PHY293 Oscillations Lecture #17

October 19, 2010

1. Extra practice problems will be posted later this week

Begin Lecture material

1. Impedance

   - In electrical circuits we know $V/I = R$, often refer R as the electrical impedance
   - Looking back at the parallels between mechanical oscillators and electrical circuits (see for example the table in the notes from the second lecture (Sept. 10)) we see that V parallels F in the mechanical case and I parallels v
   - So we should consider defining mechanical impedance as F/v, the driving force divided by the resulting velocity
   - This makes a certain amount of physical sense: higher mechanical impedance means more force necessary to get the same resulting velocity
   - But then what should we take as the impedance for a wave on a string?
   - Consider defining $Z_{\text{wave}} =$ Transverse Force/transverse velocity $= F_y/\dot{y}$
   - We saw last lecture that $F_y = -T\frac{\partial y}{\partial x}$
   - Of course the transverse velocity of elements of the string is just $\frac{\partial y}{\partial t}$
   - So for a general wave on string we'll have $y = f(x \pm c_i t)$ so we can write:

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial \mathcal{X}}{\partial t} \cdot \frac{\partial f(\mathcal{X})}{\partial \mathcal{X}} \\
&= \pm c_i \frac{\partial f(\mathcal{X})}{\partial \mathcal{X}} \\
\frac{\partial f}{\partial x} &= \frac{\partial \mathcal{X}}{\partial x} \cdot \frac{\partial f(\mathcal{X})}{\partial \mathcal{X}} \\
&= \frac{\partial f(\mathcal{X})}{\partial \mathcal{X}}
\end{aligned}$$

   - This gives us what we need to figure out the impedance:

$$Z_i = \left|\frac{F_y}{v_y}\right| = \left|\frac{-T_i\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial t}}\right| = \left|\frac{-T_i}{\pm c_i}\right| = T_i/c_i = \sqrt{T_i \mu_i}$$

   - This was just the combination of constants in our derivation of the reflected and transmitted amplitudes earlier
   - So the characteristics of the string that enter into r and t are, indeed, the mechanical impedance of the string

2. Energy Transported by Waves

   - We can compute the rate of energy transport in a wave by appealing to our formulae for the energy in discrete oscillators
   - Reverse the process of taking the continuum limit that we did when we went from a row of N discrete oscillators to and infinite number of infinitesimal chunks of a continuous string
   - Consider a small chunk of the string with density μ. Take a unit length chunk (since $\mu = M/L$) a chunk of string, of unit length will have **mass** μ
   - This chunk, oscillating up-and-down at frequency ω and having an amplitude A will have energy:

$$E = 1/2\mu\omega^2 A^2 \qquad \text{see 10.09.10 notes}$$

   - In fact we are more interested in the rate energy moves *past* this chunk
     - Before the wave arrives the chunk will have no energy
     - Once it has moved past the chunk it will have no energy
     - But it moves 'through' the chunk we can measure the rate the energy moves through the chunk: The energy flux

- Define the "Energy Flux" as Flux = Energy · velocity

$$\begin{aligned}\text{Flux} &= 1/2\,\mu\,\omega^2 A^2 \cdot c \\ &= 1/2\,\mu c\,\omega^2 A^2 \\ &= 1/2\,Z\,\omega^2 A^2\end{aligned}$$

- Where we have identified $\mu c = \mu\sqrt{\frac{T}{\mu}} = \sqrt{T\mu} = Z$
- The energy flux carried by a wave is proportional to ZA^2 (we've already seen that our incident, reflected and transmitted waves must all have the same frequency so this won't change as the waves move from one medium to another – but the impedance will and we've seen how this affects the wave amplitudes)
- Consider:

$$\begin{aligned}R \equiv \frac{\text{Reflected Flux}}{\text{Incident Flux}} &= \frac{Z_1 B^2}{Z_1 A^2} = r^2 = \frac{(Z_1 - Z_2)^2}{(Z_1 + Z_2)^2} \\ T \equiv \frac{\text{Transmitted Flux}}{\text{Incident Flux}} &= \frac{Z_2 D^2}{Z_1 A^2} = \frac{Z_2}{Z_1} t^2 = \frac{4 Z_1 Z_2}{(Z_1 + Z_2)^2}\end{aligned}$$

- R and T are now the reflected and transmitted **Flux Ratios** (not amplitude ratios – which were r and t)
- Notice however that:

$$R + T = \frac{(Z_1 - Z_2)^2 + 4 Z_1 Z_2}{(Z_1 + Z_2)^2} = \frac{(Z_1 + Z_2)^2}{(Z_1 + Z_2)^2} = 1$$

- So the energy flux incident in the initial wave (approaching the change of density in the string) that is partially reflected all ends up somewhere. The flux is either in the reflected wave or the transmitted wave. Energy is conserved

3. Group Velocity/Dispersion/Wave Packets

- All waves considered up to now (Traveling and Standing) have had a constant velocity in a given medium (ie. $c = \sqrt{\frac{T}{\mu}}$) independent of the frequency.
- Have had up to now is $\omega = ck$, ie. a proportionality between ω and k, but this is only true in non-dispersive (ideal!) media.
- In real systems there is *usually* dispersion at some level, it is just a matter of how big or small a correction it will be to the ideal/non-dispersive case.
- When the dispersion is present it leads to a wave speed that depends on ω (or k) giving frequencies that are $\omega = \omega(k)$.

4. Superposition of Traveling Waves

- Take two waves with slightly different frequencies

$$y_1 = A\cos(k_1 x - \omega_1 t) \qquad y_2 = A\cos(k_2 x - \omega_2 t)$$

- Superpose them (ie. add them together) to get:

$$y_{tot} = y_1 + y_2 = A[\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)]$$

- Using the cosine addition identity (twice): $\cos(\theta + \phi) + \cos(\theta - \phi) = 2\cos(theta)\cos(phi)$ we get:

$$y_{tot} = 2A\cos[\frac{k_1 + k_2}{2} x - \frac{\omega_1 + \omega_2}{2} t]\;\;\cos[\frac{k_1 - k_2}{2} x - \frac{\omega_1 - \omega_2}{2} t]$$

- We want to look at what this combination of waves looks like when $\omega_1 \approx \omega_2$. To do this we define the averages:

$$\omega_0 = \frac{\omega_1 + \omega_2}{2} \qquad k_0 = \frac{k_1 + k_2}{2}$$

- Plugging these into our formula we get:

$$y_{tot} = 2A\cos[k_0 x - \omega_0 t]\;\;\cos[\frac{\Delta k}{2} x - \frac{\Delta\omega}{2} t]$$

- Where we have defined $\Delta k = k_1 - k_2$ and $\Delta\omega = \omega_1 - \omega_2$ and we will consider $\Delta k \ll k_0$ and $\Delta\omega \ll \omega_0$ ... next time