PHY293 Oscillations Lecture #16

October 18, 2010

1. Looked at a Flash simulation of waves reflected/transmitted at a density change
   - http://paws.kettering.edu/ drussell/Demos/reflect/reflect.html

Begin Lecture material

1. Reflection/Transmission of Traveling waves
   - Have seen how two traveling waves can interfere to make a standing wave
   - Normally we'll have a mixture of standing waves and traveling waves
     - Since the pure standing wave solution we had **required** the two traveling waves have **exactly** the same amplitude.
     - If the amplitudes of interfering traveling waves are **not** exactly the same will lead to a standing wave plus something left over which continues to propagate as a traveling wave
   - When waves approach/encounter a boundary the incoming and outgoing amplitudes are rarely equal and we get a mixture of standing and traveling waves
   - This manifests itself as reflections from the boundaries
   - For an arbitrary boundary we want to predict the reflection and transmission
   - We can model a boundary with two different media sitting next to one another
   - In our case we'll look at two strings, joined at $x = 0$, that have different mass densities
     - For $x < 0$ we'll have μ_1 and T_1 which results in a wave-speed $c_1 = \sqrt{T_1/\mu_1}$
     - For $x > 0$ we end up with $c_2 = \sqrt{T_2/\mu_2}$
   - We'll want to solve two versions of the wave equation, simultaneously:

$$\frac{\partial^2 y}{\partial t^2} - c_1^2 \frac{\partial^2 y}{\partial x^2} = 0 \qquad \text{for } x < 0$$
$$\frac{\partial^2 y}{\partial t^2} - c_2^2 \frac{\partial^2 y}{\partial x^2} = 0 \qquad \text{for } x > 0$$

   - To figure out what is going on we need to understand what goes on at $x = 0$
   - *Aside:* Vertical component of force on a string is given by $F_y = -T \sin\theta$ (look back at the notes from October 6 to see diagrams of what is going on here)
   - We would normally have $\tan\theta = dy/dx$ but for small angles $\sin\theta \approx \tan\theta$ so we can write $F_y = -T\sin\theta = -T\frac{\partial y}{\partial x}$
   - Use this when we consider what is going on at the interface between the two types of string at $x = 0$ *: End Aside*
   - At the string interface we require:
     - The forces on the two sides of $x = 0$ balance each other: $T\frac{\partial y}{\partial x}(x^-) = T\frac{\partial y}{\partial x}(x^+)$
     - The string be continuous from below $x = 0$ to above $x = 0$: $y(x^-, t) = y(x^+, t)$
   - Where I use x^- to denote the approach to $x = 0$ from below and x^+ the approach to $x = 0$ from above
   - While we have used physical arguments (force balance and continuous string) to derive these conditions it is a general property of second-order differential equations that the 0^{th} and 1^{st} derivatives must be continuous

2. Reflection and Transmission of Traveling Waves at Boundaries
   - Last time we setup the problem of waves moving from one medium to another having different wave speeds
   - Look at the three possible wave solutions we'll have on our hybrid string (ie. string made of two different densities of strings)
   - We'll have three waves:
     - An incident wave, incoming from the left ($x < 0$) traveling to the right: $y_i = A\sin(\omega_1 t - k_1 x)$
     - A transmitted wave, outgoing to the right ($x > 0$): $y_t = D\sin(\omega_2 t - k_2 x)$
     - A reflected wave, reflected back to left ($x < 0$): $y_r = C\sin(\omega_1 t + k_1 x)$

- Establish the boundary conditions we discussed in lecture yesterday:
  (a) $y(x^-) = y(x^+)$ the string must be continuous at $x = 0$ giving:

$$A\sin(\omega_1 t) + B\sin(\omega_1 t) = D\sin(\omega_2 t)$$
$$(A + B)\sin(\omega_1 t) = D\sin(\omega_2 t)$$

  - Where we have set $x = 0$ in the wave solutions for each side of the boundary
  - Looking at this, for it to be true at all t we are forces to conclude $\omega_1 = \omega_2 \equiv \omega$
  - In this case the first constraint reduces to $A + B = D \ [I]$
  - We also note at this point that $\omega_1 = \omega_2 \equiv \omega$ implies that $k_1 = \omega/c_1$ and $k_2 = \omega/c_2$

  (b) Now we can balance the forces on either side of $x = 0$
  - To do this we must differentiate $y_- = y_i + y + r$ and $y_+ = y + t$ and equate the forces
  - Equating the forces at $x = 0$ gives us:

$$T_1 k_1[-A\cos(\omega t - k_1 x) + B\cos(\omega t + k_1 x)] = -T_2 k_2 D\cos(\omega t - k_2 x)$$

- As an aside we see why we wrote these waves in the slightly unconventional form $\sin(\omega t - kx)$ rather than the more conventional $\sin(kx - \omega t)$. In the latter formulation, setting $x = 0$ would still leave arguments like $-\omega t$ and we would have to keep track of these signs in the algebra that follows
- Setting $x = 0$

$$T_1 k_1 (B - A)\cos(\omega t) = -T_2 k_2 D\cos(\omega t)$$

- Since we've already concluded the waves on either side of the of boundary must have the same frequency we see the $\cos(\omega t)$ terms balance out in this equation leaving (after dividing through both sides by -1):

$$T_1 k_1 (A - B) = T_2 k_2 D$$

- Using the relationship between $k_i = \omega/c_i$ we can further "simplify" this to:

$$T_1/c_1 \ \ (A - B) \ \ = \ \ T_2/c_2 \ \ D \quad [II]$$

- How do we interpret the ratio T_i/c_i? It is an intrinsic property of the string. Already seen $c_i = \sqrt{T_i/\mu_i} \Rightarrow T_i/c_i = \sqrt{T_i c_i}$
- This comes up over-and-over in the study of waves propagating in media (though we're near the end in this course – there's much more we could study). So we call it $Z_i \equiv T_i/c_i = \sqrt{T_i \mu_i}$
- We'll call this the **Impedance** of the string (see why later in this lecture)
- Thus the two equations become:
  [I] $A + B = D$
  [II] $Z_1(A - B) = Z_2 D$
- Since this is two equations in three unknowns (A, B, D) we can't solve them in closed form
- All we can do is make predictions for two ratios: $r = \frac{B}{A}$ and $t = \frac{D}{A}$
- r is the ratio of reflected to incident wave amplitudes
- Similarly, t is the ratio of transmitted to incident wave amplitudes
- Physically we can think of controlling the incident amplitude (by generating the incident wave with a controllable forcing function) and then predicting the reflected and transmitted amplitudes relative to this "known" incident amplitude
- If we take $Z_1 \cdot$ [I] + [II] we get: $2Z_1 A = (Z_1 + Z_2)D$ which gives $D/A \equiv t = 2Z_1/(Z_1 + Z_2)$
- Then look at $[\mathrm{I}]/A = 1 + B/A = D/A$ but this is jut $1 + r = t$ or $R = t - 1$
- So we get $r = 2Z_1/(Z_1 + Z_2) - 1 = \frac{Z_1 - Z_2}{Z_1 + Z_2}$
- We can look at a few limiting cases to check these predictions:
  (a) If the string is connected to a fixed point at $x = 0$ this is equivalent to it being connected to a second length of string that has infinite mass density (ie. the $x > 0$ part of the string is so heavy it can't move)
  - In this case $r = \frac{Z_1 - Z_2}{Z_1 + Z_2} \to -1$ as $Z_2 \to \infty$

- The reflected wave has the same absolute amplitude but it is inverted in the process of being reflected

(b) If the two lengths of string ($x < 0$ and $x > 0$) have the same mass density then $Z_1 = Z_2$ giving $t = 1$ and $r = 0$
   - There is no reflected wave and the transmitted wave looks just like the incident wave
   - It's as if the boundary wasn't there at all – but of course it is not since we have same density on both sides of $x = 0$

(c) In the second flash demo the end of the string at $x = 0$ is free to move up and down. This is as if the second length of string is massless (ie. $Z_2 = 0$)
   - In that case we see that $r \to 1$ and $t \to 0$
   - Agreeing with what the wave looks like in the demo.

- We looked a Flash demonstrations of waves reflected in these two cases and for waves transitioning both from denser to less dense (and vice-versa) portions of the string. I worked through one of them in each class but you should see if the behaviour of the waves at $x = 0$ makes sense,to you, in these latter two cases ($Z_1 > Z_2$ and $Z_1 < Z_2$)