Solution PS#1

Q1:

a) The threshold of power: P_{th}=Nhv=$3.6\text{x}10^{-17}$W

b) Red side: $E_r=\frac{hc}{\lambda}$=1.6ev

Blue side: $E_b=\frac{hc}{\lambda}$=3.27ev

c) The rest-mass energy of an electron is

$E_e=m_ec^2=8.19\text{x}10^{-14}$J, $m_e=9.1\text{x}10^{-31}$kg

The energy of photon $E_p=E_e=\frac{hc}{\lambda}$ thus $\lambda=2.43\text{x}10^{-12}$m

Momentum $P=\frac{h}{\lambda}$ $=2.73\text{x}10^{-22}$Js/m

d) Irradiance $I(r)=\frac{100}{4\pi r^2}$ $u(r)=\frac{I}{c}=\frac{100}{4c\pi r^2}$

Energy = $\int_{cubic} u(r)dV = 8\int_0^{1.5}\int_0^{1.5}\int_0^{1.5}\frac{100}{4c\pi(x^2+y^2+z^2)}dxdydz$

$= \quad 8\int_0^{1.5}\int_0^{1.5}\frac{100}{4\pi c}\frac{1}{\left(\sqrt{x^2+y^2}\right)}\text{actan}\left(\frac{1.5}{\sqrt{x^2+y^2}}\right)dxdy$

$=6.11\text{x}10^{-7}$J (numerically)

e) The irradiance of the beam

$I=\frac{c\varepsilon E^2}{2}=\frac{c^3\varepsilon B^2}{2}$ thus $B=\sqrt{\frac{2I}{c^3\varepsilon}}=2.89\text{x}10^{-6}$T

For infinite long straight wire, $B=\frac{\mu i}{2\pi r}$, where *i* is the current

Thus $i=\frac{2\pi rB}{\mu}$=14.45mA

Q3, b) from the lorenze model of collision electrons

$$\tilde{n}^2 = (n+ik)^2 = 1 + \frac{\omega_p{}^2}{\omega_0{}^2 - \omega^2 - i\gamma\omega} \qquad (1)$$

In the medal, the electron is considered as free-electron, thus the restoring force is zero, the resonate frequency is zero, if without collision, that is $\gamma = 0$

Equation (1) turns to be $\tilde{n}^2 = 1 - \frac{\omega_p{}^2}{\omega^2}$, only when $\omega^2 > \omega_p{}^2$, the medal is not transparent to the EM waves, and there is a penetration depth, here the plasma frequency dominate the penetration. For $\omega^2 < \omega_p{}^2$, since $\tilde{n}^2 > 0$, $\tilde{n}$ is real, the metal is a transparent to the EM wave, the penetration depth is infinite….

For
$\lambda_p = 314nm, \lambda = 349nm, \quad \tilde{n}^2 \approx -0.235, \quad \tilde{n} = ik = 0.485i$
penetration depth $d = \frac{\lambda}{2\pi k} \approx 114.58nm$

For
$\lambda_p = 314nm, \lambda = 314nm, \quad \tilde{n}^2 \approx 0, \quad \tilde{n} = ik = 0$
penetration depth $d = \frac{\lambda}{2\pi k} \approx \infty$

For
$\lambda_p = 314nm, \lambda = 289nm, \quad \tilde{n}^2 > 0, \quad \tilde{n} = n > 0, \; ik = 0$
the wave can propagate in the medal infinate long, the medal is transparent medium to the light

However, if we consider the collision of electrons, equation (1) turns to be

$$\tilde{n}^2 = (n+ik)^2 = 1 - \frac{\omega_p{}^2}{\omega^2 + i\gamma\omega} = 1 - \frac{(\omega_p/\omega)^2}{1 + i(\gamma/\omega)}$$

If $\gamma/\omega >> 1$, then $\tilde{n}^2 = (n+ik)^2 \approx 1 - \frac{(\omega_p/\omega)^2}{i(\gamma/\omega)}$

$$n^2 - k^2 = 1 \qquad 2nk = \frac{(\omega_p/\omega)^2}{(\gamma/\omega)} \qquad \therefore \quad k = \left(\frac{\sqrt{1 + \omega_p{}^2/\gamma^2\omega^2} - 1}{2}\right)^{1/2}$$

Penetration depth $d = \frac{\lambda}{2\pi k}$, no matter the frequency is larger or smaller than the plasma frequency, there is always a skin-depth in metal

If $\gamma/\omega << 1$, which is the case listed for the three wavelength in this question, equation (1) turns out to be

$$\tilde{n}^2 = (n+ik)^2 = 1 - \frac{(\omega_p/\omega)^2}{1+i(\gamma/\omega)} \approx 1 - (\omega_p/\omega)^2(1 - i(\gamma/\omega))$$

$$\frac{1}{1+x} \approx 1 - x \quad \text{for } x << 1$$

$$n^2 - k^2 = 1 - (\omega_p/\omega)^2 \qquad 2nk = \frac{\omega_p{}^2\gamma}{\omega^3}$$

Thus

$$\therefore \quad k = \left(\frac{(\omega_p/\omega)^2 - 1 + \sqrt{[1-(\omega_p/\omega)^2]^2 + \gamma^2\omega_p{}^4/\omega^6}}{2}\right)^{1/2}$$

Substitute the value for wavelength into the above equation, one can get the depth for waves can propagate inside the medal is

λ	$(\omega_p/\omega)^2 - 1$	$\gamma^2\omega_p{}^4/\omega^6$	k^2	$d = \frac{\lambda}{2\pi k}$
349nm	0.235	$5.27\text{x}10^{-6}$	0.235	719.9nm
314nm	0	$2.77\text{x}10^{-6}$	$0.832\text{x}10^{-3}$	10.88μm
289nm	-0.153	$1.69\text{x}10^{-6}$	$2.7\text{x}10^{-5}$	55μm

Q5: the value is

The maximum difference in the real part of refraction index is $1.84\text{x}10^{-4}$

The percentage difference of group and phase velocity is

At $7.0\text{x}10^{14}$ Hz， 0.035%

At $9.0\text{x}10^{14}$Hz, 0.003%

2. a) we need to set a condition; one that makes sense could be that the Lorentz force becomes 10% of the force from the E-field, ie $\frac{v}{c} = 0.1$

For the component along $\vec{E}$ we can write

$$v = v_{os} \sin(\omega t) \Rightarrow a = \omega v_{os} \cos(\omega t)$$

To be correct, $F \neq ma$; instead $F \equiv \frac{dp}{dt} = \frac{d}{dt}(mv) = v \cdot \frac{dm}{dt} + m\frac{dv}{dt}$

But classically (non-relativistically) we write

$$|-eE_0| = m_0 a_0 = m_0 \omega v_{os} \qquad (m = m_e \text{ rest mass})$$

$$\Rightarrow v_{os} \equiv \frac{eE_0}{m_0 \omega}$$

Then our dimensionless parameter for qualitative comparison is

$$a_0 \equiv \frac{v_{os}}{c} = \frac{eE_0}{m_0 \omega c}$$

ie when $a_0 = 0.1$ the motion is mildly relativistic;
when $a_0 = 1$ the motion is strongly relativistic

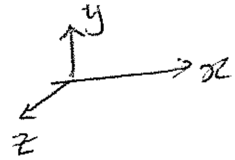
(Notes: ① if ω is small, the period T is large, so a smaller field E_0 is needed, acting over a longer time, to accelerate to high speed.

② $E^2 = m_0^2 c^4 + p^2 c^2$

a_0 can be written $a_0 = \frac{eE_0}{m\omega c} = \frac{eE_0 c / \omega}{(m_0 c^2)}$

$$\frac{eE_0 c}{\omega} = eE_0 \frac{T}{2\pi} \cdot c = (F\Delta t)c = pc$$

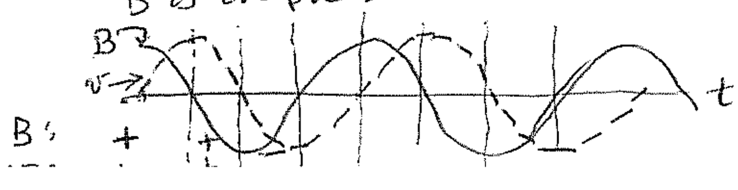
So a_0 effectively compares the kinetic energy of the electron to its rest-mass energy: $a_0 = 1$ means an energy has been given to the electron comparable to its rest-mass energy)



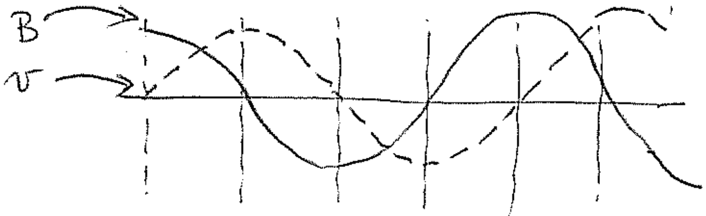
b) transverse acceleration $\hat{y}$ going as $-eE_0 \cos(\omega t)$
transverse speed $\hat{y}$ going as $-\frac{eE_0}{m_0 \omega} \sin(\omega t)$
($\pi/4$ out of phase)

$\vec{v} \times \vec{B} \sim v\hat{y} \times B\hat{z} \parallel \hat{x}$ is longitudinal force, motion

B is in phase with E, so $\underline{\text{also}}$ $\pi/4$ out of phase from v



b) cont'd



sign B:	+	−	−	+	+	+	← oscillates at ω
sign v:	+	+	−	−	+	−	← oscillates at ω
sign $v \times B$:	+	−	+	−	+	−	← product oscillates at $\underline{\underline{2\omega}}$

Thus a longitudinal force, and motion, at $\underline{\underline{2\omega}}$ is imposed.

c) $p = \gamma m_0 v \qquad \gamma \equiv (1 + p^2/m_0^2c^2)^{1/2}$

Consider linear polarization $\vec{E}$ along y, propagation $\vec{k}$ along x

$a_0 \equiv \frac{v_{os}}{c} = \frac{eE_0}{m_0 c \omega}$ dimensionless velocity-amplitude

To simplify, consider "atomic units" such that

$t \leftarrow \omega t$ ie $\omega = k = c = e = m = 1$

$x \leftarrow kx$ so $|B_z| = |E_y|$

$v \leftarrow v/c$

$p \leftarrow p/mc$

$A \leftarrow eA/mc^2$

$$\frac{d\vec{p}}{dt} = -e\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right)$$

becomes

$$\frac{d\vec{p}}{dt} = -(\vec{E} + \vec{v} \times \vec{B})$$

$$\frac{d\vec{p}_\perp}{dt} = -(\vec{E} + (\vec{v} \times \vec{B})_\perp)$$

(when electron moves along x it makes a new Lorentz force back in y)

Consider the vector potential $\vec{A}$ which gives both $\vec{E}$ and $\vec{B}$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$

For our linearly polarized light

$$\vec{A} = (0, a_0 \cos(\omega t - kx), 0) \qquad a_0 \equiv \frac{eE_0}{m_0 c\omega} = \underline{E_0 \text{ here}}$$

c) (cnt'd) Then our equation of momentum $\frac{d\vec{p}_\perp}{dt}$ becomes (i)

$$\frac{d\vec{p}_\perp}{dt} = \frac{\partial \vec{A}}{\partial t} + v_x \frac{\partial \vec{A}}{\partial x}$$

$$= \frac{d}{dt}(\vec{A})$$

the complete derivative

since $\vec{E} \equiv -\frac{\partial \vec{A}}{\partial t} = E_0 \hat{y} \sin(\omega t - kx)$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \left(0, 0, \frac{\partial A_y}{\partial x}\right)$$

$$= -\hat{z} E_0 \sin(\omega t - kx)$$

$(\omega = k = 1)$

$\vec{v}\times\vec{B} = v_x B_z$ (only v_x gives a crossproduct $\perp$)

Thus

(1) $\boxed{\frac{d\vec{p}_\perp}{dt} = \frac{d\vec{A}}{dt}}$ where $\vec{A}$ is the vector potential

For $p_\parallel = p_x$

$$\frac{dp_x}{dt} = -e\underbrace{\cancel{E_x}}_{=0} - \frac{1}{c} v_y B_z$$

$$= -v_y B_z$$

(2) $\boxed{\frac{dp_x}{dt} = -v_y E_y}$ in our units

d) From (1) above we integrate

$$\frac{d(\vec{p}_\perp - \vec{A})}{dt} = 0$$

$\Rightarrow \vec{p}_\perp - \vec{A} = \text{const}$ (a "constant of motion")

we write

$\vec{p}_\perp = \vec{A} + \vec{p}_{\perp 0}$ or starting from rest $\boxed{\vec{p}_\perp = \vec{A}}$ (3)

we combine (2) and $\frac{d\mathcal{E}}{dt} = -e(\vec{v}\cdot\vec{E})$

$$= -e v_y E_y$$

$\mathcal{E} = \gamma m_0 c^2$, so $\frac{d\gamma}{dt} = -v_y E_y$ in our units

Thus $\frac{dp_x}{dt} - \frac{d\gamma}{dt} = -v_y E_y - (-v_y E_y) = 0$

$\frac{d(p_x - \gamma)}{dt} = 0 \Rightarrow \boxed{\gamma - p_x = \alpha}$ another constant of motion (4)

d) (cont'd) $\gamma = (1 + p^2/m^2c^2)^{1/2}$

gives $\gamma^2 = 1 + p_x^2 + p_\perp^2$ in our units

$$\Rightarrow \gamma^2 - p_x^2 - p_\perp^2 = 1$$

using (4) $\gamma = (\alpha + p_x)$

$$(\alpha + p_x)^2 - p_x^2 - p_\perp^2 = 1$$

$$\alpha^2 + 2\alpha p_x - p_\perp^2 = 1$$

(5) $$\boxed{p_x = \frac{1 - \alpha^2 + p_\perp^2}{2\alpha} = \frac{1 - \alpha^2 + A^2}{2\alpha}}$$

We almost have x, if we can integrate

The total phase $\phi \equiv \omega t - kx$

Note: we can write the derivative

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + v_x \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial t} + \frac{p_x}{\gamma}\frac{\partial \phi}{\partial x}$$

$$= \omega - \frac{p_x}{\gamma}(-k)$$

$$= 1 - \frac{p_x}{\gamma} \quad \text{in our units}$$

$$= \frac{\gamma - p_x}{\gamma} = \frac{\alpha}{\gamma}$$

then $$\vec{p} = \gamma m \vec{v} = \gamma \frac{d\vec{r}}{dt} = \gamma \frac{dr}{d\phi} \cdot \frac{d\phi}{dt}$$

$$= \gamma \left(\frac{\alpha}{\gamma}\right) \cdot \frac{dr}{d\phi}$$

(6) $$\boxed{\vec{p} = \alpha \frac{d\vec{r}}{d\phi}}$$

Laboratory Frame

$t = 0$ have $p_x = p_y = 0 \Rightarrow \gamma = 1$

thus (5) $\Rightarrow \alpha = 1$ in the lab frame (conserved)

So $$p_x = \frac{1 - 1^2 + p_\perp^2}{2} = \frac{p_\perp^2}{2} = \frac{A^2}{2} = \frac{a_0^2 \cos^2(\omega t - kx)}{2}$$

$$= \frac{a_0^2}{4}(1 + \cos 2\phi)$$

d) (cont'd) from (3):

$$P_\perp = P_y = A_y = a_0 \cos\phi$$

Using (6)

$$\vec{p} = \alpha \frac{d\vec{r}}{d\phi} \Rightarrow \vec{r} = \int \vec{p}\, d\phi$$

thus the solution

$$\begin{cases} x = \frac{a_0^2}{4}\left(\phi + \frac{1}{2}\sin 2\phi\right) \\ y = a_0 \sin\phi \\ z = 0 \end{cases}$$

note that x has a term that advances with phase: the electron immediately begins to drift forward due to the relativistic effects

$$p_D \equiv \langle p_x \rangle = \frac{a_0^2}{4} \quad \text{Drift Momentum}$$

Being a little careful about velocity since the mass is constantly changing

$$\frac{v_D}{c} = \langle v_x \rangle = \frac{\langle p_x \rangle}{\langle \gamma \rangle}$$

$$\gamma^2 = 1 + p_x^2 + p_y^2$$
$$= 1 + A^2 + A^4/4 = \left(1 + \frac{A^2}{2}\right)^2$$

$$\langle \gamma \rangle = \langle (1 + A^2/2) \rangle = \langle 1 + \frac{a_0^2}{4}(1 + \cos 2\phi) \rangle$$
$$= 1 + a_0^2/4$$

(7) $$\boxed{\frac{v_D}{c} = \frac{\langle p_x \rangle}{\langle \gamma \rangle} = \frac{a_0^2/4}{1 + a_0^2/4} = \frac{a_0^2}{4 + a_0^2}}$$ Drift velocity

Transform to the average rest-frame, where now $\langle p_x \rangle = 0$ in (5), then $1 + \langle A^2 \rangle - \alpha^2 = 0$

Thus $\alpha = \left(1 + \frac{a_0^2}{2}\right)^{1/2} \equiv \gamma_0$

d) (cont'd) So 5 with this α for this frame gives

$$p_x = \frac{a_o^2}{4\gamma_o} \cos 2\phi$$

$$p_y = a_o \cos\phi$$

$$p_z = 0$$

now $\vec{p} = \gamma_o \dfrac{d\vec{r}}{d\phi}$ and we integrate

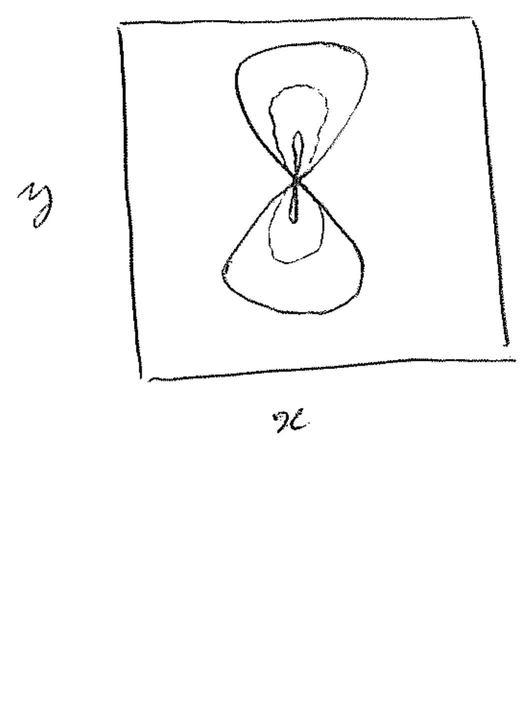
$$x = \frac{1}{2}\left(\frac{a_o}{2\gamma_o}\right)^2 \sin 2\phi$$

$$y = 2\left(\frac{a_o}{2\gamma_o}\right) \sin\phi$$

$$z = 0$$

taking $q \equiv \dfrac{a_o}{2\gamma_o}$, these satisfy the parametric equation

$$16x^2 = y^2(4q^2 - y^2)$$



in the average drift frame