PHY293 Oscillations Lecture #10

September 28, 2010

Begin Lecture material

1. Finding Normal Modes

   - Found last time that the coupled equations of motion: $[M]\ddot{\vec{x}} + [K]\vec{x} = 0$
   - Could be solved, assuming harmonic solutions giving eigen-frequencies: $\omega_1^2 = g/l \qquad \omega_2^2 = g/l + 2k/m$
   - Now find the normal modes by substituting these frequencies (one at a time) into the equations of motion:

   $$(-\omega_1^2[M] + [K])\vec{\xi}_1 = 0$$

   - Can plug in the actual values for the matrices (see last lecture) to get:

   $$\begin{bmatrix} -mg/l + mg/l + k & -k \\ -k & -mg/l + mg/l + k \end{bmatrix} \begin{bmatrix} \xi_1^a \\ \xi_1^b \end{bmatrix} = 0$$

   - Which simplifies to: $\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} \xi_1^a \\ \xi_1^b \end{bmatrix} = 0$
   - The solution to both of these relations is $\xi_1^a = \xi_1^b$.
   - This describes the motion of both pendulum bobs moving together
   - A simple representation of this is $\vec{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
   - Still have the freedom to set the amplitude (with C_1) from the initial conditions

   $$\vec{x} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t + \phi_1)$$

   - Find the other mode by substituting $\omega_2 \qquad (-\omega_2^2[M] + [K])\vec{\xi}_2 = 0$
   - Again plug in the actual values for the matrices to get:

   $$\begin{bmatrix} -mg/l - 2k + mg/l + k & -k \\ -k & -mg/l + 2k + mg/l + k \end{bmatrix} \begin{bmatrix} \xi_2^a \\ \xi_2^b \end{bmatrix} = 0$$

   - Which simplifies to: $\begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{bmatrix} \xi_2^a \\ \xi_2^b \end{bmatrix} = 0$
   - The solution to both of these relations is $\xi_2^a = -\xi_2^b$.
   - This describes the motion of the pendulum bobs moving opposite one another
   - A simple representation of this is $\vec{\xi}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
   - The full solution for the second normal mode is: $\vec{x} = C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t + \phi_2)$

2. Orthogonality and Projections

   - We want to show that normal modes are orthogonal
   - Remind ourselves how to write a vector sum out of two orthogonal vectors
   - Express an arbitrarily complicated (allowed!) motion as the sum of two orthogonal modes
   - We *won't* explain why normal modes must be orthogonal (back to algebra notes for that)
   - For $\vec{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{\xi}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ we not that $\vec{\xi}_1 \cdot \vec{\xi}_2 = 0$
   - This will be a general property of our solutions that: $\vec{\xi}_i[M]\vec{\xi}_j = 0 \quad \text{for } i \neq j$

- This is also true for $[K]$
- Arises because these are the eigenvectors of a real symmetric matrix
- Also the eigenvalues will be positive so the frequencies are real
- Projections: We know that if $\vec{a} \perp \vec{b}$ then $\vec{a} \cdot \vec{b} = 0$
- This allows us to write an arbitrary vector $\vec{u} = u_a\vec{a} + u_b\vec{b}$
- We can find the coefficients of the projection by taking the inner product

$$\vec{a} \cdot \vec{u} = u_a\vec{a} \cdot \vec{a} + u_b\vec{a} \cdot \vec{b}$$

- But the second term is 0 and the first term is the norm of $\vec{a}$ which we'll write as $||\vec{a}||^2$ thus we get $u_a = \frac{\vec{a} \cdot \vec{u}}{||\vec{a}||^2}$
- For example: $\vec{a} = (1, 2)$ and $\vec{b} = (2, -1)$ then $\vec{u} = (3, 2)$ can be written as:

$$u_a = 1/5(1, 2) \cdot (3, 2) = 7/5 \quad \text{and} \quad u_b = 1/5(2, -1) \cdot (3, 2) = 4/5$$

- Giving $\vec{u} = 1.4\vec{a} + 0.8\vec{b}$, the unique decomposition in this basis
- Also do this implicitly every time we work in $\hat{x} - \hat{y}$ coordinates

$$\vec{u} = (5, 4) \Rightarrow \hat{x} \cdot \vec{u} = 5 \text{ and } \hat{y} \cdot \vec{u} = 4 \Rightarrow \vec{u} = 5\hat{x} + 4\hat{y}$$

- This is simpler because, in addition to being orthogonal $(\hat{x}, \hat{y})$ are also unit vectors – this is an orthonormal basis
- The lesson is that it can be convenient to have normalised eigenvectors when we want to make projections
- In our simple example we found: $\omega_1^2 = g/l; \omega_2^2 = g/l + 2k/m \Rightarrow \vec{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \vec{\xi}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- With $[M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$ this gives: $\vec{\xi}_1[M]\vec{\xi}_2 = (1, 1)\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = m(1, 1)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$
- Our eigenvectors are orthogonal

3. Solving a more complicated example
   - Lets have a look at $m_B = 2m_A \equiv 2m$
   - The characteristic equation is still:

$$[M]\ddot{\vec{x}} + [K]\vec{x} = 0$$

   - To get the characteristic frequencies we'll need to solve:

$$\det\left|-\omega^2\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} + \begin{bmatrix} mg/l + k & -k \\ -k & 2mg/l + k \end{bmatrix}\right| = 0$$
$$\det\begin{vmatrix} -\omega^2 m + mg/l + k & -k \\ -k & -2\omega^2 m + 2mg/l + k \end{vmatrix} = 0$$
$$\det\begin{vmatrix} \omega^2 - \omega_0^2 - k/m & k/m \\ k/m & 2\omega^2 - 2\omega_0^2 - k/m \end{vmatrix} = 0$$

   - Where in the last step we've defined $\omega_0^2 = g/l$ and divided each term through by $-m$
   - Expanding the determinant we get:

$$(\omega^2 - \omega_0^2 - k/m)(2\omega^2 - 2\omega_0^2 - k/m) - (k/m)^2 = 0$$

   - Expand and gather powers of ω^2 (the characteristic frequencies we are solving for)

$$\omega^4 - \omega^2(2\omega_0^2 + 1.5k/m) + (\omega_0^4 + 1.5\omega_0^2 k/m) = 0$$

- This is a quadratic equation for ω^2 with solutions:

$$\begin{aligned}\omega^2 &= \frac{1}{2}\left[(2\omega_0^2 + 1.5k/m) \pm \sqrt{(2\omega_0^2 + 1.5k/m)^2 - 4(\omega_0^4 + 1.5\omega_0^2 k/m)}\right] \\ &= \omega_0^2 + 3/4k/m \pm \frac{1}{2}\sqrt{9/4k^2/m^2} \\ &= \omega_0^2 + 0.75k/m \pm 0.75k/m\end{aligned}$$

- This minor arithmetic miracle that leads to $\omega_1 = \omega_0 = g/l$ and $\omega_2 = \sqrt{\omega_0^2 + 1.5k/m}$
- Not too different from the equal mass case
- Now find the corresponding eigenvectors. For $\omega_1 = g/l$ we have:

$$(-\omega_1^2[M] + [K])\vec{\xi_1} = 0$$

$$\begin{bmatrix} -m\omega_0^2 + m\omega_0^2 + k & -k \\ -k & -2m\omega_0^2 + 2\omega_0^2 + k \end{bmatrix} \begin{bmatrix} \xi_1^a \\ \xi_1^b \end{bmatrix} = 0$$

- This gives the two equations:

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} \xi_1^a \\ \xi_1^b \end{bmatrix} = 0$$

- Which again gives an eigenvector $\vec{\xi_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where both pendulum bobs swing together – the spring doesn't change length
- To find the second eigenvector that has $\omega_2^2 = g/l + 1.5k/m$ we have:

$$(-\omega_2^2[M] + [K])\vec{\xi_2} = 0$$

$$\begin{bmatrix} -m\omega_0^2 - 1.5k/m + m\omega_0^2 + k & -k \\ -k & -2m\omega_0^2 - 3k + 2\omega_0^2 + k \end{bmatrix} \begin{bmatrix} \xi_2^a \\ \xi_2^b \end{bmatrix} = 0$$

- This gives the two equations:

$$\begin{bmatrix} -0.5k & -k \\ -k & -2k \end{bmatrix} \begin{bmatrix} \xi_2^a \\ \xi_2^b \end{bmatrix} = 0$$

- The solution to both of these equations is $\xi_2^a = -2\xi_2^b$ so one representation of the second eigenvector is $\vec{\xi_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
- The mass difference between the pendulum bobs has produced a more interesting solution
- We still have a normal mode where the two bobs swing opposite one another, but in this case the light mass moves **twice** as far, in the opposite direction, as the heavy mass. When the two bobs are heading towards (or away) from each other at their neutral point (fastest speed) they will have equal magnitudes of momentum ("2m" moving at v while "m" is moving - $-2v$). In one class I said equal but opposite average momentum, but in fact it is equal magnitudes of 'peak momentum', when they are moving fastest.
- The complete solution becomes:

$$\vec{x}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_0 t + \phi_1) + C_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \cos(\sqrt{\omega_0^2 + 1.5k/m}t + \phi_2)$$

- Notice that the eigenvectors are orthogonal:

$$\vec{\xi_1}[M]\vec{\xi_2} = (1,1)m \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = m(1,1) \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 0$$

- Note that we have to use $[M]$ in the inner product as this carries the information that one bob is twice as massive as the other
- Could also show $\vec{\xi_1}[K]\vec{\xi_2} = 0$

- However:

$$\vec{\xi}_1[M]\vec{\xi}_1 = [1,1]m \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = m[1,1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3m$$
$$\vec{\xi}_2[M]\vec{\xi}_2 = [2,-1]m \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = m[2,-1] \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 6m$$

- So these eigenvectors are **not** normalised (viz $\vec{\xi}_1 = \frac{1}{\sqrt{3m}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{\xi}_2 = \frac{1}{\sqrt{6m}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ are normalised).