Q(3) a) plasma frequency $\omega_p = \sqrt{\frac{N_e e^2}{m \varepsilon_0}}$

$N_e = 1.5 \times 10^{28} / m^3$

$e = 1.6 \times 10^{-19} C$

$m = 9.11 \times 10^{-31} kg$

$\sigma = 6.8 \times 10^{7}$

$\therefore \nu_p = \frac{1}{2\pi} \sqrt{\frac{N_e e^2}{m \varepsilon_0}} = 1.1 \times 10^{15} Hz$

$\sigma = \frac{N_e e^2 \tau}{m} \Rightarrow \tau = \frac{m \sigma}{N_e e^2}$

$\Rightarrow \tau = 1.61 \times 10^{-13} / s$

Q4: The dispersion relationship between k and ω is -

$$k = \frac{\omega}{c}[1+\alpha(\omega)]^{\frac{1}{2}}$$

while
$$\alpha(\omega) = \frac{a}{\omega_0^2-\omega^2+i2\beta\omega} = \frac{a(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)+4\beta^2\omega^2} - i\frac{2a\beta\omega}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$$

$$a = \frac{Ne^2}{2\varepsilon_0 M} = 9.5\times10^{27}\,s^{-2} \quad N = 6\times10^{23}/m^3 \quad \omega_0 = 2\pi\frac{c}{\lambda_0} = 6.28\times10^{15}\,Hz$$

$$\beta = 10^{13}\,s^{-1}$$

compare α with 1, $\alpha(\omega)$ is in the order around $10^{-3} \ll 1$

$$\therefore [1+\alpha(\omega)]^{\frac{1}{2}} \approx 1+\frac{1}{2}\alpha(\omega)$$

$$\therefore k = \frac{\omega}{c}+\frac{\omega}{2c}\alpha(\omega) \approx k_R + ik_I$$

$$k_R = \frac{\omega}{c}+\frac{\omega}{2c}\frac{a(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2} \qquad k_I = \frac{\omega}{2c}\frac{2a\beta\omega}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$$

Phase velocity $v_p = \frac{\omega}{k} = \frac{\omega}{\frac{\omega}{c}+\frac{\omega}{2c}Re[\alpha(\omega)]} = \frac{c}{1+Re[\alpha(\omega)]/2}$ $\quad Re[\alpha(\omega)] \sim$ real part of $\alpha(\omega)$

group velocity $v_g = \frac{d\omega}{dk} = \frac{1}{\left(\frac{dk}{d\omega}\right)} \neq \cancel{\frac{c}{1+\frac{Re}{2}[\alpha(\omega)]}}$

$$= \frac{c}{1+\frac{1}{2}Re[\alpha(\omega)]+\frac{\omega}{2}Re[\alpha'(\omega)]} = \frac{v_p}{1+\frac{v_p}{2\lambda}Re[\alpha'(\omega)]}$$

where
$$Re[\alpha'(\omega)] = \frac{-2a\omega}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2} + \frac{-a(\omega_0^2-\omega^2)[2(\omega_0^2-\omega^2)(-2\omega)+8\beta^2\omega]}{[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2]^2}$$

$$RE = \frac{2a\omega(\omega_0^2-\omega^2)^2-8\omega^3a\beta^2-8a\omega\beta^2(\omega_0^2-\omega^2)}{[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2]^2}$$

$$Re[\alpha(\omega)] = \frac{a(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)+4\beta^2\omega^2}$$

$$\cancel{\frac{v_p - v_g}{v_p} = \frac{v_p - \frac{v_p}{1+\frac{v_p}{2\lambda}Re[\alpha'(\omega)]}}{\frac{v_p}{1+\frac{v_p}{2\lambda}Re[\alpha'(\omega)]}}}$$

$$\frac{v_p - v_g}{v_p} = \frac{v_p - \frac{v_p}{1+\frac{v_p}{2\lambda}Re[\alpha'(\omega)]}}{v_p} = \frac{\frac{v_p}{2\lambda}Re[\alpha'(\omega)]}{1+\frac{v_p}{2\lambda}Re[\alpha'(\omega)]} = \frac{v_p Re[\alpha'(\omega)]}{2\lambda + v_p Re[\alpha'(\omega)]}$$

for $\lambda = 320nm$, $Re[\alpha(\omega)] = 0.077$,

$Re[\alpha'(\omega)] = 1.1\times10^{-15}$

$v_p = 0.996c$

$v_g = 0.946c$

$\frac{v_p - v_g}{v_p} = 5\%$

for $\lambda = 500nm$.

$Re[\alpha(\omega)] = 0.045$

$Re[\alpha'(\omega)] = 2.8\times10^{-17}$

$v_p = 0.999c$

$v_g = 0.998c$

$\frac{v_p - v_g}{v_p} \approx 0.1\%$

The absorption cofficient is $2k_I$ because intensity is amplitud squared. Thus

$$k(\omega) = \frac{\omega}{c} Im[\alpha(\omega)] = \frac{2c^{-1} a \beta \omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}.$$

The maximum of this is located at ω_0, which can be derived methematically

by $\frac{dk}{d\omega} = 0 = 2\omega + \frac{4\omega^3(\omega_0^2-\omega^2) - 8\beta^2\omega^3}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2} \Rightarrow \omega = \pm\omega_0, \pm i\omega_0$

of course, only the real positive frequeny is meaningful.

$\therefore$ ~~$2k(\omega_0) =$~~

$$A(\omega_0) = 2k_I = \frac{2c^{-1}a\beta\omega_0^2}{4\beta^2\omega_0^2} = a/2\beta c = 1.6\times10^5 m^{-1}$$

Q5: absorption

$$a(\nu) = \frac{Ne^2}{4\pi\varepsilon_0 mc}\frac{\delta\nu_0}{(\nu-\nu_0)^2+(\delta\nu_0)^2} \qquad \delta\nu_0 = \frac{\gamma}{4\pi}$$

at $\nu=\nu_0$, consider the oscillator strength, that is the fraction of oscillators response to frequency ν_0 is

$$a(\nu_0) = \frac{Ne^2}{4\pi\varepsilon_0 mc}\cdot\frac{f}{\delta\nu_0} = 1400\ m^{-1}$$ from the lineshape.

and $2\delta\nu_0 = 10^{14}\ s^{-1} \Rightarrow \cancel{\delta\nu_0} = 5\times10^{13}\ s^{-1}$ $\quad e = 1.6\times10^{-19}C$

$N \approx 10^{23}/m^3$, $m = 9.11\times10^{-31}kg$ $\quad \varepsilon_0 = 8.85\times10^{-12}$

$$\therefore f = \frac{1400\times5\times10^{13}\times4\pi\varepsilon_0 mc}{Ne^2} = 0.1331$$

$$\tau = \frac{1}{\beta} = \frac{1}{2\pi\delta\nu_0} \approx 3.18\times10^{-15}\ s \quad \sim \text{life time.}$$

The real part $\quad n = n_b + \dfrac{Ne^2}{4m\varepsilon_0\omega_0}\dfrac{\omega_0-\omega}{(\omega-\omega_0)^2+\frac{\gamma^2}{4}} \qquad n_b = 1.55$

$\omega = \omega_0 - \frac{\gamma}{2}$ max, $\quad \omega = \omega_0 + \frac{\gamma}{2}$ min $\Leftarrow \frac{dn}{d\omega} = 0$

$$\cancel{\Delta n_{max}} = \frac{Ne^2}{\cancel{4\pi\varepsilon_0 mc\cdot 2\nu_0\gamma}} \qquad \Delta n_{max} = \frac{Ne^2}{2m\varepsilon_0\omega_0\gamma} = \frac{Ne^2}{4\pi\nu_0\gamma m\varepsilon_0}$$

$$\approx 3.6\times10^{-4}$$

$$V_\phi = \frac{c}{n(\omega)} \qquad V_g = \frac{d\omega}{dk} \qquad \frac{V_\phi - V_g}{V_g} = \frac{\omega}{n(\omega)}\frac{dn(\omega)}{d\omega}$$

$$\because \frac{dn(\omega)}{d\omega} = \frac{Ne^2}{4m\varepsilon_0\omega_0}\left\{\frac{2(\omega-\omega_0)^2}{[(\omega-\omega_0)^2+\frac{\gamma^2}{4}]^2} - \frac{1}{(\omega-\omega_0)^2+\frac{\gamma^2}{4}}\right\} \qquad \omega_0 = 2\pi\nu_0 \quad \Delta\gamma = 4\pi\delta\nu_0$$

$$\frac{V_\phi - V_g}{V_g} = \frac{Ne^2\nu}{4m\varepsilon_0 n(\omega)\nu_0}\left\{\frac{2(\omega-\omega_0)^2}{[(\omega-\omega_0)^2+\frac{\gamma^2}{4}]^2} - \frac{1}{(\omega-\omega_0)^2+\frac{\gamma^2}{4}}\right\} \quad \cdots\cdots (*)$$

for $\nu = 7 \times 10^{14}\,Hz$, $\nu_0 = 6.8 \times 10^{14}\,Hz$.

$$\left| \frac{V_\phi - V_g}{V_g} \right| \simeq 0.035\% ,$$

$\Big\{$ first calculat $n(\omega)\big|_{\bar{\nu} = 7 \times 10^{14} Hz}$.
then substitute into equation (*)

for $\nu = 9 \times 10^{14}\,Hz$.

$$\left| \frac{V_\phi - V_g}{V_g} \right| = 0.003\%$$