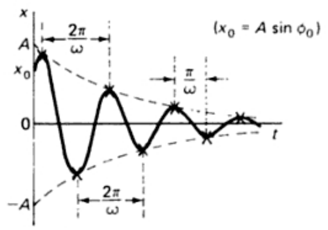
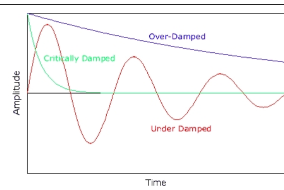
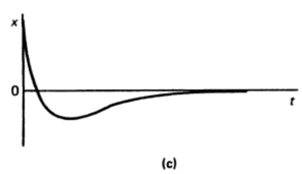
PHY293 Oscillations (Review Ch. 1-3)

September 24, 2010

1. Review of Forced Damped SHO

   (a) Given : $\ddot{x} + \gamma\dot{x} + \omega_0^2 = a_0\omega_0^2\cos(\omega t)$

   - And two initial conditions (like $x(0) = x_i$ and $\dot{x}(0) = v_i$
   - We find a solution $x(t) = x_c(t) + \mathcal{X}(t)$
   - Where the complementary solutions are

I	$\gamma < 2\omega_0$ Under-damped		$x_c(t) = C_1 e^{-\gamma t/2}\cos(\omega' t + C_2)$ $\omega' = \sqrt{\omega_0^2 - \gamma^2/4}$
II	$\gamma = 2\omega_0$ Critically damped		$x_c(t) = (C_1 + C_2 t)e^{-\gamma t/2}$
III	$\gamma > 2\omega_0$ Over-damped	 Figure 13-1 Overdamped oscillations.	$x_c(t) = C_1 e^{-\alpha_1 t} + C_2 e^{-\alpha_2 t}$ $\alpha_{1,2} = -\gamma/2 \pm \sqrt{\gamma^2/4 - \omega_0^2}$

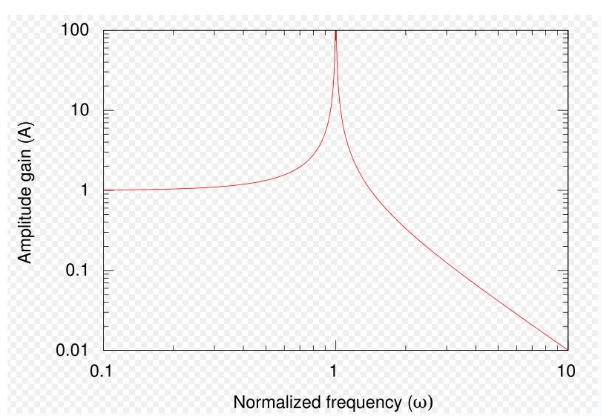
   - In all three cases $C_{1,2}$ are determined by x_i and v_i
   - Independent of the drive frequency ω or its amplitude a_0 these contributions die away like $e^{-\gamma t/2}$ (or faster).
   - The steady state solution has an amplitude and phase independent of time

$$\mathcal{X}(t) = a(\omega)\cos(\omega_t - \delta)$$

   - Where:

$$a(\omega) = \frac{a_0\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \qquad \tan\delta = \frac{\omega\gamma}{(\omega_0^2 - \omega^2)}$$

   - This will be the solution for any γ
   - But for $\gamma^2 > 2\omega_0^2$ there is no resonance
   - This solution is independent of initial conditions



- The steady state solution can be de-composed into its two phases

$$\begin{aligned}\mathcal{X}(t) &= G\cos(\omega t) + H\sin(\omega t)\\ &= a(\omega)[\cos\delta\cos(\omega t) + \sin\delta\sin(\omega t)]\end{aligned}$$

(b) There are three frequency regimes for forced oscillations

low ω	Resonant ω	High ω
$\delta \to 0$	$\delta = \pi/2$	$\delta \to \pi$
$a \to a_0$	$a = a_0 Q \frac{1}{\sqrt{1-1/4Q^2}}$	$a \to a_0 \frac{\omega_0^2}{\omega^2}$
Stiffness dominated	spring/damping balanced	Inertia dominated

(c) Velocity Response

- Resonance at $\omega = \omega_0$

$$\dot{\mathcal{X}}(t) = -\omega a(\omega)\sin(\omega t - \delta) \equiv -v(\omega)\sin(\omega t - \delta)$$

$$v(\omega) = \frac{a_0\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2/\omega^2 + \gamma^2}}$$

- For low ω, $v \to 0$; for high ω, $v \to a_0\omega_0^2/\omega$
- At resonance the reactive elements of the system cancel each other out
- The system becomes easy to **drive**: viscosity/damping and restoring force/spring work together resulting in resonance

(d) Power Absorbed by an oscillator

- Power averaged over a complete cycle is only absorbed by out-of-phase component of solution

$$< P(\omega) >=< P_{\max} > \frac{\gamma^2}{(\omega_0^2 - \omega^2)^2/\omega^2 + \gamma^2}$$

- Resonant at $\omega = \omega_0$, $< P_{\max} >= \frac{1}{2} m a_0^2 \omega_0^3 Q$
- At high Q: $< P >= P_{\max}\frac{\gamma^2/4}{(\omega_0-\omega)^2+\gamma^2/4}$
- In either case the full width at half maximum of the power curve is γ and $Q = \omega_0/\gamma$

(e) Summary of Resonance

	Amplitude	Velocity	Power
Frequency	$\omega = \omega' = \sqrt{\omega_0^2 - \gamma^2/2}$	$\omega = \omega_0$	$\omega = \omega_0$
Peak Value	$a_m = \frac{a_0 Q}{\sqrt{1-\frac{1}{4Q^2}}}$	$v_m = a_0\omega_0 Q$	$< P_m >= \frac{1}{2} m a_0^2\omega_0^3 Q$
Comments	$\omega \gg \omega_0$ $a \to a_0\omega_0^2/\omega^2$ $\delta \to \pi$	reactive elements cancel Easy to drive	FWHM = $\gamma = \omega_0/Q$ Narrow for large Q