APPENDIX: Some Helpful Formulae and Equations

Classical Model

$$\frac{\epsilon_I}{\epsilon_o} = 2n\kappa = \frac{Ne^2}{m\epsilon_o}\left(\frac{\gamma\omega}{\left(\omega^2-\omega_o^2\right)^2+\gamma^2\omega^2}\right) \qquad \frac{\epsilon_R}{\epsilon_0} = n^2-\kappa^2 = 1 + \frac{Ne^2}{2m\epsilon_0\omega_0}\left(\frac{\omega_0-\omega}{(\omega-\omega_0)^2+\frac{\gamma^2}{4}}\right) \quad \text{for } \omega\approx\omega_0$$

Diffraction Optics

$$U_p = -\frac{ik}{4\pi}e^{-i\omega t}\iint_A U_A \frac{e^{ikr}}{r}(\cos(\hat{n},\vec{r})+1)\,dS \qquad U_P = \frac{1}{4\pi}\iint_A\left[U\frac{\partial}{\partial n}\frac{e^{ikr}}{r}-\frac{e^{ikr}}{r}\frac{\partial U}{\partial n}\right]dS$$

$$U_P(u,v) = -\frac{i}{\lambda z}e^{ik|PS|}\exp\left\{\frac{ik}{2z_a}[x_m^2+y_m^2]\right\}\iint_A U_o(x_o,y_o)\exp\{-i[ux_o+vy_o]\}\,dx_o\,dy_o$$

$$U_P(x,y,z) = -\frac{iU_o}{\lambda z z'}e^{ik|PS|}\iint_A \exp\left\{\frac{ik}{2z_a}[(x_o-x_m)^2+(y_o-y_m)^2]\right\}dx_o\,dy_o \qquad x_m = \frac{zx'+z'x}{z+z'}; \quad y_m = \frac{z'y+zy'}{z+z'}; \quad z_a = \frac{zz'}{z+z'}$$

Fresnel, sufficient: $z^3 \gg \frac{\pi}{4\lambda}((x-x_o)^2+(y-y_o)^2)^2$

Fresnel, rectangular apertures: $U_P = \frac{U_{P_o}}{(1+i)^2}[C(s)+iS(s)]\Big|_{u_1}^{u_2}\times[C(s)+iS(s)]\Big|_{v_1}^{v_2}$

Fresnel, circular apertures: $U_P = -iU_o\int_o^{\psi_o}e^{i\psi}Q(\psi)\,d\psi$ Fresnel zone radii: $r_n = \sqrt{n\lambda z_a}$

Fraunhofer, sufficient: $z_a \gg \frac{\pi(x_0^2+y_0^2)}{\lambda}$ $u = \frac{kx_m}{z_a}$ and $v = \frac{ky_m}{z_a}$

Special Functions

$$\text{sinc}(\alpha) \equiv \frac{\sin(\pi\alpha)}{\pi\alpha} \qquad comb\left(\frac{x_o}{L}\right) \equiv |L|\sum_{-\infty}^{\infty}\delta(x_o-nL)$$

$$\text{Gaus}\left(\frac{x_o}{L}\right) \equiv e^{-\pi(x_o/L)^2} \qquad \text{rect}\left(\frac{x_o}{L}\right) = \begin{cases}1 & |x_o|\le\frac{L}{2}\\ 0 & \text{else}\end{cases}$$

$$\text{cyl}\left(\frac{r}{R}\right) = \begin{cases}1 & \frac{r}{R}\le 1\\ 0 & \text{else}\end{cases} \qquad \text{step}\left(\frac{x_o}{L}\right) = \begin{cases}0 & \frac{x_o}{L}\le 0\\ 1 & \frac{x_o}{L}>0\end{cases}$$

Function	Fourier Transform
rect (x)	$\text{sinc}\left(\frac{u}{2\pi}\right)$
δ (x)	1
comb(x)	$\text{comb}\left(\frac{u}{2\pi}\right)$
Gaus (x)	$\text{Gaus}\left(\frac{u}{2\pi}\right)$
step (x)	$\frac{1}{2}\delta\left(\frac{u}{2\pi}\right)+\frac{1}{iu}$
cyl (r)	$\left(\frac{J_1(u')}{u'}\right)$

where $u \equiv \frac{kx}{z}$; $u' \equiv \frac{kr}{z}$, the radial-only analogue

Fourier Transform Relations

If $\mathcal{F}\{g(x,y)\} = G(u,v)$ and $\mathcal{F}\{h(x,y)\} = H(u,v)$ then

$$\mathcal{F}\{g(ax,by) = \frac{1}{|ab|} G\left(\frac{u}{a},\frac{v}{b}\right) \qquad \mathcal{F}\{g(x-a,y-b)\} = G(u,v)\, e^{-i(ua+vb)} \qquad \mathcal{F}\{g * h\} = G(u,v)\; H(u,v)$$

$$T(x_o,y_o) = e^{ikn\Delta_o} \exp\left\{-\frac{ik}{2f}\left(x_o^2+y_o^2\right)\right\}$$ transmission (aperture) function for a lens

$$\int_0^a x\exp\left(-iux\right)dx = -\frac{a\exp\left(-iua\right)}{iu} + \frac{\exp\left(-iua\right)-1}{u^2} \qquad \frac{I(x,y)}{I_0} = \frac{\ell_x^2\,\ell_y^2}{\lambda^2 z^2}\,\text{sinc}^2\left[\ell_x\frac{x}{\lambda z}\right]\text{sinc}^2\left[\ell_y\frac{y}{\lambda z}\right]$$

Gaussian Beams etc.

$$\Psi_{l,m} = \frac{C_{l,m}}{w(z)} H_l\left[\frac{\sqrt{2}x}{w(z)}\right] H_m\left[\frac{\sqrt{2}y}{w(z)}\right]\exp\left[-\frac{x^2+y^2}{w^2(z)}\right]\exp\left[-\frac{ik\left(x^2+y^2\right)}{2R(z)}\right]e^{-i(l+m+1)\phi(z)}e^{ikz}$$

$$w^2(z) = \frac{\lambda z_o}{\pi}\left(1+\frac{z^2}{z_o^2}\right);\quad R(z) = \frac{z^2+z_o^2}{z};\quad \tan\phi = \frac{z}{z_o} \qquad \frac{r_1}{L} >> \frac{\lambda}{r_2}$$

$$\tan\frac{\theta}{2} = \frac{w(z)}{z} \cong \frac{w_o\cdot\frac{z}{z_o}}{z} = \frac{\lambda}{\pi w_o} \cong \frac{\theta}{2} \quad \text{for small} \quad \frac{\lambda}{\pi w_o}$$

$$g_1 = \left(1-\frac{L}{R_1}\right) \qquad g_2 = \left(1-\frac{L}{R_2}\right) \qquad z_o^2 = \frac{L^2 g_1 g_2(1-g_1 g_2)}{(g_1+g_2-2g_1 g_2)^2} \qquad 0 < g_1 g_2 < 1$$

$$w_1^2 = \frac{L\lambda}{\pi}\left(\frac{g_2}{g_1(1-g_1 g_2)}\right)^{1/2} \quad \text{and} \quad w_2^2 = \frac{L\lambda}{\pi}\left(\frac{g_1}{g_2(1-g_1 g_2)}\right)^{1/2}$$

$$\int_{z_1}^{z_2} k_{\text{eff}}\,dz = k(z_2-z_1) - (l+m+1)\left[\tan^{-1}\left(\frac{z_2}{z_0}\right) - \tan^{-1}\left(\frac{z_1}{z_0}\right)\right] + \frac{\phi_2+\phi_1}{2} = q\pi$$

$$\omega_q^{l,m} = \frac{q\pi c}{nL} + \frac{c}{nL}\left[(l+m+1)\left(\tan^{-1}\left(\frac{z_2}{z_0}\right) - \tan^{-1}\left(\frac{z_1}{z_0}\right) - \frac{\phi_2+\phi_1}{2}\right)\right] = \left[q + \frac{(l+m+1)\cos^{-1}(\sqrt{g_1 g_2})}{\pi}\right]\frac{c\pi}{nL}$$

Airy pattern: $$U(r,\theta) = -i\,U_A\frac{\exp(ik|PS|)}{z}\exp\left(\frac{ik}{2z_a}\left[x_m^2+y_m^2\right]\right)\frac{J_1\left(\frac{kRr}{z}\right)}{\frac{kRr}{z}} \propto \frac{J_1\left(kR\sin(\theta)\right)}{kR\sin(\theta)}$$

Airy disk diameter: $\Delta r = 1.22\,\lambda z/R$; $\Delta\theta = 1.22\,\lambda/R$ $\quad f = \frac{\pi\sqrt{R}}{1-R}$

$$\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{i\lambda}{\pi\; n\; w^2(z)} \qquad q' = \frac{Aq+B}{Cq+D}$$

Free space:	$\begin{bmatrix}1 & L\\ 0 & 1\end{bmatrix}$	*Thin lens, focal length f*:	$\begin{bmatrix}1 & 0\\ -\frac{1}{f} & 1\end{bmatrix}$
Re*fraction at plane*:	$\begin{bmatrix}1 & 0\\ 0 & \frac{n_1}{n_2}\end{bmatrix}$	*Curved dielectric*:	$\begin{bmatrix}1 & 0\\ \frac{(n_1-n_2)}{n_2 R} & \frac{n_1}{n_2}\end{bmatrix}$

lensmaker's equation: $$\frac{1}{f} = \frac{n_2-n_1}{n_1}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$\frac{P(E_1)}{P(E_2)} = \exp\left\{-(E_1 - E_2)/k_BT\right\}$$

$$\rho(\omega) = \sigma_{BB}\langle E(\omega)\rangle = \frac{n^3\omega^3}{\pi^2 c^3}\frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{k_BT}\right)-1} = \frac{4h}{\lambda^3}\frac{1}{\exp\left(\frac{\hbar\omega}{k_BT}\right)-1}$$

$I = c\int_0^\infty \rho_{BB}(\omega)\, d(\omega) = Z_{SB}T^4$ $\quad Z_{SB} = 5.67\text{x}10^{-8}\ \text{W/m}^2\text{K}^4$ $\quad h = 6.6 \text{ x } 10^{-34}$ J•s $\quad k_B = 1.34 \text{ x } 10^{-23}$ J•K^{-1}

$$\gamma(\omega) = \frac{\left(N_j - \frac{g_j}{g_i}N_i\right)3\pi^2c^2}{n^2\omega^2 t_{sp}}\, g(\omega) \qquad A_{j\to i} = \frac{e^2|x_{ij}|^2}{3\pi\,\varepsilon_o c^3}\hbar^{-1} n\,\omega_o{}^3 g_i \qquad B_{i\to j} = \frac{\pi e^2\ |x_{ij}|^2}{3n^2\varepsilon_o}\hbar^{-2}g_j$$

$$\gamma(\omega) = -\alpha(\omega) = -\frac{4\pi\kappa}{\lambda} = -\frac{k}{n_B}\chi_I \qquad g_i B_{i\to j} = g_j B_{j\to i}$$

$$\chi_I(\omega) = -\frac{\left(N_j - \frac{g_j}{g_i}N_i\right)3\pi^2c^3 g(\omega)}{(n_B\omega^3 t_{sp})} \qquad \chi_R(\omega) = -\frac{2\left(N_j - \frac{g_j}{g_i}N_i\right)3\pi^2c^3g(\omega)}{n_B\omega^3t_{sp}}\frac{\omega_o - \omega}{\Delta\omega} = \frac{2(\omega_o - \omega)}{\Delta\omega}\chi_I$$

$$t_c = n_B\frac{L}{c\pounds} \qquad t_c = \frac{n_B}{c\left[\zeta - L^{-1}\ln\sqrt{R_1R_2}\right]} \qquad \gamma_t = \zeta - \frac{1}{L}\ \ell n\sqrt{R_1R_2} = \frac{n_B}{ct_c}$$

$$\Delta N_t = \left(N_j - \frac{g_j}{g_i}N_1\right)_t = \frac{n^2\omega^2 t_{sp}}{g(\omega_q)3\pi^2c^2}\left(\zeta - \frac{1}{L}\ \ell n\sqrt{R_1R_2}\right) \qquad \int_{-\infty}^{\infty} g(\omega)\,d\omega = 1, \quad \textit{thus}\quad g(\omega_o)\Delta\omega \cong 1$$

Doppler: $g(\omega) = g_o\exp\left(-4\ln 2\ \frac{(\omega-\omega_o)^2}{(\Delta\omega_D)^2}\right)$ *where* $g_o = \frac{c}{\omega_o}\left(\frac{M}{2\pi k_B T}\right)^{1/2}$ *and FWHM* $\Delta\omega_D = 2\omega_o\sqrt{2k_BT\ \frac{\ln 2}{Mc^2}}$

$$P = \Delta N_t V W_{21}\hbar\omega \qquad P = P_S\left(\frac{R}{R_t} - 1\right) \qquad P_S = \frac{\Delta N_t}{t_{sp}}\hbar\omega V \qquad P' = \frac{N_o}{2}\hbar\omega\frac{V}{t_2}$$