A Primer on Dispersion in Waveguides

Waves are waves The linear wave equation for sound waves, as for light waves and others, is:

$$\nabla^2\Phi - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\Phi = 0 \qquad [1]$$

For sound waves, this can be used to solve for the scalar pressure-amplitude p(x,y,z;t) of the acoustic wave, or for the oscillation-amplitude $\vec{x}$(x,y,z;t), or velocity-amplitude $\vec{v}$(x,y,z;t), which are vector fields. Since the oscillation and velocity can be found from the pressure field, we usually solve for that.

The velocity and oscillation amplitudes can be found from the pressure gradient $\vec{\nabla}p(x,y,z,t)$, which gives the force term for the equation of motion of the gas in which the sound propagates. The waveguide surfaces can exert a force perpendicular to the face, but nothing in the sliding direction parallel to the surface. So the two boundary conditions for this second-order differential equation are that the first derivative and second derivative of the pressure must be zero along the direction perpendicular to the surfaces.

Take the direction along the waveguide to be *z*, and take the transverse directions to be *x* (L_x = 5cm) and *y* (L_y = 15cm). A general plane wave can be written as:

$$p(\vec{r},t) = p_o \exp\left\{ i\left(\omega t - \vec{k}\cdot\vec{r} + \phi\right)\right\} \qquad [2]$$

Where ω is the frequency of the wave, and $\vec{k}$ is the *wave-vector*, having magnitude 2π/λ (where λ is the wavelength) and a direction perpendicular to the surfaces of constant phase, *i.e.*, in the direction of propagation. The magnitude alone of the wave-vector is known as the *wavenumber*. The argument $\left(\omega t - \vec{k}\cdot\vec{r} + \phi\right)$ is called the *total phase*. (Note that this is *phasor notation*.)

Show that this solution to the wave equation in the waveguide has the general form:

$$p(x,y,z,t) = p_o \cos(k_x x)\cos(k_y y)\cos(\omega t - k_z z + \phi) \qquad [3]$$

where ϕ is arbitrary, provided that the constants k_x, k_y, satisfy:

$$k_x = m\frac{\pi}{L_x} \quad and \quad k_y = n\frac{\pi}{L_y}; \quad where \;\; m,n \in Z \qquad [4]$$

The k_z is then determined from the relation for these perpendicular components:

$$k^2 = k_x^2 + k_y^2 + k_z^2 \qquad [5]$$

which becomes

$$k_z = \pm\sqrt{k^2 - \left(m\frac{\pi}{L_x}\right)^2 - \left(n\frac{\pi}{L_y}\right)^2} \qquad [6]$$

Only the last term in equation [3] evolves in time. The total phase, ($\omega t - k_z z + \phi$), is constant if

$$z = \frac{\omega}{k_z}t \qquad [7]$$

Thus we see that surfaces of constant phase move down the waveguide at speed ω/k_z , which is therefore called the *phase speed*, v_ϕ. The positive and negative signs for k_z correspond to waves moving in opposite directions.

Equation [4] points out that the different solutions we can have, for a wave propagating in a waveguide, can be labelled by an ordered pair *(m, n)* of the indices. These different *ways* of propagating are termed *modes* of the waveguide, even as there are normal modes of oscillation of a stretched wire or rectangular drum-head.[1]

It's also easy to figure out these modes geometrically, for a given wavenumber k. Propagating a plane wave obliquely down the waveguide at an arbitrary angle θ from the z-axis, reflections at each wall can be constructed. After reflection from two opposite-facing walls, the reflection joins the source wave, again propagating in the original direction. If the reflection joins the source wave on the next (or subsequent) cycle exactly in-phase, there can be constructive interference; any other possibility will mean that multiple reflections trailing behind will eventually cancel out the wave entirely. You can picture this easily, using the LabVIEW program "Mode Conditions.vi' on the lab PC for this experiment, or on the website for the Photonics Lab; this software permits you to change the wavenumber and angle of propagation.

Dispersion relations and ω–k plots Returning to our development, our original plane wave in equation [2] propagates in the most ordinary way with a phase speed equal to the free-space wave speed c_s. Thus $c_s = \omega/k$, which we can put into [6] to get:

$$k_z = \pm\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(m\frac{\pi}{L_x}\right)^2 - \left(n\frac{\pi}{L_y}\right)^2} \qquad [8]$$

For a given mode, is there a wavelength (wavenumber) for any arbitrary frequency? See that in [8], there is a minimum value of ω, for any mode *(m, n)*, and for frequencies lower than this, k_z is imaginary. The meaning of an imaginary wavenumber can be seen if we go back to the fuller phasor notation for [3], which is:

$$p(x,y,z,t) = p_o\cos(k_x x)\cos(k_y y)\exp\{i(\omega t - k_z z + \phi)\} \qquad [9]$$

If the frequency has too low a value, it leads to an imaginary wavenumber, and for this wavenumber there is no longer a propagating solution — only an exponentially growing solution or an exponentially decaying solution. Although both cases are needed for a general solution over a finite length, the growing solution will normally pose energy problems as it propagates off to infinity. Therefore, in general terms there is a *cutoff frequency* for a given mode, and below this frequency the solution dies evanescently. this cutoff frequency is given by:

$$\omega = \pm c\sqrt{\left(m\frac{\pi}{L_x}\right)^2 + \left(n\frac{\pi}{L_y}\right)^2} \qquad [10]$$

(Note that the two signs of ω and likewise of k_z do not combine to give four solutions; only two are unique, and correspond to right-going and left-going waves. This can be seen from the fact that $\cos(\omega t - k_z z + \phi)$ is an even-valued function.)

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1 (In vector-space terms t can be shown that these are orthogonal functions which span the space of solutions to this differential problem: differential equation plus boundary conditions.)

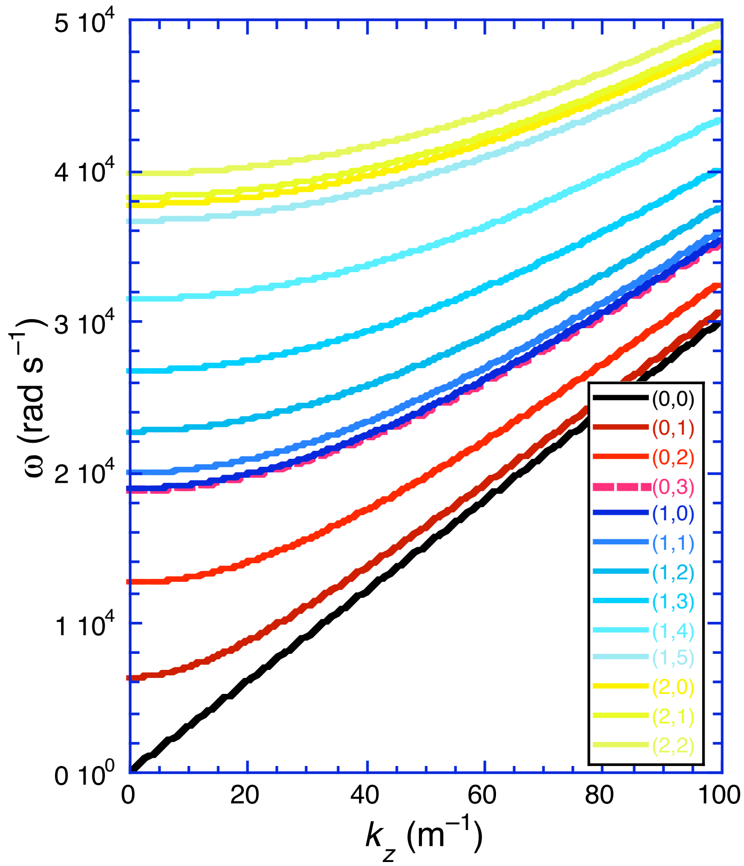

Fig. 1: ω–*k* dispersion curve; modes are labelled in order of increasing cut-off frequency.

We can plot ω vs. k_z for different modes, as in Fig. 1 at left:

$$\omega = c\sqrt{k_z^2 + \left(m\frac{\pi}{L_x}\right)^2 + \left(n\frac{\pi}{L_y}\right)^2} \qquad [11]$$

The lowest-frequency mode, $(m,n) = (0,0)$, propagates as does a wave in free-space: it exhibits no *dispersion*. The other modes, however, show different phase speeds for different frequencies of wave.

Since the phase speed for a given wave-number k_z is given by ω/k_z, it is found graphically as the slope of a line from the origin to that (k, ω) point on the curve for a given mode. So, mode (0,0) has a single, well-defined phase speed, equal to the speed of sound in free space. Upper-modes are asymptotic to (0,0) at large k_z (or, equivalently, high frequencies), as you can see simply from the limits of [11], and the phase speeds all go to the same value c_s as k_z gets large (*i.e.,* short wavelengths, high frequencies).

But note that as k_z goes to zero (*i.e.,* as the frequency approaches the cut-off value), the phase speeds of *all* upper modes go to infinity!

Behaviour of pulses Phase-fronts for waves of different frequencies move forward at different speeds, in this system. Of course, a plane wave with a single well-defined frequency exists for *all* times, and cannot be turned on or off; you will know already from Fourier analysis (or from Heisenberg's uncertainty principle) that, for any wave, shaping a carrier wave with an envelope that turns on or off to make a pulse of finite duration will lead to not just one frequency but a spread of frequencies. As this pulse propagates in a particular waveguide mode, the various constituent plane waves will propagate at different phase speeds, and thus change their phase relationship to each other. This depends on the shape of the curve in Fig. 1 for that particular mode. Consequently, the pulse will re-shape itself as it propagates, and become something else.

At the outset, I noted that the wave equation good for sound is also good for light waves, longer-wavelength electromagnetic waves, such as microwaves in metallic waveguides, and a number of other linear waves. The main practical distinction is that the speed c and the boundary conditions will change — for instance, the electrostatic scalar potential $\phi(x,y,z,t)$ must be constant for microwaves at a conductive wall. Since other experiments in the lab will look at dispersion of optical pulses, let's use electromagnetic waves as the continuing example. It's still correct to continue with the acoustic-pressure amplitude in place of the electric field amplitude shown below.

Consider a pulse whose spectrum has a centre frequency ω_o. A simple example would be a symmetric pulse-amplitude envelope, like a gaussian pulse, imposed on a carrier-wave of frequency ω_o. We can invert the ω–k dispersion curve to give $k_z(\omega)$ rather than $\omega(k_z)$ (as you'll see if you exchange axes in Fig. 1) and make a Taylor-series expansion around that frequency:

$$k(\omega)=k(\omega_o)+k'\cdot(\omega-\omega_o)+\frac{1}{2}k''\cdot(\omega-\omega_o)^2 \qquad [12]$$

$$\text{where}\quad k'\equiv\left.\frac{\partial k}{\partial\omega}\right|_{\omega=\omega_o}\qquad k''\equiv\left.\frac{\partial^2 k}{\partial\omega^2}\right|_{\omega=\omega_o}$$

For a gaussian pulse, we can write the electric field $E_o(t)$ and its Fourier transform $\tilde{E}_o(\omega)$:

$$E_o(t)=e^{-(t/\tau)^2}e^{i\omega_o t}=\exp\left(\Gamma_o t^2+i\omega_o t\right),\qquad \tilde{E}_o(\omega)=\exp\left[-\frac{(\omega-\omega_o)^2}{4\Gamma_o}\right] \qquad [13]$$

The Fourier transform in time is of course *invertible* — it contains all the information that $E_o(t)$ does, and is an equally good representation of the electric field. So if we let the pulse propagate in z, we can do that equally well letting the Fourier transform evolve in z. Our Fourier transform was in time, so the behaviour in theindependent variable z is just as it was in equation [2], and we have the chance to recognize that the wavenumber k (sometimes called the *propagation constant* β) depends on the frequency ω. Using our Taylor-series expansion:

$$\begin{aligned}\tilde{E}(z,\omega)&=\tilde{E}_o(\omega)\cdot\exp[-ik(\omega)z]\\&=\exp\left[-ik(\omega_o)z-ik'z\cdot(\omega-\omega_o)-\left(\frac{1}{4\Gamma_o}+\frac{ik''z}{2}\right)\cdot(\omega-\omega_o)^2\right]\end{aligned} \qquad [14]$$

Phase velocity and group velocity This equation above is a complete answer of how a pulse will alter, propagating in a system that has dispersion. It is, however, convenient to convert this Fourier transform back into the time-domain to see explicitly how the waveform in time $E(z,t)$ propagates in space:

$$E(z,t)\equiv\int_{-\infty}^{\infty}\tilde{E}(z,\omega)\cdot e^{i\omega t}d\omega \qquad [15]$$

$$E(z,t)\equiv\frac{e^{i[\omega_o t-k(\omega_o)z]}}{2\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{(\omega-\omega_o)^2}{4\Gamma(z)}+i(\omega-\omega_o)(t-k'z)\right]d(\omega-\omega_o) \qquad [16]$$

$$\text{where}\quad \frac{1}{\Gamma(z)}\equiv\frac{1}{\Gamma_o}+2ik''z \qquad [17]$$

This can be integrated formally, but it's easy enough to do by inspection, noticing that the integral is much the same as the one we will have done in [13] to get the electric field into the frequency domain, but with these changes $\Gamma_o \leftrightarrow -1/[4\Gamma(z)]$, $t \leftrightarrow (\omega-\omega_o)$ and $\omega_o \leftrightarrow (t-k'z)$:

$$\begin{aligned}E(z,t)&=\exp\left[i(\omega_o t-k(\omega_o)z)\right]\cdot\exp\left[-\Gamma(z)\cdot(t-k'z)^2\right]\\&=\exp\left[i\omega_o\left(t-\frac{z}{v_\phi(\omega_o)}\right)\right]\cdot\exp\left[-\Gamma(z)\cdot\left(t-\frac{z}{v_g(\omega_o)}\right)^2\right]\end{aligned} \qquad [18]$$

$$\text{where} \quad v_\phi(\omega_o) \equiv \frac{\omega_o}{k(\omega_o)} \qquad v_g(\omega_o) \equiv \frac{1}{k'(\omega_o)} \tag{19}$$

This formula [19] we can see as representing a carrier wave, in the first term, and a gaussian-amplitude envelope in the second term. But now, surfaces of constant phase (constant = $\phi_{total} = t - z/v_\phi$) — the phase fronts — move no longer at the speed of the wave in free space, but at the new phase speed v_ϕ given by equation [19].

Likewise, the pulse envelope no longer moves at the speed of the wave in free space, but instead at the *group velocity* v_g, also given in [19].

Group velocity dispersion Something more interesting still can be learned by collecting the phase and amplitude terms separately in the exponentials in [18]:

$$\begin{aligned} E(z,t) &= \exp\left[i(\omega_o t - k(\omega_o)z)\right] \cdot \exp\left[-(\mathrm{Re}[\Gamma(z)] + i\,\mathrm{Im}[\Gamma(z)]) \cdot (t - k'z)^2\right] \\ &= \exp\left[i\left\{(\omega_o t - k(\omega_o)z) - \mathrm{Im}[\Gamma(z)] \cdot (t - k'z)^2\right\}\right] \cdot \exp\left[-\mathrm{Re}[\Gamma(z)] \cdot (t - k'z)^2\right] \\ &= e^{i\phi_{total}(z,t)} \cdot \exp\left[-\mathrm{Re}[\Gamma(z)] \cdot (t - k'z)^2\right] \end{aligned} \tag{20}$$

The second term is the amplitude term, and determines the pulse shape; the first phasor term has unit amplitude and gives the field oscillations. Its argument is the total phase of the evolving wave. The instantaneous frequency of the wave is the *rate of change of this total phase*, which is not ω_o. Instead, in this case the instantaneous frequency is:

$$\begin{aligned} \omega(z,t) &\equiv \frac{\partial}{\partial t}\phi_{total}(z,t) \\ &= \frac{\partial}{\partial t}\left\{\omega_o t - k(\omega_o)z - \mathrm{Im}[\Gamma(z)] \cdot (t - k'z)^2\right\} \\ &= \omega_o - 2 \cdot \mathrm{Im}[\Gamma(z)] \cdot (t - k'z) \end{aligned} \tag{21}$$

So, at one fixed position z in space, as the pulse travels past the frequency isn't constant — it changes linearly in time. For our acoustic waveguide experiment, this makes a sliding tone, like the chirp of a bird's song, which you can possibly detect by ear, and certainly measure with the equipment provided. By analogy, this is termed a *frequency-chirped pulse*.

The value of the chirp is easy to find, and depends on the *group velocity dispersion* (GVD), given by $k''(\omega_o)$.

$$\frac{1}{\Gamma(z)} \equiv \frac{1}{\Gamma_o} + 2ik''z \tag{22}$$

Write $\Gamma(z) = a(z) - i\,b(z)$ in its real and imaginary parts; then rationalize the denominator:

$$\frac{1}{\Gamma(z)} = \frac{1}{a(z) - ib(z)} = \frac{a(z)}{a^2(z) + b^2(z)} + i\frac{b(z)}{a^2(z) + b^2(z)} \tag{23}$$

Equating the real and imaginary parts of [22] and [23], we can identify:

$$\begin{aligned} a(z) &= \frac{\Gamma_o}{1 + \left(2k''z\Gamma_o\right)^2} \\ b(z) &= \frac{2k''z\Gamma_o^2}{1 + \left(2k''z\Gamma_o\right)^2} = \frac{2k''z}{(1/\Gamma_o^2) + \left(2k''z\right)^2} \end{aligned} \tag{24}$$

So the instantaneous frequency is given by

$$\omega(z,t) = \omega_o + 2(t - k'z)\frac{2k''z}{(1/\Gamma_o^2) + (2k''z)^2} = \omega_o + \frac{4(t-k'z)k''z}{(1/\Gamma_o^2) + (2k''z)^2} \qquad [25]$$

And from the other term of equation [22], the half-width of the pulse, measured at 1/*e* of the maximum, is:

$$\begin{aligned}\tau(z) &= \frac{1}{\sqrt{a(z)}} \\ &= \sqrt{\frac{1+(2k''z\Gamma_o)^2}{\Gamma_o}}\end{aligned} \qquad [26]$$

The FWHM can then be calculated:

$$\tau(z) = 2\ln 2\sqrt{\frac{1+(2k''z\Gamma_o)^2}{\Gamma_o}} \qquad [27]$$

This is very interesting: for this gaussian pulse, the *pulsewidth now depends on the distance z propagated*. The pulse starts with its original duration, but then as it propagates at the group velocity it stretches out further and further. For an animated illustration of this, see the demonstration on the lab webpage (TravelChirp.mov).

In summary, note that $k(\omega_o)$ determines the *phase velocity*, $k'(\omega_o)$ determines the *group velocity*, and $k''(\omega_o)$ determines the spreading of the wavepacket, the *group velocity dispersion*.

$$\begin{aligned} k &\equiv k(\omega)\big|_{\omega=\omega_o} = \frac{\omega_o}{v_\phi(\omega_o)} \equiv \frac{\omega_o}{\textit{phase velocity}} \\ k' &\equiv \frac{dk}{d\omega}\bigg|_{\omega=\omega_o} = \frac{1}{v_g(\omega_o)} \equiv \frac{1}{\textit{group velocity}} \\ k'' &\equiv \frac{d^2k}{d\omega^2}\bigg|_{\omega=\omega_o} = \frac{d}{d\omega}\left(\frac{1}{v_g(\omega)}\right)\bigg|_{\omega=\omega_o} \equiv \textit{group velocity dispersion} \end{aligned} \qquad [28]$$