# PHY293 Oscillations Lecture #10

October 1, 2010

1. Midterm is Thursday morning at 9:30 in McCaul examination centre

Begin Lecture material

1. Initial Value Problems
   - Up to now we've studied how to find normal frequencies and their corresponding normal modes, or eigenvectors ($\vec{\xi}_1, \vec{\xi}_2$)
   - If we had 3 masses on springs then the state vector would become:

$$\vec{x} = \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix}$$

   - The mass and spring matrices would be 3x3
   - The characteristic equation would give three normal frequencies and there would be three, $1 \times 3$ element, eigenvectors.
   - For $N$ masses we'd get a solution like:

$$\vec{x}(t) = \Sigma_{n=1}^{N} C_n \vec{\xi}_n \cos(\omega_n t + \phi_n)$$

   - where the $\vec{\xi}_n$ would have $N$ components
   - Instead of writing $C_n \vec{\xi}_n \cos(\omega_n t + \phi_n)$ could expand back to:

$$A_n \vec{\xi}_n \sin(\omega_n t) + B_n \vec{\xi}_n \cos(\omega_n t)$$

   - Note that these are equivalent because:

$$C_n \vec{\xi}_n \cos(\omega_n + \phi_n) = C_N \vec{\xi}_n \cos(\omega_n t)\cos(\phi_n) - C_n \vec{\xi}_n \sin(\omega_n t)\sin(\phi_n)$$

   - in this equation we can just make the association $A_n = -C_n \sin(\phi_n)$ and $B_n = C_n \cos(\phi_n)$ to get from the $C_n, \phi_n$ initial value constants to the $A_n, B_n$ w$A_n, B_n$ constants
   - You might be wondering "Why go to all this trouble?"
   - The answer lies in another question: "How do we determine $C_n, \phi_n$ or $A_n, B_n$ from the initial conditions?"
   - We'll see that the $A_n, B_n$ are easier to determine
   - Usually given initial conditions in the form $\vec{x}(t=0) = \vec{x}_0$ and $\vec{v}(t=0) = \dot{\vec{x}}(t=0) = \vec{v}_0$
   - For $t = 0$ the $A_n, B_n$ form of the solution looks like:

$$\vec{x}(0) = \Sigma_{n=1}^{N} [A_n \vec{\xi}_n \sin(0) + B_n \vec{\xi}_n \cos(0)]$$

   - But the first terms in this sum all vanish ($\sin(0) = 0$) and the $\cos$ terms in the second part of the sum are all 1 ($cos(0) = 1$) so we are left with

$$\vec{x}(0) = \Sigma_{n=1}^{N} B_n \vec{\xi}_n$$

   - Similarly we have:

$$\begin{aligned} \vec{x}(0) &= \Sigma_{n=1}^{N} [\omega_n A_n \vec{\xi}_n \cos(0) - B_n \omega_n \vec{\xi}_n \sin(0)] \\ &= \Sigma_{n=1}^{N} \omega_n A_n \vec{\xi}_n \end{aligned}$$

   - Given our initial conditions vectors we are left with:

$$\vec{x}_0 = \Sigma_{n=1}^{N} B_n \vec{\xi}_n \qquad \text{and} \qquad \vec{v}_0 = \Sigma_{n=1}^{N} \omega_n A_n \vec{\xi}_n$$

   - So this form of the solution where the $\sin$ and $\cos$ are written explicitly leads to decoupled equations for the initial positions and velocities of all the elements in the coupled oscillator system
   - For $N$ masses we'll have two equations for $N$-dimensional vectors (the eigenvectors are $N$ dimensional

- We have $2N$ unknowns (the $A_i$ and $B_i$) but $2N$ constraints so we can solve, in principle
- In fact the orthogonality of the eigenvectors makes it straightforward to do this in practice as well
- Consider:

$$\vec{\xi}_1 \cdot \vec{x}_0 = \vec{\xi}_1 \cdot \Sigma_{n=1}^{N} B_n \vec{\xi}_n = \Sigma_{n=1}^{N} B_n \vec{\xi}_1 \cdot \vec{\xi}_n$$

- But the RHS vanishes unless $n = 1$ because $\vec{\xi}_1$ is orthogonal to all the other eigenvectors.
- Turns out I may have dropped a sum too soon in this derivation for one of the classes. If you have any questions please ask.
- So this simplifies to:

$$B_1 = \vec{\xi}_1 \cdot \vec{x}_0 / ||\vec{\xi}_1||^2 = \vec{\xi}_1 [M] \vec{x}_0 / ||\vec{\xi}_1||^2$$

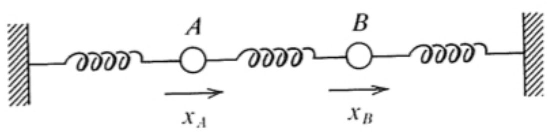
- Similarly can show:

$$A_1 = \frac{1}{\omega_1} \vec{\xi}_1 \cdot \vec{v}_0 / ||\vec{\xi}_1||^2 = \frac{1}{\omega_1} \vec{\xi}_1 [M] \vec{v}_0 / ||\vec{\xi}_1||^2$$

- In fact can repeat this for all $n$ (ie. dotting in all eigenvectors, one at a time) to show:

$$A_n = \frac{1}{\omega_n} \vec{\xi}_n \cdot \vec{v}_0 / ||\vec{\xi}_n||^2 \qquad \text{and} \qquad B_n = \vec{\xi}_n \cdot \vec{x}_0 / ||\vec{\xi}_n||^2 = \vec{\xi}_n [M] \vec{x}_0 / ||\vec{\xi}_n||^2$$

2. Example of Initial Value machinery in action
   - Consider a system of two masses $m$ on left (A) and $2m$ (B) on right
   - They are joined to fixed walls and each other with identical springs with stiffness $k$




   - Find
     (a) Natural frequencies
     (b) Normal modes/eigenvectors
     (c) Full solution for $\vec{v}_0 = (0, 0)$ and $\vec{x}_0 = (1, 0)$; that is both masses stationary at $t = 0$, the right-most one at its point of equilibrium and the left-most one displaced, to the right, by 1 m. For simplicity we'll take $s = 1$ N/m and $m = 1$ kg
   - Find the full spring and mass matrices by looking at free-body diagrams for each mass separately
   - These give:

$$\begin{aligned} m\ddot{x_A} &= -kx_A + k(x_B - x_A) \\ 2m\ddot{x_B} &= -k(x_B - x_A) - kx_B \end{aligned}$$

   - Which we can re-arrange on the LHS to:

$$\begin{aligned} m\ddot{x_A} + 2kx_A - kx_B &= 0 \\ 2m\ddot{x_B} + 2kx_B - kx_A &= 0 \end{aligned}$$

   - We can re-write these equations of motion in matrix form as:

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \ddot{\vec{x}} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \vec{x} = 0$$

   - Find the natural frequencies by assuming harmonic solutions $\ddot{\vec{x}} = -\omega^2 \vec{x}$ and finding determinant

$$\begin{aligned} \det \left| -\omega^2 \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \right| &= 0 \\ \det \begin{vmatrix} 2k - \omega^2 m & -k \\ -k & 2k - 2\omega^2 m \end{vmatrix} &= 0 \\ (2k - m\omega^2)(2k - 2m\omega^2) - (-k)^2 &= 0 \\ (2m^2)\omega^4 - (6km)\omega^2 + 3k^2 &= 0 \end{aligned}$$

- Once again we have a quadratic equations for $\omega^2$ which has solutions

$$\begin{aligned}\omega^2 &= \frac{1}{2(2m^2)}\left[-(-6mk) \pm \sqrt{(-6mk)^2 - 4(2m^2)(3k^2)}\right] \\ &= k/2m(3 \pm \sqrt{3})\end{aligned}$$

- This gives two roots: $\omega_1 = \sqrt{k/m}\sqrt{\frac{3-\sqrt{3}}{2}} \approx 0.80\text{s}^{-1}$ and $\omega_1 = \sqrt{k/m}\sqrt{\frac{3+\sqrt{3}}{2}} \approx 1.54\text{s}^{-1}$
- Numerical values already assume $\sqrt{k/m} = 1\ \text{s}^{-1}$ from the values we were given for the spring constant and the mass
- Now find the normal vectors
- Substitute the eigenfrequencies back into the equations of motion:

$$\begin{aligned}(-\omega_1^2[M] + [S])\vec{\xi}_1 &= 0 \\ [-\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\frac{k}{2m}(3-\sqrt{3}) + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}]\begin{bmatrix} \xi_1^a \\ \xi_1^b \end{bmatrix} &= 0 \\ s\begin{bmatrix} -\frac{3-\sqrt{3}}{2} + 2 & -1 \\ -1 & -(3-\sqrt{3}) + 2 \end{bmatrix}\begin{bmatrix} \xi_1^a \\ \xi_1^b \end{bmatrix} &= 0 \\ \begin{bmatrix} \frac{1+\sqrt{3}}{2} & -1 \\ -1 & \sqrt{3}-1 \end{bmatrix}\begin{bmatrix} \xi_1^a \\ \xi_1^b \end{bmatrix} &= 0\end{aligned}$$

- As usual this gives us two constraint equations that have the same solution
- Here the solution is $-\xi_1^a + (\sqrt{3}-1)\xi_1^b = 0$ or $\vec{\xi}_1 = \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.73 \\ 1 \end{bmatrix}$
- In this normal mode the masses move in the same direction
- The lighter (left mass at $x_A$) moves only $73\%$ as far as the heavier mass
- Now find the second normal mode

$$\begin{aligned}(-\omega_2^2[M] + [K])\vec{\xi}_2 &= 0 \\ [-\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\frac{k}{2m}(3+\sqrt{3}) + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}]\begin{bmatrix} \xi_2^a \\ \xi_2^b \end{bmatrix} &= 0 \\ s\begin{bmatrix} -\frac{3+\sqrt{3}}{2} + 2 & -1 \\ -1 & -(3+\sqrt{3}) + 2 \end{bmatrix}\begin{bmatrix} \xi_2^a \\ \xi_2^b \end{bmatrix} &= 0 \\ \begin{bmatrix} \frac{1-\sqrt{3}}{2} & -1 \\ -1 & -\sqrt{3}-1 \end{bmatrix}\begin{bmatrix} \xi_2^a \\ \xi_2^b \end{bmatrix} &= 0\end{aligned}$$

- Again this gives us two constraint equations that have the same solution
- Here the solution is $-\xi_2^a - (\sqrt{3}+1)\xi_2^b = 0$ or $\vec{\xi}_2 = \begin{bmatrix} -\sqrt{3}-1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2.73 \\ 1 \end{bmatrix}$
- In this normal mode the masses move in opposite directions
- The lighter (left mass at $x_A$) moves 2.73 times as far as the heavier mass in each stroke
- Are now in a position to use the initial conditions to find the full solution
- Recall that our solution takes the form:

$$\vec{x}(t) = A_1\vec{\xi}_1\sin(\omega_1 t) + B_1\vec{\xi}_1\cos(\omega_1 t) + A_2\vec{\xi}_2\sin(\omega_2 t) + B_2\vec{\xi}_2\cos(\omega_2 t)$$

- Find $A_{1,2}$ and $B_{1,2}$ from the initial conditions $\vec{x}_0 = \begin{bmatrix} 1//0 \end{bmatrix}$ and $\vec{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- Recall that the $A_n$ arise from $A_n = \frac{1}{\omega_n}\vec{\xi}_n \cdot \vec{v}_0/||\vec{\xi}_n||^2$ but with $\vec{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ all projections will be 0, so all $A_n = 0$

- The $B_n$ are less trivial
$$B_n = \vec{\xi_n} \cdot \vec{x}_0 / ||\vec{\xi_n}||^2 \Rightarrow B_1 = \vec{\xi_1}[M]\vec{x}_0$$
- Note that in this expression the $[M]$ will bring in factors $m$ on the top and bottom. Even if we weren't given $m = 1$ kg they would cancel between the projection (on the top) and the normalisation of $\vec{\xi}$ on the bottom
- First $||\vec{\xi_n}||^2 = \xi_1[M]\xi_1 = (\sqrt{3}-1, 1) \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix} = 2.54$
- Giving: $B_1 = \frac{1}{2.54}(\sqrt{3}-1, 1) \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\sqrt{3}-1}{2.54} = 0.29$
- Similarly $B_2 = -0.29$
- So finally we get:

$$\begin{aligned} \vec{x}(t) &= 0.29 \begin{bmatrix} 0.73 \\ 1 \end{bmatrix} \cos(0.80t) - 0.29 \begin{bmatrix} -2.73 \\ 1 \end{bmatrix} \cos(1.54t) \\ &= \begin{bmatrix} 0.21 \\ 0.29 \end{bmatrix} \cos(0.80t) + \begin{bmatrix} 0.79 \\ -0.29 \end{bmatrix} \cos(1.54t) \end{aligned}$$

- Recall that these are two separate functional expressions for $x_A$ and $x_B$ our two mass positions in our state vector
- These are plotted in red (for $x_A$) and blue (for $x_B$) on the next page
- At least one sanity checks is in order: $x_A$ starts at +1m, $x_B$ starts at 0.

$X_A(t)$

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In[12]:= Plot[ 0.29*0.73 * Cos[0.80 t] - 0.29 * (-2.73) Cos[1.54 t], {t, 0, 100},
         PlotPoints → 100, PlotStyle → {{Thickness[0.007], Hue[1.0]},}]
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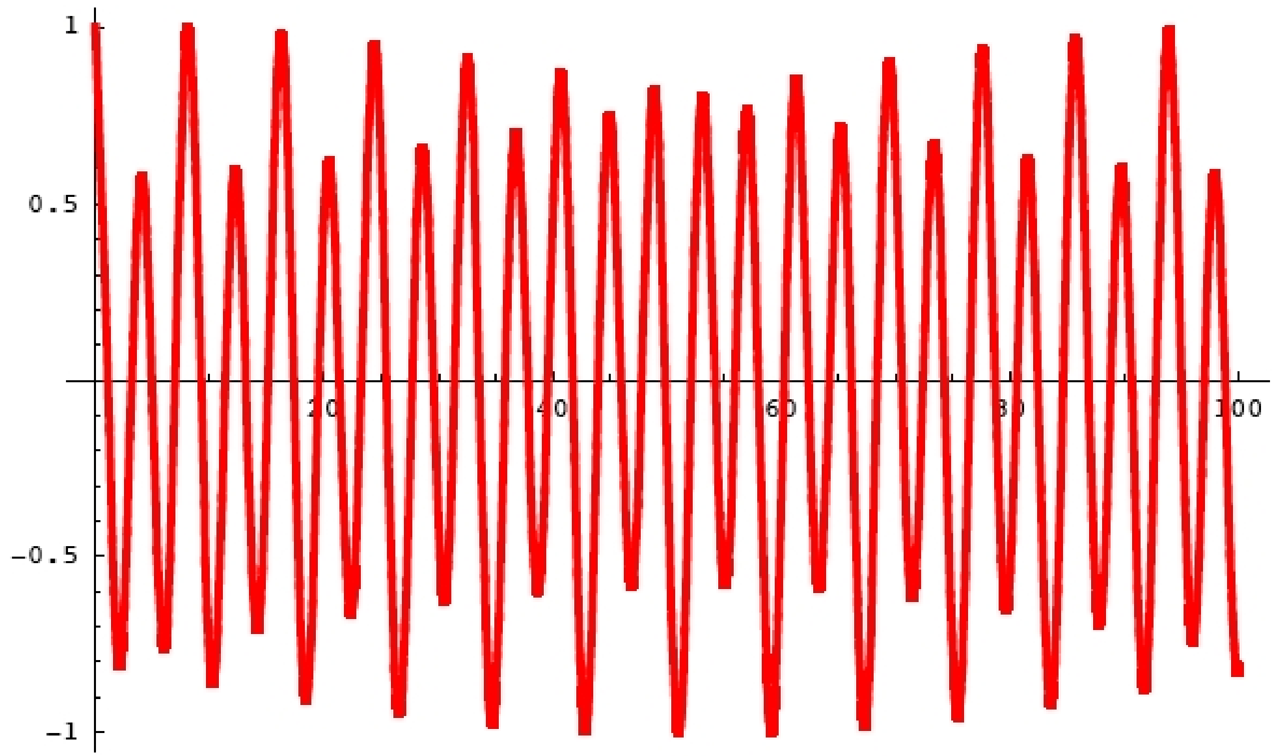

$X_B(t)$

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In[10]:= Plot[ 0.29*1 * Cos[0.80 t] - 0.29 * 1* Cos[1.54 t], {t, 0, 100},
          PlotPoints → 200, PlotStyle → {{Thickness[0.007], Hue[0.6]},}]
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