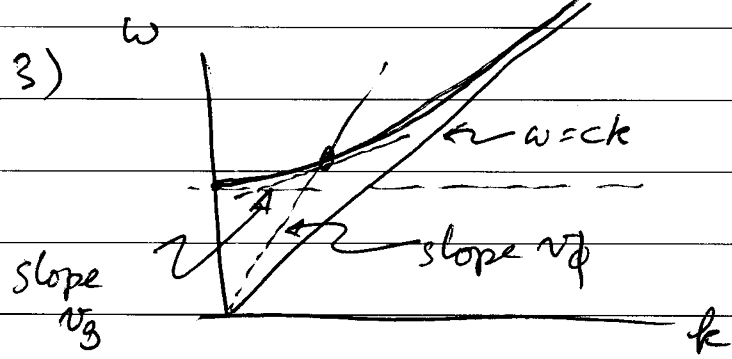
3)



$$\omega^2 = \omega_p^2 + c^2 k^2$$

$$k = 0 \Rightarrow \omega = \omega_p$$

$$k \to \infty \Rightarrow \omega \to ck$$

(asymptote)

greatest curvature $\left(\frac{d^2 k}{d\omega^2}\right)$ or GVD around $\omega \simeq \omega_p$

a) $$v_\phi = \frac{\omega}{k} = \frac{\sqrt{\omega_p^2 + c^2 k^2}}{k} = \sqrt{\frac{\omega_p^2}{k^2} + c^2}$$

$$= c\sqrt{1 + \frac{\omega_p^2}{k^2 c^2}}$$

$k \to \infty \Rightarrow v_\phi \to c$

$k \to 0^+ \Rightarrow v_\phi \to \infty$

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\left(\omega_p^2 + c^2 k^2\right)^{1/2}$$

$$= \frac{1}{2}\left(\omega_p^2 + c^2 k\right)^{-1/2} \cdot 2c^2 k$$

$$= \frac{c^2 k}{\omega}$$

$$= c^2 \frac{1}{v_\phi}$$

i.e. $v_\phi v_g = c^2$

b) $\lambda_0 = 500\,\text{nm} \Rightarrow \omega_p = \frac{2\pi c}{500\,\text{nm}}$ (leave in this

$\lambda_0 = 250\,\text{nm} \Rightarrow \omega = \frac{2\pi c}{250\,\text{nm}}$ form for now)

$$k^2 = \frac{\omega^2 - \omega_p^2}{c^2} = \frac{\left(\frac{2\pi c}{250\,\text{nm}}\right)^2 - \left(\frac{2\pi c}{500\,\text{nm}}\right)^2}{c^2}$$

$$= (2\pi)^2 \left\{\frac{1}{(250\,\text{nm})^2} - \frac{1}{(500\,\text{nm})^2}\right\}$$

$$= \left(\frac{2\pi}{250\,\text{nm}}\right)^2 \left\{1 - \frac{1}{(2)^2}\right\}$$

$$= 0.75\left(\frac{2\pi}{250\,\text{nm}}\right)^2$$

ie $k = \sqrt{0.75} \cdot \frac{2\pi}{250\,\text{nm}}$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi n}{\lambda_0} \qquad \left(\lambda = \frac{\lambda_0}{n}\right)$$

$=$

ie $\lambda = \frac{250}{\sqrt{0.75}}$ nm $=$ 288.7 nm

$n = \sqrt{0.75} = 0.87$ less than vacuum

(Thus reflection from a plasma or metal can be by total internal reflection in vacuum or air)

This wavelength propagates, with a longer-than air/vacuum wavelength.

c) $\lambda_0 = 500\,\text{nm} \Rightarrow \omega = \omega_p$

$\Rightarrow k = 0$

$\Rightarrow \lambda = \infty$

an infinite wavelength means the whole body of plasma oscillates together with no spatial differences

$e^{i(kz-\omega t)} = e^{-i\omega t}$ (no spatial dependence)

This also means $v_\phi = \frac{\omega}{k} = \infty$ infinite phase speed

(which is why theres no spatial dependence)

NB: $v_g = 0$ here – the whole plasma oscillates but no information or energy is transmitted. In fact, theres no EM wave (no spatial dependence means no currents, no magnetic field). This is an electrostatic wave (E-only).

d) $k^2 = \dfrac{\omega^2 - \omega_p^2}{c^2} = (2\pi)^2 \left\{ \dfrac{1}{(750\text{ nm})^2} - \dfrac{1}{(500\text{ nm})^2} \right\}$

$$= \left(\frac{2\pi}{750\text{nm}}\right)^2 \left\{ 1 - \left(\frac{750}{500}\right)^2 \right\}$$

$$= \left(\frac{2\pi}{750\text{nm}}\right)^2 \left\{ 1 - \left(\frac{3}{2}\right)^2 \right\}$$

$$= -1.25 \left(\frac{2\pi}{750\text{nm}}\right)^2 < 0$$

$k^2 < 0 \Rightarrow k \in \mathrm{Im} = \pm i K \quad (K > 0)$

Thus the wave goes as $e^{i(iKz - \omega t)}$

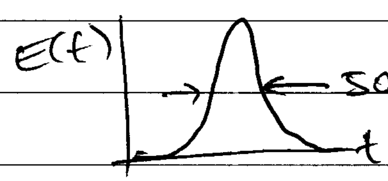
$$= e^{\pm Kz} e^{-i\omega t}$$

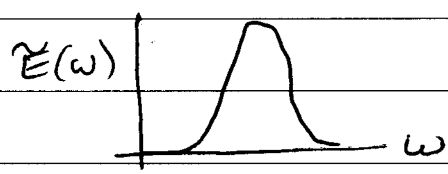
The solutions are formally exponentials that decay or grow - decay for absorption, grow for gain. Without a laser, there is just decay:

$$E \propto e^{-Kz} e^{-i\omega t}$$

So the oscillating wave dies away evanescently. The wavelength is formally imaginary-valued.

c) We ignore reflection losses, but the pulse will see group velocity dispersion as it propagates through the ~~thin~~ ~~thick~~ 2 cm layer of plasma (2,250 wavelengths in ~~air~~ vacuum - how many wavelengths in plasma?)





~~(in vacuum, $\omega\lambda = 2\pi c \Rightarrow$ $\Delta\omega \cdot \lambda + \omega\Delta\lambda = 0$ $\Delta\omega = \frac{-\omega\Delta\lambda}{\lambda}$)~~

$\Delta\nu \Delta t \gtrsim$ "1" depends on def'n of "Δ" width (FWHM? RMS? other?)

transform-limited gaussian $\Delta\nu\Delta t \simeq 0.44$

$$\Delta \nu = \frac{0.44}{\Delta t} = \frac{0.44}{(50 \times 10^{-15} s)} = 8.8 \times 10^{12} s^{-1} \quad (Hz)$$

The spreading of the pulse can be found from the GVD

$$\omega^2 = \omega_p^2 + c^2 k^2$$

differentiate implicitly:

$$2\omega\, d\omega = 2c^2 k\, dk$$

$$\frac{1}{v_g} = k' = \frac{dk}{d\omega} = \frac{\omega}{c^2 k}$$

$$k'' = \frac{d}{d\omega}\left(\frac{dk}{d\omega}\right) = \frac{1}{c^2 k} - \frac{\omega}{c^2 k^2} \cdot \frac{dk}{d\omega}$$

$$= \frac{1}{c^2 k} - \frac{\omega}{c^2 k^2} \cdot \frac{\omega}{c^2 k}$$

$$= \frac{1}{c^2 k}\left(1 - \frac{\omega^2}{c^2 k^2}\right) = \frac{1}{c^2 k}\left(-\frac{\omega_p^2}{c^2 k^2}\right)$$

(from dispersion relation)

$$\omega_p^2 = k_p^2 c^2$$

$$= -\frac{1}{c^2 k}\left(\frac{k_p^2}{k^2}\right)$$

$$= -\frac{\lambda}{2\pi c^2}\left(\frac{\lambda}{\lambda_p}\right)^2$$

$$k''\Big|_{\lambda = 400\,nm} = -\frac{400\,nm}{2\pi\,(3 \times 10^8\, m s^{-1})^2}\left(\frac{400\,nm}{500\,nm}\right)^2$$

$$= 4.5 \times 10^{-25}\, m^{-1} s^2 = 0.45\, m^{-1} (ps)^2$$

$$\tau(z) = \tau_0 \sqrt{1 + (2k'' z \Gamma_0)} \qquad \Gamma_0 = \frac{1}{\tau_0^2}$$

FWHM

$$50 fs = 2\ln 2 \cdot \tau_0$$

$$\tau_0 = 36 fs$$

$$\Rightarrow \Gamma_0 = 770 (ps)^{-2}$$

$$\tau(2cm) = 36 fs \sqrt{1 + (2 \cdot 0.45\, ps^2 \cdot 0.02 m \cdot 770 (ps)^{-2})^2}$$

$$= 36 fs \sqrt{1 + (13.86)^2} = 14\, \tau_0 = 500 fs$$

$$\tau_{FWHM} = 2\ln 2\, \tau(2cm) = 700 fs.$$

4. (a) Construction: need a birefringent crystal, cut from bulk material so that it presents its two axes, exhibiting different indices of refraction, transverse to the axis chosen for the crystal length, and perpendicular to each other. Thus one can choose the polarization of E at different angle α between the two axes. In this way the E-field vector will decompose over two components, one along each axis. Label these axes by their indices: typically n_o (ordinary) and n_e (extraordinary).

Since the indices are different, waves with polarizations along each axis will travel at different phase-speeds, and the *optical path-lengths* differ for each polarization although they travel the same physical distance L through the crystal. Thus they acquire different net phase during their travel, and acquire a phase-difference between the two of:

$$\Phi_{total} = (\omega t - kz)$$

$$\Delta\phi = k_e L - k_o L = \frac{2\pi n_e}{\lambda_0}L - \frac{2\pi n_o}{\lambda_0}L = (n_e - n_o)\frac{2\pi}{\lambda_0}L$$

For a half-wave plate (HWP), the length L is cut so as to make this phase-change a value π (*i.e.,* 1/2*2π). Thus:

$$\pi = \Delta\phi = (n_e - n_o)\frac{2\pi}{\lambda_0}L$$

i.e., the optical path-length difference is one-half wave:

$$(n_e - n_o)L_{HWP} = \frac{\lambda_0}{2}$$

$$L_{HWP} = \frac{\lambda_0}{2(n_e - n_o)}$$

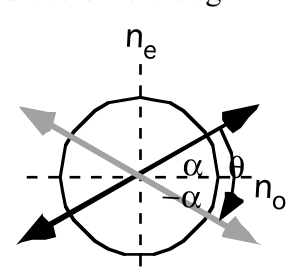
Similarly, for a quarter-wave plate (QWP) the phase change is π/2 (*i.e.,* 1/4*2π) and the optical path-length difference is one-quarter of a wave:

$$(n_e - n_o)L_{QWP} = \frac{\lambda_0}{4}$$

$$L_{QWP} = \frac{\lambda_0}{4(n_e - n_o)}$$

(This much detail is not required, it's just meant to be helpful)

To rotate through an arbitrary angle θ, we use a HWP. If the E-field vector is brought in at an arbitrary angle α to one of the axes, then after propagation the two components will have a relative phase-difference of π — one will be multiplied by (–1) relative to the other. Without loss of generality, this is a mirror image of the polarization in the axis from which α was measured (see diagram).



Thus going from α to – α gives a net angle-change of θ=2 α. Thus, to rotate by θ, bring the polarization in at an angle θ /2.

In Jones calculus, HWP:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

vector rotated by α from x-axis:

$$R(\alpha)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

Then the HWP operation:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ -\sin\alpha \end{bmatrix} = R(-\alpha)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus $\theta = \alpha - (-\alpha) = 2\alpha$.

4. b) For the Pockels effect, we fix L and adjust

$$\Delta n = \Delta n(V)$$

$$\Delta\phi = k_e L - k_o L = (n_e - n_o)\frac{2\pi}{\lambda_0}L$$

$$= \Delta n \frac{2\pi}{\lambda_0}L = (n_0^3 r_{63} E)\frac{2\pi}{\lambda_0}L$$

but $EL = \Delta V$, where E is a constant (uniform) field.

$$\Delta\phi = n_0^3 r_{63}\frac{2\pi}{\lambda_0}\Delta V \quad \text{(regardless of crystal length)}$$

Use: set two crossed polarizers (90° rotation between them) to make a normal OFF=CLOSED state (no transmission). Now add the PC in between, eventually to defeat the crossed polarizers: Set the PC fast and slow axes at $\alpha=\pi/4$ relative to polarizers, *i.e.*, $\theta=\pi/2$ for the rotation of the polarization after the PC if it is made to be a HWP by the right voltage.

Length: is irrelevant (but one usually chooses a convenient length to be about the same as the crystal width).

For ΔV: For the best ON=OPEN state, make the PC a HWP — all the light will then be transmitted. So

$$\pi / 2 = \Delta\phi = (n_0^3 r_{63} E)\frac{2\pi}{\lambda_0}L$$

i.e.,

$$\Delta V = \frac{\lambda_0}{2 n_0^3 r_{63}}$$

for $\lambda_0 = 1$ μm,

$$\Delta V = \frac{1\times 10^{-6}\, m}{2(1.52)^3\, 23.3\times 10^{-12}\, m/V} = 6.11\ kV$$

Note that shorter wavelengths require reduced voltage — a physical shift of distance $\lambda_0/2$ between the phase fronts is a smaller and easier to create for shorter λ_0, with wavefronts closer together.