + (M_B g/l + k)x_B - kx_A = 0$$

   - Can write these in a vector form using a **state vector**: $\vec{x} = \begin{bmatrix} x_A \\ x_B \end{bmatrix}$
   - Note that these are both positions along the x-axis, just a notation that keeps track of the *state* of the system
   - Using this notation we can re-write the equations of motion as:

$$\begin{bmatrix} m_A & 0 \\ 0 & m_B \end{bmatrix} \frac{d^2}{dt^2} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + \begin{bmatrix} m_A g/l + k & -k \\ -k & m_B g/l + k \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = 0$$

   - Defining $[M] = \begin{bmatrix} m_A & 0 \\ 0 & m_B \end{bmatrix}$ as the **Mass matrix**
   - And $[K] = \begin{bmatrix} m_A g/l + k & -k \\ -k & m_B g/l + k \end{bmatrix}$ as the **Spring matrix** – generalised to include all restoring forces – get:

$$[M]\ddot{\vec{x}} + [K]\vec{x} = 0$$

   - This alot like $m\ddot{x} + kx = 0$ from SHO in first week of course
   - Solved that equation by:
     (a) Assuming solutions of the form $x = C\cos(\omega t + \delta)$
     (b) Substitute into the equation of motion to find ω_0
     (c) Using the initial conditions to determine C, δ
   - Do the same again here, but matrices ⇒ we'll need some linear algebra, rather than just a simple division, to find frequencies

$$m\ddot{x} + kx = 0 \longrightarrow -m\omega_0^2 C\cos(\omega_0 t + \delta) + kC\cos(\omega_0 t + \delta) = 0$$
$$(-m\omega_0^2 + k)C\cos(\omega_0 t + \delta) = 0$$

   - Has the trivial solution $C = 0$ – we ignored this one
   - The dynamic solution has $\omega_0^2 = k/m$ ... and we never looked back
   - Try a similar approach for our new equations
     (a) Assume $\vec{x} = C\vec{\xi}\cos(\omega t + \phi)$ so $\ddot{\vec{x}} = -\omega^2 \vec{x}$

(b) Substituted into equations of motion:

$$[M]\ddot{\vec{x}} + [K]\vec{x} = 0$$
$$-\omega^2[M]\vec{x} + [K]\vec{x} = 0$$
$$(-\omega^2[M] + [K])\vec{x} = 0$$

- This equation represents two simultaneous, coupled, equations in two unknowns
- Has the trivial solution again $\vec{x} = 0$ – nothing moves
- To find the non-trivial solutions we need to take the determinant of the coefficient matrix and set it equal to 0

$$\det(-\omega^2[M] + [K]) = 0$$

- This gives the **characteristic equation** or frequency equation for this coupled system
- Will give us constraints for the **normal frequencies** ω_1 and ω_2
- These are the two natural frequencies of this coupled system, or the **eigenfrequencies**

(c) Now determine the nature of the motion associated with each eignfrequnecy
- This is a step that was not needed for the SHO case
- At ω_1 we'll have $\vec{x} = C_1\vec{\xi}_1 \cos(\omega_1 t + \phi_1)$
- At ω_2 we'll have $\vec{x} = C_2\vec{\xi}_2 \cos(\omega_2 t + \phi_2)$
- Knowing ω_1 and ω_2 we can substitute, one at a time, into the equations of motion to find $\vec{\xi}_1$ and $\vec{\xi}_2$
- These will be the eigenmodes or eigenvectors associated with each eigenfrequency

(d) The general solution is then the sum of all harmonic responses

$$\vec{x}(t) = C_1\vec{\xi}_1 \cos(\omega_1 t + \phi_1) + C_2\vec{\xi}_2 \cos(\omega_2 t + \phi_2)$$

- Where $C_{1,2}$ and $\phi_{1,2}$ are determined from the initial conditions

- Try this new formalism on our simple system (equal masses) that we guessed the solution for in the previous lecture

$$[M] = \begin{bmatrix} m_A & 0 \\ 0 & m_B \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad [K] = \begin{bmatrix} m_A g/l + k & -k \\ -k & m_B g/l + k \end{bmatrix} = \begin{bmatrix} mg/l + k & -k \\ -k & mg/l + k \end{bmatrix}$$

(a) Assume harmonic response $[M]\ddot{\vec{x}} = -\omega^2[M]\vec{x}$
- Makes the equations of motion $-\omega^2[M] + [K])\vec{x} = 0$

(b) To solve this need the determinant: $\det|-\omega^2[M] + [K]| = 0$

$$\det\left|-\omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} + \begin{bmatrix} mg/l + k & -k \\ -k & mg/l + k \end{bmatrix}\right| = 0$$
$$\det\begin{vmatrix} -\omega^2 m + mg/l + k & -k \\ -k & -\omega^2 m + mg/l + k \end{vmatrix} = 0$$
$$(-\omega^2 m + mg/l + k)^2 - k^2 = 0$$

- This is the eigenfrequency equation and it has solutions for $-\omega^2 + mg/l + k = \pm k$
- Thus we have two equations for ω^2 (one for the $+$ root and the other for the $-$ root) giving:

$$-\omega_1^2 m + mg/l = 0 \quad \text{or} \quad \omega_1^2 = g/l \quad \text{for } + \text{ root}$$
$$-\omega_2^2 m + mg/l + 2k = 0 \quad \text{or} \quad \omega_2^2 = g/l + 2k/m \quad \text{for } - \text{ root}$$

- These are just the frequencies we found from our guessed solutions in last lecture
- But this method would have given us answers even if $m_A \neq m_B$ or $l_A \neq l_B$ etc.

3. Normal Modes

- We have shown that our linear algebra formalism recovers the same natural frequencies as our guessed solution
- Have not determined what *kind* of motion is associated with each frequency
  - These will be the normal modes of the system
  - They are related to the q_i variables we guessed for our solution in the last lecture
- In the first mode the two pendula swing together
- $\omega_1 = \sqrt{g/l}$ doesn't depend on spring strength $\rightarrow$ spring length doesn't change

$$\vec{x} = \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} C_1 \cos(\omega_1 t + \phi_1)$$

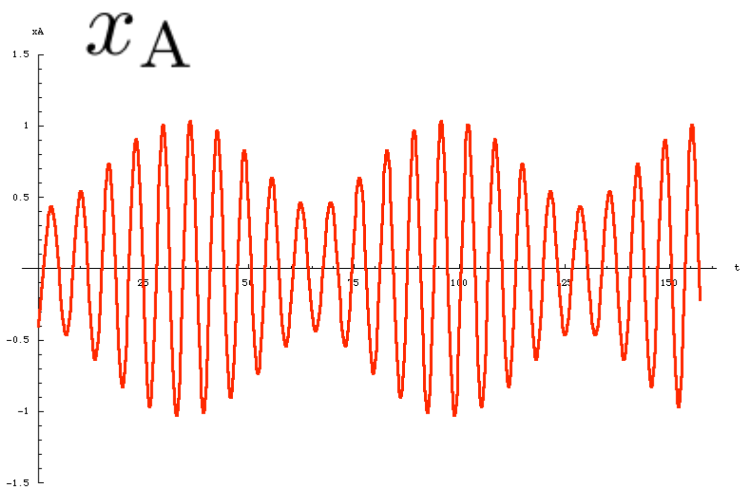
- Notice that with this result for $\vec{\xi}$ if the first bob moves 1 cm to the left the second one also moves 1 cm to the left.
- The second mode has the two pendula opposing one another
- $\omega_2 = \sqrt{g/l + 2k/m}$ a higher frequency where the spring is doubly extended for each movement of one pendulum bob

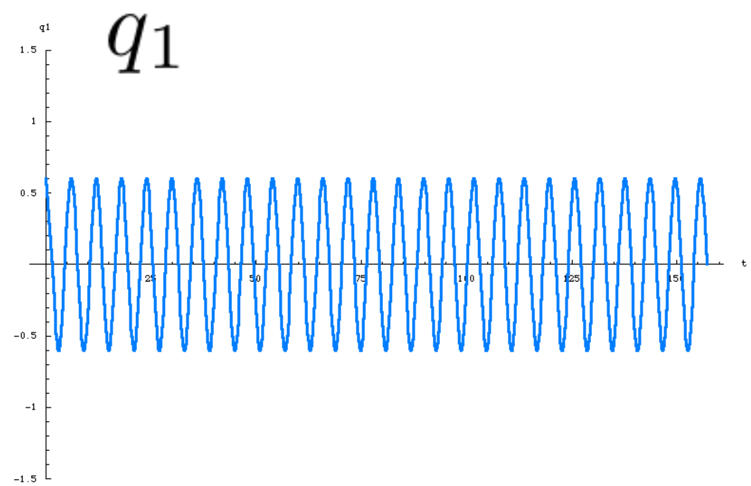
$$\vec{x} = \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} C_2 \cos(\omega_1 t + \phi_2)$$

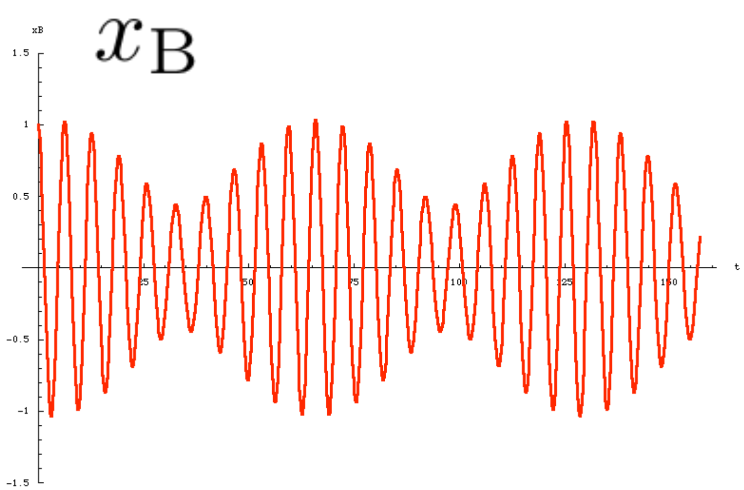
- This solution is anti-symmetric: If the left bob moves 1 cm to the right then the right-one moves 1 cm to the left
- General solution is just sum of these two modes (with arbitrary amplitudes ($C_{1,2}$) and possibly different phases ($\phi_{1,2}$))
- Superposition works because equations of motion are linear, which is a result of only considering linear restoring force(s))
- Normal modes are special because
  - They are associated with a single eigenfrequency, or natural frequency
  - They are orthogonal to one another, which simplifies calculations (energy, power, etc.)
  - An excitation in one mode **will not** transfer to another mode, because energy is **not** transferred between modes
  - They are a complete set, capable of describing all excitations of a system

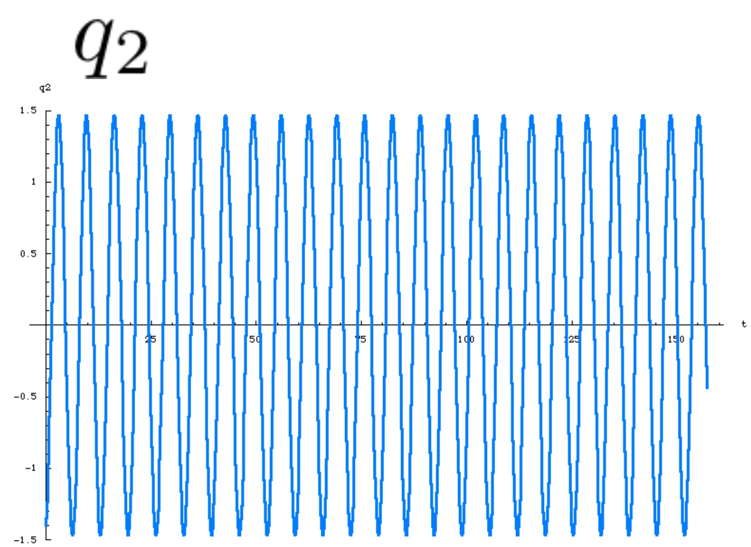
mass vs. normal coordinates

(both normal modes excited)









mass vs. normal coordinates

(one normal mode excited)