Q1: see text for definitions, significance.
This question was pretty well done. The main shortcomings were:

1) not giving a definition (e.g., "the dielectric function leads to the index of refraction", which is an OK siginificance but not a definition of the dielectric function). Those affected, try to start a sentence: "The dielectric function IS..."

2) not including a significance -- why this is important in optics, what important thing does it lead to, what historical problem did it solve, etc.

Note that Kramers-Kronig relates *functions*, in our case functions for the real and imaginary index of refraction vs. wavelength or frequency. It is not correct to say that it ties *a particular index* to *a particular absorption*.

I think the average mark for this question could go up by 20% on the final, if folks pay careful attention. Long answers are not needed, but precise answers are.

Q2 a) The Lorentz model is a model of the atom as a simple harmonic oscillator, with the nucleus as a fixed point and the electron moving under a restoring force, with applied fields represented as waves.

It is useful because it is an excellent approximation which explains the interaction of light ~~without~~ with matter without complicated interactions with the nucleus.

It was introduced to explain spectroscopic observations of absorption and emission at specific wavelengths.

b) [NOTE: There is an error in the paper. The expression for the coefficient should have read

$$K = \frac{e^2}{4\pi\varepsilon_0 r^3}$$

not $K = \frac{e^2}{4\pi\varepsilon_0 r^2}$

However, full marks are awarded whether you used the expression in the midterm or not]

A linear restoring force means the particle undergoes simple harmonic motion with angular frequency at which it resonates

$$\omega^2 = \frac{K}{m}$$

$$= \frac{e^2}{4\pi\varepsilon_0 m r^2}$$

$$\omega = 3.18 \times 10^{11} \text{ rad/s}$$

The frequency is then

$$f = \frac{\omega}{2\pi} = 5.06 \times 10^{10} \text{ Hz}$$

Q2 c) The linewidth is given by the damping coefficient, γ (3)

For dipole $\quad P(t) = -e\,x(t)$

$$x(t) = A e^{-\frac{\gamma}{2}} \cos \omega t$$

$$\Rightarrow \quad \frac{d^2 p(t)}{dt^2} = -Ae\omega^2 e^{-\frac{\gamma}{2}t} \cos \omega t$$

$$P_{rad} = \frac{1}{4\pi\varepsilon_0} \frac{2}{3c^3} \left(-Ae\omega^2 e^{-\frac{\gamma}{2}t} \cos \omega t \right)^2$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{2}{3c^3} A^2 e^2 \omega^4 e^{-\gamma t} \cos^2 \omega t$$

$$\frac{dE}{dt} = -\langle P_{rad} \rangle = -\frac{1}{4\pi\varepsilon_0} \frac{2e^2}{3c^3} \omega^4 A^2 e^{-\gamma t} \frac{1}{2}$$

Also, for a damped oscillator

$$E = \frac{1}{2} k \,(\text{Amplitude})^2$$

$$= \frac{1}{2} m\omega^2 A^2 e^{-\gamma t}$$

$$\frac{dE}{dt} = -\gamma E$$

$$= -\gamma \frac{1}{2} m\omega^2 A^2 e^{-\gamma t}$$

Comparing the two terms for $\frac{dE}{dt}$,

$$-\frac{1}{4\pi\varepsilon_0}\frac{2e^2}{3c^3}\omega^4 A^2 e^{-\gamma t}\frac{1}{2} = \frac{1}{2}m\omega^2 A^2 e^{-\gamma t}$$

$$\Rightarrow \qquad \gamma = \frac{e^2\omega^2}{6\pi\varepsilon_0 m c^3} = \frac{(1.6\times10^{-19})^2(3.18\times10^{11})^2}{6\pi(8.85\times10^{-12})(9.11\times10^{-31})(3\times1}$$

$$= 0.63$$

The maximum index change is at $\omega = \omega_0 - \frac{\Delta\omega}{2}$

$$= \omega_0 - \frac{\gamma}{2}$$

with

$$\Delta n = \frac{Ne^2}{4m\varepsilon_0\omega_0}\frac{\omega_0 - (\omega_0 - \frac{\Delta\omega}{2})}{(\omega_0 - (\omega_0 - \frac{\Delta\omega}{2}))^2 + \frac{(\Delta\omega)^2}{4}}$$

$$= \frac{Ne^2}{4m\varepsilon_0\omega_0}\frac{\frac{\Delta\omega}{2}}{2(\frac{\Delta\omega}{2})^2}$$

$$= \frac{Ne^2}{4m\varepsilon_0\omega_0}\frac{1}{\Delta\omega}$$

$$N = 6\times10^{19}/cm^3 = 6\times10^{25}/m^3$$

$$\Delta n = \frac{6\times10^{25}\times(1.6\times10^{-19})^2}{4\times(9.11\times10^{-31})\times(8.85\times10^{-12})\times(3.18\times10^{11})}\frac{1}{\times 0.}$$

$$= 2.38\times10^{17}$$

[NOTE: These values are unphysical because of the mistake in the formula for part b)]

Q3 a) In the Jones calculus, a matrix formalism, we write this basis

$$\{\vec{\alpha}, \vec{\beta}\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$$

orthogonality:

$$\vec{E}_1 \cdot \vec{E}_2^* = 0$$

here means: $\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 0 \Rightarrow$ orthogonal

b) $$\{\vec{\sigma}+, \vec{\sigma}-\} = \left\{\begin{bmatrix}1\\-i\end{bmatrix}, \begin{bmatrix}1\\i\end{bmatrix}\right\}$$

<u>equally valid</u>: $\vec{\alpha} = \begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\left\{\begin{bmatrix}1\\-i\end{bmatrix} + \begin{bmatrix}1\\i\end{bmatrix}\right\} = \frac{1}{2}\{\vec{\sigma}+ + \vec{\sigma}-\}$

$$\vec{\beta} = \begin{bmatrix}0\\1\end{bmatrix} = \frac{i}{2}\left\{\begin{bmatrix}1\\-i\end{bmatrix} - \begin{bmatrix}1\\i\end{bmatrix}\right\} = \frac{i}{2}\{\vec{\sigma}+ - \vec{\sigma}-\}$$

our current basis gives the old basis, and all that that basis provided, as a linear combination

likewise

$$\vec{\sigma}+ = \begin{bmatrix}1\\-i\end{bmatrix} = \left\{\begin{bmatrix}1\\0\end{bmatrix} - i\begin{bmatrix}0\\1\end{bmatrix}\right\} = \vec{\alpha} - i\vec{\beta}$$

$$\vec{\sigma}- = \begin{bmatrix}1\\i\end{bmatrix} = \left\{\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\1\end{bmatrix}\right\} = \vec{\alpha} + i\vec{\beta}$$

Thus <u>each</u> basis is expressed in terms of the other, therefore they are equivalent & so equally valid

<u>orthogonal</u>

$$\vec{\sigma}+ \cdot \vec{\sigma}-^* = \begin{bmatrix}1 & i^*\end{bmatrix}\begin{bmatrix}1\\-i\end{bmatrix} = 1 - i i^* = 1 - i(-i) = 0$$

c) circular polarizer

plan: convert LHC to the correct linear, then pass through polarizer & use 2nd QWP to convert back to LHC

$\begin{bmatrix}1\\i\end{bmatrix}$ <u>Use</u> ① QWP $\begin{bmatrix}1 & 0\\0 & -i\end{bmatrix}$ fast vertical gives $\begin{bmatrix}1 & 0\\0 & -i\end{bmatrix}\begin{bmatrix}1\\i\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$ lin pol 45°

② lin pol 45° $\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ gives $\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

③ QWP fast horiz: $\begin{bmatrix}1 & 0\\0 & i\end{bmatrix}$ gives $\begin{bmatrix}1 & 0\\0 & i\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\i\end{bmatrix}$

Thus we have

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

③ QWP horiz ② pol 45° ① QWP vert

LHC $\begin{bmatrix} 1 \\ i \end{bmatrix}$ passes OK. Must show RHC $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ does <u>not</u>

① $\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

② $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$ RHC does <u>not</u> pass

4(a) $\quad \omega = \dfrac{2\pi c}{\lambda}$ (λ free space value, (standard))

$k = \dfrac{2\pi n}{\lambda}$ (ie $2\pi/\lambda'$ λ' in material)

$k = \dfrac{\omega}{c} n$ Thus a simpler formula for n is useful. Eventually we want $k = k(\omega)$

$$n^2(\lambda) = 1 + \frac{B_1}{1 - (C_1/\lambda^2)}$$ note $C_1/\lambda^2 \ll 1$ in our range

$$= 1 + B_1 \left\{ 1 + \frac{C_1}{\lambda^2} + \left(\frac{C_1}{\lambda^2}\right)^2 + \dots \right\}$$

$$\simeq 1 + B_1 + \frac{B_1 C_1}{\lambda^2}$$

$$= (1 + B_1) \left\{ 1 + \frac{B_1 C_1}{(1+B_1)} \frac{1}{\lambda^2} \right\}$$ note λ is in μm in the formula

thus

$$n(\lambda) = \sqrt{1+B_1} \left\{ 1 + \frac{B_1 C_1}{(1+B_1)} \frac{1}{\lambda^2} \right\}^{1/2}$$ 2nd term small

$$\cong \sqrt{1+B_1} \left\{ 1 + \frac{1}{2} \frac{B_1 C_1}{(1+B_1)} \frac{1}{\lambda^2} \right\}$$

thus

$$\boxed{n(\lambda) \simeq 1.47 \left\{ 1 + 2.84 \times 10^{-3} \frac{1}{\lambda^2} \right\}}$$ λ in μm

changing variables

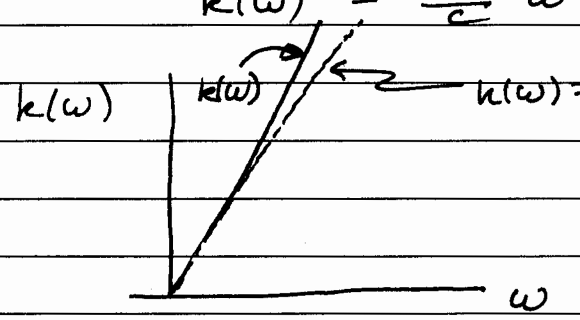
$$\omega = \frac{2\pi c}{\lambda} \quad \text{or} \quad \frac{2\pi c}{\lambda \times 10^{-6}}$$ if λ in μm

$$\frac{1}{\lambda^2} \rightarrow \left(\frac{\omega}{2\pi c}\right)^2$$ λ in μm, ω in 10^6 rad s^{-1} (~~rad s^{-1}~~)

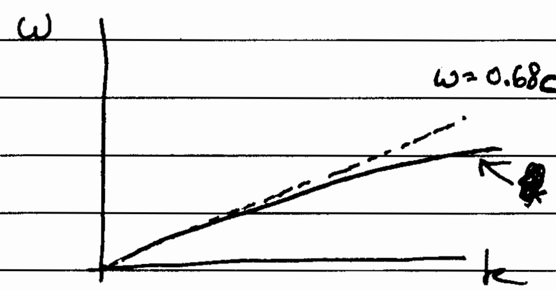
$$n(\omega) = 1.47 \left\{ 1 + 2.84 \times 10^{-3} \left(\frac{\omega}{2\pi c}\right)^2 \right\}$$ ω in 10^6 rad s^{-1}

$$k(\omega) = \frac{\omega}{c} n(\omega) = \frac{1.47\,\omega}{c} \left\{ 1 + 2.84 \times 10^{-3} \left(\frac{\omega}{2\pi c}\right)^2 \right\}$$

$$k(\omega) = \frac{1.47}{c}\omega + 2\pi \cdot 4.17 \times 10^{-3} \left(\frac{\omega}{2\pi c}\right)^3$$

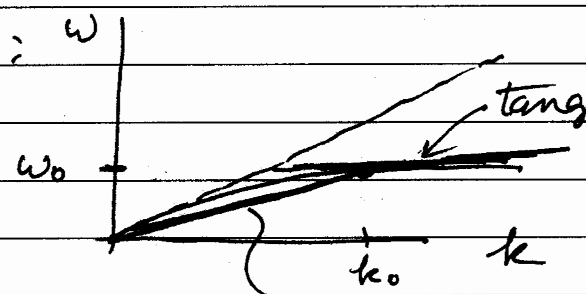


thus



4 (a) cnt'd Our ω-k curve drops away from the phase speed nominal, $v = 0.68c$ at the background index $n \simeq 1.47$ in figure

Exaggerating:



(b) $v_\phi \equiv \frac{\omega}{k(\omega)}$ where $\omega \leftrightarrow \lambda = 0.5\ \mu m$ (doesn't depend on $\Delta\lambda$)

$$= \frac{c}{n(\omega)}$$

$$= \frac{c}{1.47\{1 + 2.84\times10^{-3}\left(\frac{1}{0.5}\right)^2\}} = \frac{c}{1.47\{1 + 1.14\times10^{-2}\}}$$

$$\boxed{v_\phi = 0.673c \quad \text{at } \lambda = 0.5\ \mu m}$$

$$v_g \equiv \left(\frac{dk(\omega)}{d\omega}\right)^{-1}$$

$$k(\omega) = \frac{1.47}{c}\omega + 2\pi\, 4.17\times10^{-3}\left(\frac{\omega}{2\pi c}\right)^3$$

$$k'(\omega) = \frac{1.47}{c} + \frac{3\times4.17\times10^{-3}}{c}\left(\frac{\omega}{2\pi c}\right)^2$$

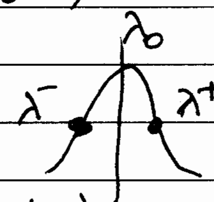
$$= \frac{1.47}{c} + \frac{3\times4.17\times10^{-3}}{c}\left(\frac{1}{\lambda}\right)^2$$

$$= \frac{1}{c}\left(1.47 + 5.0\times10^{-2}\right)$$

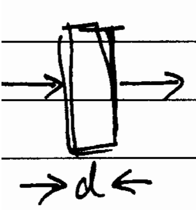
$$= \frac{1.52}{c}$$

$$\boxed{v_g = \frac{c}{1.52} = 0.658c}$$

c) a 25 fs pulse has a range of frequencies (the "bandwidth")



We can calculate the 2nd derivative of $k(\omega)$, or estimate from the difference in times of transit for $\lambda + \frac{\Delta\lambda}{2}$ and $\lambda - \frac{\Delta\lambda}{2}$



Time of transit $t = \frac{d}{v_g}$

Spread of transit times: $\Delta t = d\,\frac{d}{d\omega}\left(\frac{1}{v_g}\right)\cdot\Delta\omega$

$$= d\cdot k''(\omega)\cdot\Delta\omega$$

4(c) cnt'd

$$k''(\omega) = \frac{d}{d\omega}\left\{ \frac{1.47}{c} + \frac{3 \times 4.17 \times 10^{-3}}{c}\left(\frac{\omega}{2\pi c}\right)^2 \right\}$$

$$= \frac{2 \times 3 \times 4.17 \times 10^{-3}}{2\pi c^2} \frac{\omega}{2\pi c}$$

$$= \frac{3.98 \times 10^{-3}}{c^2}\left(\frac{1}{\lambda}\right) \qquad \lambda \text{ in } \mu m$$

$$= \frac{9.96 \times 10^{-3}}{c^2}$$

So, $\Delta t = 5\,cm \; \frac{9.96 \times 10^{-3}}{c^2} \Delta\omega \qquad \Delta\omega$ in 10^6 rad s^{-1}

$\Delta\nu \Delta t \simeq 1/2$ for transform-limited

$\Delta\omega \Delta t \simeq \pi$

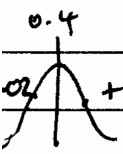
$\Delta\omega \simeq \pi / 25 \times 10^{-15}\,s = 1.26 \times 10^{14}$ rad s^{-1}

$= 1.26 \times 10^{8} \times \underbrace{10^6 \text{ rad s}^{-1}}$

$\Delta t = 0.05\,m \; \frac{9.96 \times 10^{-3}}{(3 \times 10^8)^2} \; 1.26 \times 10^8$ (for λ in μm, ω in Mrad s^{-1})

$= 7 \times 10^{-13}\,s$ ie 700 fs.

The short pulse stretches a lot, relatively speaking!



(Can also use $v_g(0.4\,\mu m + 0.02\,\mu m)$ & $v_g(0.4\,\mu m - 0.02\,\mu m)$ and find difference in transit times $\simeq$ 520 fs.)