ADVANCED UNDERGRADUATE LABORATORY

EXPERIMENT 10, FTS

Fourier Transform Spectroscopy

Introduction

The Fourier Transform

Two functions defined by

$$f(x) = \int_{-\infty}^{\infty} g(s) e^{i2\pi sx} ds \tag{1}$$

and

$$g(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi sx} dx \tag{2}$$

are said to constitute a Fourier transform pair. $f(x)$ and $g(s)$ are called Fourier transforms of each other. If $f(x)$ and $g(s)$ are even functions then the imaginary parts of (1) and (2) disappear and thus

$$f(x) = \int_{-\infty}^{\infty} g(s) \cos(2\pi sx) ds \tag{3}$$

and

$$g(s) = \int_{-\infty}^{\infty} f(x) \cos(2\pi sx) dx \tag{4}$$

Equations (3) and (4) are called cosine Fourier transforms. Cosine Fourier transforms are more commonly introduced in the following way.

If $f(x)$ is defined only for $0 < x < \infty$ then $f(x)$ can be represented by

$$f(x) = 2 \int_{0}^{\infty} g(s) \cos(2\pi sx) ds \tag{5}$$

and

$$g(s) = 2 \int_{0}^{\infty} f(x) \cos(2\pi sx) dx \tag{6}$$

Equations 5 and 6 constitute a cosine Fourier transform pair. An analogous set of equations define the sine Fourier transform.

The Fourier Transform and the Michelson Interferometer

Consider a beam of light incident on the beam splitter with an electric field given by

$$E(x, \sigma) = E_0(\sigma) e^{i(2\pi \sigma x - \omega t)} \tag{7}$$

where $\sigma = 1/\lambda$ is the inverse wavelength. σ is related to the wave number by $\sigma = k/2\pi$. The flux density $B(\sigma)$ is proportional to $E_0^2(\sigma)$. In this derivation we will ignore all proportionality constants since we will eventually only be interested in the variation in a signal and not in its absolute magnitude.
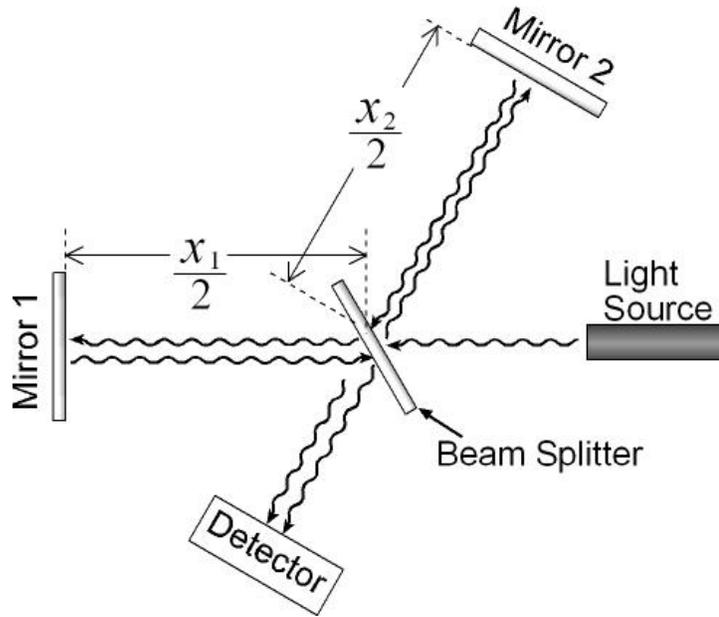
Figure 1. Schematic of a Michelson Interferometer.

In the Michelson interferometer, after amplitude splitting there are two beams which have traveled distances x_1 and x_2. They recombine to form a resultant electric field $E_R(x_1, x_2, \sigma)$. Each beam has undergone one reflection from, and one transmission through the beam splitter so both should have the same amplitude. The resultant electric field, aside from a proportionality factor to account for losses is

$$E(x_1, x_2, \sigma) = E_0(\sigma)[e^{i(2\pi\sigma x_1 - \omega t)} - e^{i(2\pi\sigma x_2 - \omega t)}] \tag{8}$$

The flux for a particular value of σ is given (within a multiplying factor) by the square of the electric field:

$$I_R = E_R E_R^* \tag{9}$$

which, written explicitly, becomes

$$I_R(x_1, x_2, \sigma) = E_0^2(\sigma)[2 + e^{i2\pi\sigma(x_1 - x_2)} + e^{-i2\pi\sigma(x_1 - x_2)}] \tag{10}$$

or

$$I_R(x_1, x_2, \sigma) = 2E_0^2(\sigma)[1 + \cos(2\pi\sigma\delta)] \tag{11}$$

where $\delta = x_1 - x_2$. The total flux at any path difference δ is obtained by integrating over σ:

$$I_R(\delta) = \int_0^\infty I(\delta, \sigma) d\sigma \tag{12}$$

$$I_R(\delta) = 2\int_0^\infty E_0^2(\sigma) d\sigma + 2\int_0^\infty E_0^2(\sigma) \cos(2\pi\sigma\delta) d\sigma \tag{13}$$

Equation 13, which gives $I_R(\delta)$ will have a constant part and a varying part. The constant part could be determined by measuring $I_R(0)$ when the path difference is zero or by measuring $I_R(\infty)$ which is the constant level to which the signal tends when the path difference becomes very large (see the reference Bell p.41 for further information on this point). In actual fact, if the beam splitter in the interferometer does not split the incoming beam into two equal components then there will be a constant flux, I, which is independent of the path difference and this will add another constant term. If we agree to interpret $I_R(\delta)$ as the fluctuation of the interferogram from its final constant level and not from zero then equation 13 becomes

$$I_R(\delta) = 2\int_0^\infty E_0^2(\sigma)\cos(2\pi\sigma\delta)d\sigma \tag{14}$$

Again, ignoring proportionality constant we may replace $2E_0{}^2(\sigma)$ by $B(\sigma)$:

$$I_R(\delta) = \int_0^\infty B(\sigma)\cos(2\pi\sigma\delta)d\sigma \tag{15}$$

But this is just one of a cosine Fourier transform pair and so we may also write:

$$B(\sigma) = \int_0^\infty I_R(\delta)\cos(2\pi\sigma\delta)d\delta \tag{16}$$

So within a multiplicative constant the incident power spectrum can be computed by a cosine Fourier transform of the interferogram.

The interferogram $I_R(\delta)$ as observed in this experiment consists of a rapidly oscillating component modulated by a slowly varying envelope. In order to obtain $B(\sigma)$ one would have to sample $I_R(\delta)$ several times per rapid oscillation and this would involve an unreasonable amount of work and computation. In the special case when $B(\sigma)$ is symmetric about some central value σ_0, one may obtain $B(\sigma)$ from the Fourier transform of the envelope of $I_R(\delta)$. This is true because, if $B(\sigma)$ is symmetric about some central value σ_0, then $B(\sigma)$ is just an even function $B_e(\sigma)$ shifted to the right by σ_0. The Fourier transform of $B(\sigma)$ is given by

$$I_R(\delta) = e^{i2\pi\sigma_0\delta}I(\delta) \tag{17}$$

where $I(\delta)$ is the Fourier transform of the even function $B_e(\sigma)$.

$I_R(\delta)$ is the observed interferogram and it is real. $B_e(\sigma)$ also is real since it is just the flux density spectrum $B(\sigma)$ shifted so that it is symmetric about $\sigma = 0$. $I(\delta)$ is the Fourier transform of $B_e(\sigma)$ which is a real and even function and thus $I(\delta)$ must be real. Equation 17 then becomes

$$I_R(\delta) = \cos(2\pi\sigma_0\delta)I(\delta) \tag{18}$$

where $I_R(\delta)$ is the observed interferogram and $\cos(2\pi\sigma_0\delta)$ is the rapidly oscillating component. $I(\delta)$, the modulating envelope is the Fourier transform of $B_e(\sigma)$ which is $B(\sigma)$ shifted so that it is symmetric about the origin. $B_e(\sigma)$ is also the Fourier transform of $I(\delta)$.

The shape of $B(\sigma)$ is given by the shape of $B_e(\sigma)$:

$$B_e(\sigma) = \int_0^\infty I(\delta)\cos(2\pi\sigma\delta)d\delta \tag{19}$$

where $I(\delta)$ is the envelope of the observed interferogram. The central wavelength of $B(\sigma)$ is obtained from $\cos(2\pi\sigma_0\delta)$, the rapidly oscillating component of the observed interferogram.

Experiment

Adjustment of the Michelson Interferometer

Disengage the motor drive from the interferometer. The motor drive is disengaged when the magnetic clutch switch is set to off. The mercury lamp is powered by a DC current. If it doesn't illuminate within about 30 seconds of turning it on, you can tip the column of mercury horizontally briefly to complete the circuit, then allow the column to return to its vertical position. Use the glass rod on top to do the tipping. Do not touch the metal supports holding the lamp, as there is a small risk of electric shock.

    a) With only a lens and a ground glass screen on the optical bench, use the mercury lamp and try to find fringes. Move the ground glass screen to bring any fringes into focus and if none are found then proceed to (b). If fringes are observed then proceed to (d).

b) If no fringes are observed, remove the ground glass screen and check by eye for very narrow fringes. If these are observed then proceed to (d) but if not then proceed to (c).

c) If no fringes are observed, it means that the mirrors are not properly aligned. The mirrors may be aligned by looking at the image of a set of crosshairs in both mirrors simultaneously. One way is to insert a frosted glass screen at the input to the interferometer, in addition to having the ground glass screen. Using a laser, one should be able to see interference patterns even when far away from zero path difference (why?). Attempt to centre the pattern on the ground glass screen. This roughly aligns the mirrors. Another way is to remove the ground glass screen in the input beam as well as the lens in the output beam. A set of crosshairs mounted in the centre of a 5 cm square aluminium plate is provided. Locate the holder and place it in the input beam and illuminate it from the side with a desk lamp. You should see two bright images of the crosshairs and a fainter one in the mirrors. Adjust the thumbscrews on the fixed mirror until the two brighter images fall exactly on top of one another (you will still see the third fainter image separately). Remove the crosshairs and lamp and replace the ground glass screen, lens and the mercury lamp in the input beam. Fringes should now be observed.

d) In this section and those that follow, the use of a ground glass screen in the output beam may be useful as a screen onto which the fringes may be focused. The use of this tool is left as an option to the student. Adjust the thumbscrews until circular fringes are observed.

e) As you face the output arm of the interferometer, the moveable mirror moves to the right when the hand crank turns clockwise. When the hand crank turns counter clockwise we must rely on a weight which pulls the mirror back. If the mirror holder sticks when the hand crank is turning counter clockwise (the ring pattern is stationary) then it will be necessary to pull back on the large metal shaft holding the mirror. Whenever measurements are taken, the mirror should be moving to the right. Use the hand crank to decrease the path difference. Should the rings collapse into the centre or expand from it? See Appendix II if you are uncertain. As the path difference becomes smaller the ring pattern will become larger, i.e., there will be fewer and larger rings. It may take many turns of the hand crank to accomplish this.

f) Decrease the path difference until there are only a few rings on the screen. At this point it may be useful to use the fine adjustments available through the piezoelectric crystals on which one of the mirrors is mounted. Turn on the power supply to the crystals on which one of the mirrors is mounted. Turn on the power supply to the crystals and adjust the voltages so that a clear ring pattern is maintained. Keep increasing the size of the rings until the ground glass screen is completely filled successively with the different colours of the mercury spectrum. You are now near zero path difference. If you pass zero path difference then the ring pattern will expand from the centre.

g) To find zero path difference, replace the mercury source with a desk lamp which is a white light source. The output beam will probably be white and not coloured. Slowly keep decreasing the path difference until coloured fringes appear. It may be advantageous to remove the ground glass screen in the output beam now if you have been using it and observe with your eye alone. When you have found zero path difference and the mirrors are perfectly aligned, the whole field of view will be alternately completely white and black or at least all of one colour. To obtain this condition you will probably have to make fine adjustments with the piezoelectric crystals.

Obtaining an Interferogram

a) Put an interference filter in the input arm and illuminate it with parallel white light. The transmission characteristics of interference filters change if the incident light is not parallel. The wavelength of maximum transmission shifts to lower wavelengths and the transmission curve broadens. You will be measuring the transmission characteristics of filters and comparing them to measurements made using a monochromator, so it is important that the light be parallel.

b) When you are near zero path difference the rings are so large that the whole field of view is covered by a circle which is one ring. Focus this circle onto the ground glass screen. Move the mirror and note the change in intensity of the circle. Actually there are two overlapping circles, one fainter than the other and the fainter one changes very little in intensity. What gives rise to this fainter circle? How does it affect the mathematical treatment of the results (see equation 13)

c) The next step is best accomplished without the lens or ground glass screen in the output arm. Move the mirror slowly and adjust the piezoelectric crystals so that the pattern is centred and the whole field of view becomes alternately bright and dark.

d) Figure 2 shows the position of the piezoelectric crystals on the fixed mirror as you look into the interferometer. The crystals which are actually behind the mirror are numbered to correspond with the numbers on the coaxial cables. Note that crystal 3 has the same motion as the top thumbscrew.
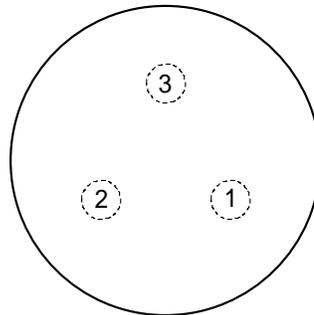
Figure 2. Location of the piezoelectric crystals on the fixed mirror.

e) Insert the lens and photomultiplier and bring the circle to a focus in a plane containing the pinhole.

f) Connect the high voltage to the photodiode and connect the output to the chart recorder. Make sure the photodetector is set to amplifier not detector. A voltage of order mV should be seen. Adjust the chart recorder using zero adjust knob on the recorder and the voltage zero on the photodetector to centre the needle on the recorder.

g) With the mirror moving, turn on the chart motor to start the chart moving down the page.

h) Try to maximize the sensitivity of the scale used. Remember, the sensitivity dial can produce fine and course adjustments.

i) Obtain an interferogram and repeat this procedure for at least one more interference filter. When taking data, be sure to include the side-lobes as far out as there is signal. (Do not stop at the first node.) In the analysis program, you may want to enter the side-lobes as alternating in sign, assuming there is a π phase shift at every node.

Analysis of the Data

You may use a variety of programs to analyze the data. There a program on faraday called "fourier". Also, a sample Matlab program is described in Appendix I and is available for download on the course web site.

The first step is to create the data-set (Step 1) from your data. The values for this first variable will be the $I(\delta)$ values taken from the envelope of the $I(\delta)$ curve. The values of the other variable, δ, will be the corresponding distances that the mirror has moved from zero path difference.

For the second variable (Step 2), one requires knowledge of the distance that the mirror moves when there is one rapid oscillation within the envelope of the interferogram. You should know how to determine an estimate of this, knowing that there are 40 threads per inch (1 inch = 2.540 cm) on the screw that drives the moveable mirror and also that the outer gear ratio is 150 to 1. However, since even a small error will shift the central maximum of $B(\sigma)$ by a relatively large amount, it is permissible to use the value of $\lambda_{central}$ from the curves in Appendix III as if they were given quantities.

To make calculation of δ easier, the first value from the envelope of $I(\delta)$ should be for $\delta=0$. Subsequent readings from the envelope should be taken at equally spaced values of δ. One need not take readings from the envelope at every rapid oscillation since this would be too time consuming. Every third or fifth oscillation will be sufficient depending on how rapidly the envelope is changing. Note that $I(\delta)$ will usually change sign every time the envelope goes to zero!

If one follows the procedure in the paragraph above, one need only define and enter values for one variable which gives the values of $I(\delta)$. The other variable may be calculated. For example, if $\lambda_{central}=500$ nm, and if values of $I(\delta)$ were entered for every third rapid oscillation, then δ could be defined by $\delta=3\times500\times(\text{datapoint}-1)$.

Using the Program

The input spectrum which has been sampled at regular intervals in the value of the independent variable δ, consists of a non periodic set of real numbers. Since the transmission curve of the filter is assumed to be symmetric, then the Fourier transform of $I(\delta)$ is an even function.

Output from the Program

The number of values of the transformed variable FT is equal to the number of $I(\delta)$ values in your data set (or half the number if you have dropped the upper half of the transformed spectrum).

An examination of a table of values or a graph o the output shows that while the results should agree reasonably well with the transmission curve of the interference filter, the number of "usable" points, which means the number of points in the output data which can be compared to the filter transmission curve data is limited to about four or five. In addition, your calculated transmission curve will be artificially wide in the wings and also be non-zero away from the central maximum. The limited number of "usable" points, the artificial broadening and the non-zero asymptote are not a fault of the program but are a consequence of the limited amount of information in your data set. The Fourier transform program needs more information and fortunately you have more

The Problem of "Usable" Points

Recall that the number of points in the output from the program is equal to the number of points in your data set. However, the number of "usable" points results from the restricted range of δ for which you have entered values of $I(\delta)$ and not from the spacing between the values of $I(\delta)$ in your data set.

If you have sampled the spectrum every x units, the Fourier Transform will have a period of $1/x$. For example, if x (i.e. δ) goes from 0 to 10 and you enter 10 points which give a spacing of x of 1 then $1/x$ will have a period of 1 and also 10 points which give a spacing of $1/x$ of 0.1 Suppose you now keep the same range for x, 0 to 10, but you increase the number of points to 20 by decreasing the spacing in x to 0.5. The period for $1/x$ will now be 2 which is larger than before but since there will be 20 values for the $1/x$ the spacing for $1/x$ will remain 0.1 which is the same as before and this leaves unchanged the

number of "usable" points or the number of points which can be compared to the transmission curve of the filter.

The additional information that you have which allows you to generate more "usable" points is that $I(\delta) = 0$ for large values of δ. For example, if you wish for times as many "usable" or non-zero values of FT then you must extend the range of δ by a factor of four and enter values of $I(\delta)$ equal to zero in the extended range. However, one must not extend the range too far because this can also influence the artificial broadening and the non-zero asymptote.

The Problem of Artificial Broadening and the Non-Zero Asymptote

You have assumed that the envelope of the $I(\delta)$ data is continuous curve but the data you have entered is really a histogram. You can produce more closely spaced data points by fitting a smooth curve through your data points and have the computer generate points from that curve.

Final Results

Photocopy the appropriate transmission curves from Appendix III and superimpose on them the transmission curves for the interference filters that you obtain from the program. The transmission curves in Appendix III were obtained using a scanning monochromator.

Questions

1. If you see a ring shaped pattern when you look into the interferometer does this mean that there is coherence between the two beams that travel through different paths in the interferometer? Is the beam of light entering the interferometer coherent?
2. The fringes in the Michelson interferometer are called fringes of equal inclination. What does the term "equal inclination" refer to?
3. Consider the situation where the mirrors are exactly parallel. If the lens and detector were removed from the output arm of the interferometer and replaced by a white screen, would interference fringes be seen on the screen? Why or why not? Does your answer depend on whether or not the separation between the mirrors is zero?
4. As the distance between the mirrors becomes smaller what happens to the ring pattern? Why?
5. In your interferogram, what distance, d, has the moving mirror moved when the pen on the chart recorder has moved from the peak of one rapid oscillation to the peak of the next rapid oscillation?
6. [Answer this question if you have used the red (632.8 nm) filter with the narrow (3.6 nm) bandwidth.] The transmission curve for this filter is approximately Gaussian. For a Gaussian, $I(\delta)$ is given by

$$I(\delta) = ke^{-\left(\frac{\pi}{2}\right)^2\left(\frac{\Delta\lambda}{\lambda_0}\right)^2\left(\frac{\delta}{\lambda_0}\right)^2}$$

where k is a constant, $\Delta\lambda$ is the full width at $1/e$ points in the transmission curve, and λ_0 is the central wavelength of the transmission curve. When $I(\delta)$ falls to $1/e$ of its maximum at $\delta = \delta_e$ then from the equation above,

$$\frac{\Delta\lambda}{\lambda_0} = \frac{2}{\pi(\delta_e/\lambda)}$$

Find δ_e in units of λ_0 from your interferogram and calculate $\Delta\lambda$ by assuming a value for λ_0. Does this value agree with your *FT* data?
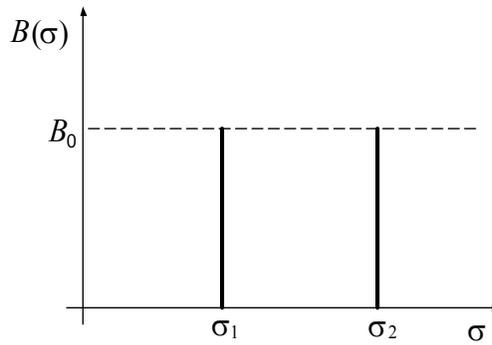7. Consider a two line source as in Figure 3.

Figure 3. Intensity profile for two discrete lines.

$B(\sigma_1) = B_0$

$B(\sigma_2) = B_0$

$B(\sigma) = 0$ for $\sigma \neq \sigma_1$ and $\sigma \neq \sigma_2$.

a. Show that $I(\delta) = 2B_0 \cos\left[2\pi\delta\dfrac{(\sigma_1 + \sigma_2)}{2}\right]\cos\left[2\pi\delta\dfrac{(\sigma_1 - \sigma_2)}{2}\right]$.

b. Sketch the shape of $I(\delta)$.

c. If σ_2 increased by 100 wavenumbers and σ_1 decreased by 100 wavenumbers, how would this affect the sketch in (b)?

d. If the intensity of the two lines were not the same, for example, if $B(\sigma_1) = B_0$ and $B(\sigma_2) = B_0/2$, how would the sketch in (b) change?

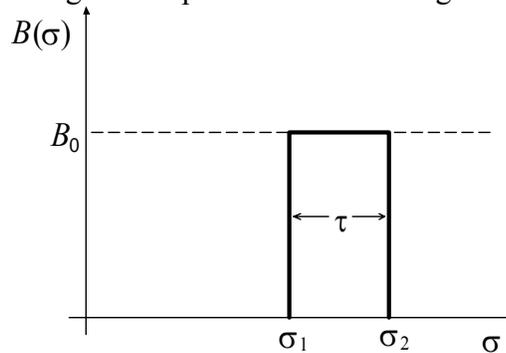8. Consider a source with a rectangular shape such as that in Figure 4.

Figure 4. Rectangular intensity profile.

$B(\sigma) = 0$ for $\sigma < \sigma_1$

$B(\sigma) = B_0$ for $\sigma_1 \leq \sigma \leq \sigma_2$

$B(\sigma) = 0$ for $\sigma > \sigma_2$.

Call $\sigma_1 - \sigma_2 = \tau$

and $\dfrac{\sigma_1 - \sigma_2}{2} = \sigma_0$.

a. Show that $I(\delta) = B_0\tau\left(\dfrac{\sin \pi\delta\tau}{\pi\delta\tau}\right)\cos 2\pi\delta\sigma_0$.

b. Sketch the shape of $I(\delta)$.

c. What effect does the width of the spectral line, τ, have on the shape of your sketch in (b)?

9. Show that for a filter which has a transmission curve which is either a rectangle or a triangle with a width at half maximum of $\Delta\lambda$, the interferogram will have its first node for a path difference $\delta = \dfrac{\lambda_0^2}{\Delta\lambda}$. Hint: if $\Delta\lambda << \lambda_0$ then $\Delta\sigma \cong \dfrac{\Delta\lambda}{\lambda_0^2}$.

9

From one of your interferograms, find δ at the first node in terms of λ_0 (ie. the number of rapid oscillations) and hence determine $\Delta\lambda$. How does this value compare with your *FT* data?

References

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Appendix I: Fourier Analysis in the Matlab Development Environment

By Mike Pritchard 2003

Matlab has excellent Fourier analysis algorithms which are designed for discrete data.
Note: For help on any command, type "help commandname" in the Command window.

Suggested Matlab Algorithm for Fourier Spectroscopy Analysis (available for download from the course web-site as `FTSanalysis.m`)

1. Define a vector containing the data points corresponding to the data spaced at intervals of `n` times the central wavelength:

```
lambda_central = 499.6;
I = [13.4, 13.2, 12.6, 11.7, 10.5, 9.0, 8.5, 6.0, 4.5, 3.1, 1.9, 1.0, 0.3, 0.5,
0.8, 0.8, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3];
```

2. Now define a vector of the same length corresponding to the values of delta at each data point:

```
n = 3;  %initialized n
delta = (0:n:length(I)*n-1)*lambda_central;
```

3. The following steps fit the data to a cubic-spline and then interpolate to an `N_newpts`-point vector called `Icsi`:

```
N_newpts = 101;
cs = spline (delta, [0 I 0]);
dd = linspace (delta(1), delta(end), N_newpts);
I2 = ppval (cs, dd);
```

4. Now we do the Discrete Fourier Transform (DFT), using the command `fft`. If your data has been set up in a decreasing sense as in Step 1, this will work. If not, refer to the help associated with the `fft` command. Note: If `N` is longer than `N_newpts`, the vector will be zero-padded; this is a good thing. Also, we drop off the upper half of the matrix to eliminate the data aliased into the frequency regime greater than the Nyquist frequency.

```
N = 500;
I_fft = (fft(I2,N));
I_fft = I_fft(1:N/2);
```

5. Take the real part of the DFT and normalize it to get a transmission spectrum out of a total of 1:

```
transmission = abs(I_fft)/ max(abs(I_fft));
figure(2), clf;
plot (transmission);
```

6. We also have to define the frequency vectors corresponding to this new spectral information. This will define the wavenumbers that correspond to each element of the `I_fft` vector.

```
nyquist = 1/n/lambda_central;
wavenumbers = [0]; % initialized wavenumbers
wavenumbers = sort((0:N/2-1)/N*2*nyquist);
wavenumbers = wavenumbers (1:N/2); %changed wavenumber to wavenumbers
```

7. These frequency vectors can be converted to wavelength:

```
for (j = 1:N/2)
        lambda_lower (j) = 1/(1/lambda_central + wavenumbers(j));
        lambda_greater (j) = 1/(1/lambda_central - wavenumbers(j));%-
1/(1/lambda_central+ wavenumbers(250));
end
```

8. Now we're ready to plot the Fourier transform. First we plot the values lower than the central wavelength, and then we can superimpose the wavelengths greater than the central wavelength.

```
figure(3), clf;
plot (lambda_lower, transmission, 'o');
hold on;
plot (lambda_lower, transmission, '-');
plot (lambda_greater, transmission, 'o');
plot (lambda_greater, transmission, '-');
```

Make sure that your final results are independent of "N_newpts" and "N".