Problem Set 4

Q1 a) Hecht 12.9

The coherence length is $L_c = 0.26\ \lambda\ L / R$

$= 0.26 \times 589.3\text{E-9} \times 1 / 5\text{E-5}$

$= 3.06$ mm

b) Hecht 12.15

Visibility = $|\ \text{sinc}\ (a\ \pi\ b / l\ \lambda\ )| = 0.85$

=> $(a\ \pi\ b / l\ \lambda) = 0.97$

$b = l\ \lambda\ 0.97 / a\ \pi = 1.5 \times 500\text{E-9} \times 0.97 / (0.5\text{E-3} \times \pi)$

$= 0.463$ mm

c) Hecht 12.17

Define the x-axis along the axis of the two pinholes so that $\mathbf{k.r} = kx \tan \alpha$ for each source. Then the Cittert-Zernicke theorem gives:

$$
\begin{aligned}
|\Gamma(r_1, r_2)| &\propto |\int_{-\infty}^{\infty} I(r) e^{i \frac{k \cdot r_i}{L}(r_i - r)} dr| \\
&= |\int_{-\infty}^{\infty} I_{0,1} \delta(r - r_1) e^{i \frac{\mathbf{k} \cdot \mathbf{r_1}}{L} \Delta r} + I_{0,2} \delta(r - r_2) e^{i \frac{\mathbf{k} \cdot \mathbf{r_2}}{L} \Delta r} dr| \\
&= |I_{0,1} e^{ik\alpha_1 a} + I_{0,2} e^{ik\alpha_2 a}|.
\end{aligned}
$$

with $I_{0,1} = I_{0,2}$ and the Visibility proportional to the above mutual coherence function, we can write

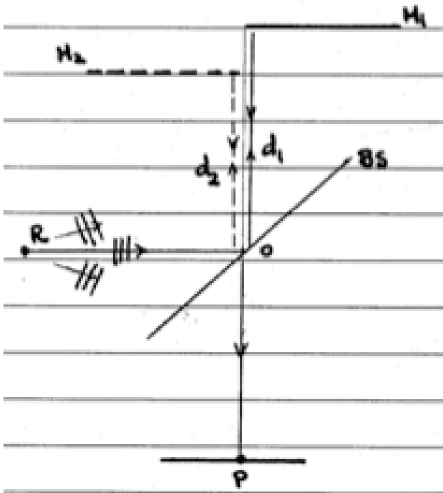
$$
V^2 \propto |e^{ik\alpha_1 a} + e^{ik\alpha_2 a}|^2 = 2(1 + \cos(ka(\alpha_1 - \alpha_2))) = 0
$$

which is minimum for $\cos(k\ a(\alpha_1 - \alpha_2)) = -1$

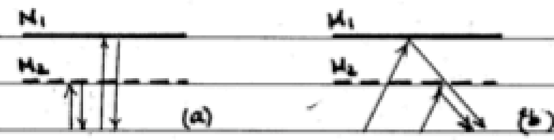
⇨ $a(\alpha_1 - \alpha_2) = ½\ m$ for odd m integers

Q2 a)

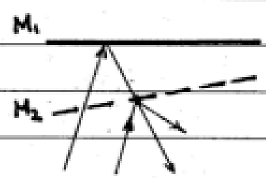
Let us consider first the case when M_1 and M_2 mirrors are perpendicular to each other. To clarify the situation we redraw the Michelson's interferometer scheme as below. The redrawn scheme shows that a situation similar to reflection from a dielectric slab is occuring.



To explain the appearance of the ringlike interferences patterns we have to consider that rays from the source fall on BS at angles different from 45° as well.

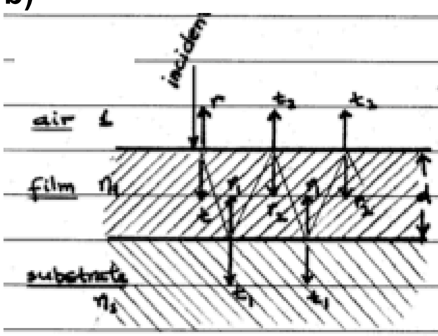


To observe the interference of the rays falling on the BS at angles ≠ 45°, the eye of the observer has to focus at infinity, or likewise the source (extended) should be put at the focus of a lens, and the interferometer at the other side of the lens. This situation (of perpendicular M_1 and M_2) has cylindrical symmetry about the RO axis of the picture. Bundles of parallel rays, falling on BS at different angles, fulfill different interference conditions as can be seen from (a) and (b) pictures. Bundles of parallel rays coming toward P are then focused on the retina by the eyeball and should result in circular interference patterns. If M_1 is not perpendicular to M_2 then we have a wedge like situation



The interference pattern can be observed if the eye is focused on the surface of M_2 where reflected rays meet. In this case the interference pattern is made up of stripes lined in the direction out of the page.

Q 2
b)



$n_1 > n_s > 1$

have: $r = \frac{1-n_1}{1+n_1}$ (since $n_1>1$ we have a phase change of 180° as the wave goes from less dense to dense medium)

$r_1 = \frac{n_1-n_s}{n_1+n_s}$ (no phase change)

$r_2 = \frac{n_1-1}{n_1+1} = -r$

$t = \frac{2}{n_1+1}$; t_1 = (do not need this) ; $t_2 = \frac{2n_1}{n_1+1} = n_1 t$

For each pass through the film, the normally incident ray changes the phase by e^{ikd} where d is the film thickness. Now $kd = n_1 \frac{2\pi}{\lambda_0} d = \frac{\pi}{2} \frac{d}{\left(\frac{\lambda_0}{4n_1}\right)}$
If $d = \frac{\lambda_0}{4n_1}$ we have a quarter-wave film $\Rightarrow e^{ikd} = e^{i\pi/2} = i$.

The reflected ray has an amplitude factor (where subindex c is for coating)

$$r_c = r + t\,ir_1it_2 + t\,ir_1ir_2ir_1it_2 + t\,ir_1ir_2ir_1ir_2ir_1it_2 + \cdots$$

$$= r + tt_2 i^2 r_1 + tt_2 i^4 r_1^2 r_2 + tt_2 i^6 r_1^3 r_2^2 + \cdots$$

$$= r + \frac{tt_2}{r_2}(i^2 r_1 r_2) + \frac{tt_2}{r_2}(i^4 r_1^2 r_2^2) + \frac{tt_2}{r_2}(i^6 r_1^3 r_2^3) + \cdots$$

$$= r + \frac{tt_2}{r_2}(i^2 r_1 r_2)^1 + \frac{tt_2}{r_2}(i^2 r_1 r_2)^2 + \frac{tt_2}{r_2}(i^2 r_1 r_2)^3 + \cdots$$

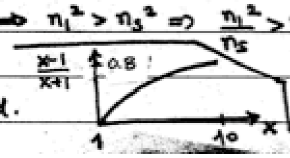
$$= r + \frac{tt_2}{r_2} i^2 r_1 r_2 \left[1 + (i^2 r_1 r_2) + (i^2 r_1 r_2)^2 + \cdots\right]; \text{ since } i^2 r_1 r_2 = -r_1(-r) = r_1 r$$

$$= r - r_1 t t_2 \left[1 + (r_1 r) + (r_1 r)^2 + \cdots\right] = r - r_1 n_1 t^2 \frac{1}{1-r_1 r} = \frac{r - r_1 r^2 - r_1 n_1 t^2}{1 - r_1 r}$$

$$= \frac{r - r_1[\underbrace{r^2 + n_1 t^2}_{=1}]}{1 - r_1 r} = \frac{r - r_1}{1 - r_1 r} = \frac{n_s - n_1^2}{n_s + n_1^2} = \frac{1 - \frac{n_1^2}{n_s}}{1 + \frac{n_1^2}{n_s}}$$

This amplitude factor has to be compared with the amplitude factor of the bare substrate $r_b = \frac{1-n_s}{1+n_s}$; Since $n_1 > n_s \Rightarrow n_1^2 > n_s^2 \Rightarrow \frac{n_1^2}{n_s} > n_s$

we get $|r_c| > |r_b|$. The reflectance is increased.



Q 2 c) We require the resolution to distinguish between 589.59 and 588.99, a difference of 0.60 nm

$$(\Delta\lambda)_{res} = \lambda_0^2 / \text{Finesse} \times 2n_f\, d = 0.60E\text{-}9$$

We require a free spectral range over the visible spectrum~ 400 nm

$$(\Delta\lambda)_{FSR} = \lambda_0^2 / 2n_f\, d = 4E\text{-}7$$

$$n_f d = \lambda_0^2 / 3E\text{-}7 \times 2$$
$$= (5E\text{-}7)^2 / 3E\text{-}7 \times 2$$
assuming central wavelength in middle of spectral range
$$= 416.67 \text{ nm},$$
which is very small – it's unusual to have FSRs as big as the whole visible spectrum.

The finesse required for the desired resolution is then

$$\text{Finesse} = \lambda_0^2 / (\Delta\lambda)_{res} \times 2n_f\, d$$
$$= (5E\text{-}7^2) / 0.6E\text{-}9 \times 2 \times 416.67\ E\text{-}9$$
$$= 500, \text{ quite reasonable}$$

$$\text{sqrt}(F) = (2 \times \text{Finesse} / \pi)$$
$$= 318.47$$

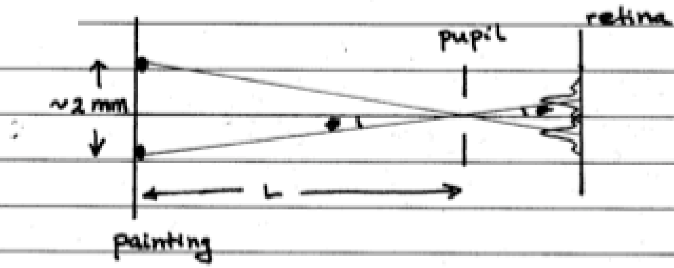
$$\text{sqrt}(F) = 2r/(1-r^2)$$

$$(1-r^2)\ 318.67 = 2r$$

$$318.47\ r^2 + 2r - 318.47 = 0$$
Solve with quadratic formula, formally get:

r = 0.997, or –1.003 (2nd is non-physical)
A mirror of 99.7% reflectivity is quite possible for a range of wavelengths, perhaps 40nm, but maybe not the whole visible range. I'm not sure metallic coatings have such a high value, though they can span a bigger range. It's certainly a demanding design criterion…

Q3 a)



Take the pupil size to be $D_p \sim 5\,mm$. The angular radius of the Airy disk is $\sin\theta_a = 1.22 \frac{\lambda}{D_p}$. From the picture, if $\theta < \theta_a$ then Airy disks of different colors overlap on the retina and color mixing occurs. $\tan\theta \approx \frac{1mm}{L}$; also $\sin\theta_a \approx 1.22 \frac{480nm}{5mm}$ for blue lights $\Rightarrow \sin\theta_a \approx 1.2\times10^{-4} \Rightarrow \theta_a \approx 1.2\times10^{-4}$ rad. since $\theta < \theta_a \Rightarrow \tan\theta \approx \sin\theta$ and $\sin\theta \approx \frac{1mm}{L} < \sin\theta_a \approx 1.2\times10^{-4} \Rightarrow L \approx 8.5\ m$

Q3 b) The angular resolution of the Hubble telescope is

$$\sin\theta_a = 1.22 \frac{550nm}{2.4m} \approx 2.8\times10^{-7} \Rightarrow \theta \approx 2.8\times10^{-7}\ rad$$

$$\theta_a = 2.8\times10^{-7} \frac{180°}{\pi} \approx 1.6\times10^{-5} \text{ degrees of arc} \approx 1.6\times10^{-5}\times3600 \text{ sec of arc} = 0.058 \text{ sec of arc.}$$

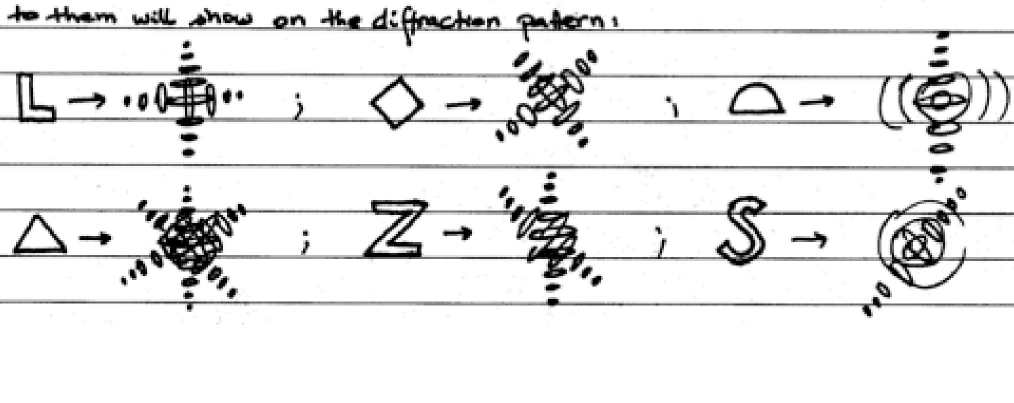
The condition of resolving two objects on the Moon's surface is

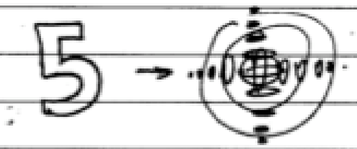
$$\frac{d/2}{L} = \tan\theta \approx \sin\theta > \sin\theta_a \approx 2.8\times10^{-7} \Rightarrow \underline{d > 215m}$$ for

an Earth-Moon distance of $L \approx 3.844\times10^8 m$, and d is the objects' separation.

On the other hand, to resolve two objects on the Moon's surface by bare eye, they should be $d > 2L\sin\theta_a = \underline{92\ km}$ appart.

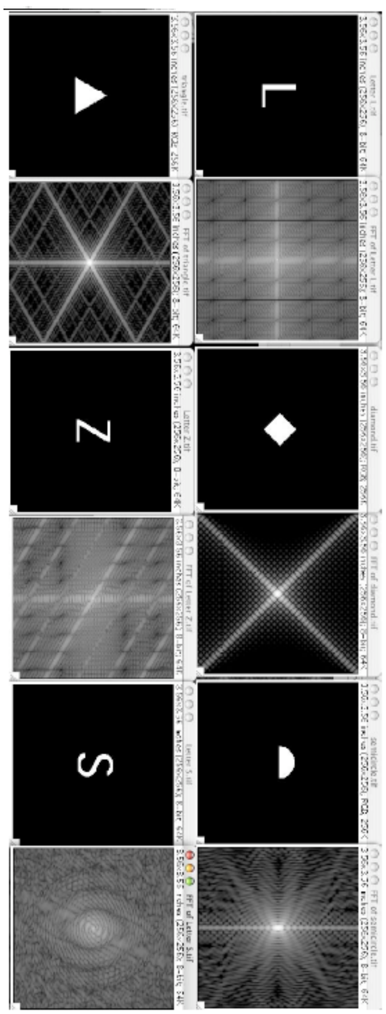
Q3 c) Whenever straight edges are present, diffraction lines perpendicular to them will show on the diffraction pattern:





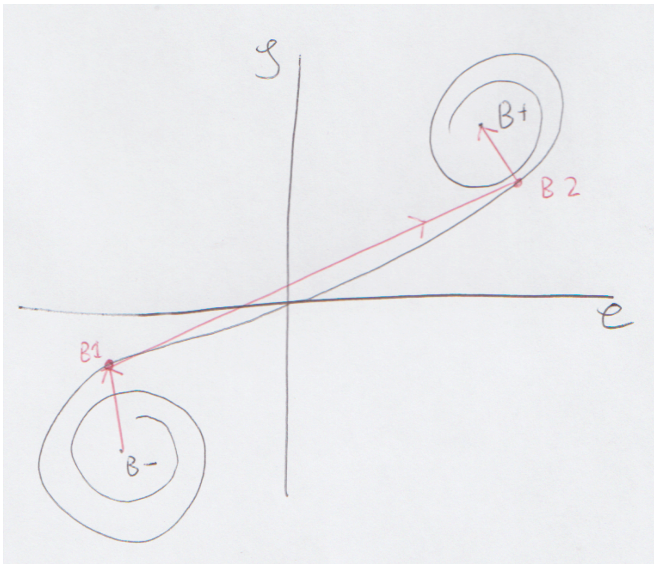
Note: Exact diffraction patterns corresponding to the above shapes can be drawn using the computer. You can use a free software: ImageJ (a Java image processor) available for Mac, PCs, Linux, etc. Please visit: http://rsb.info.nih.gov/ij/

These figures are Fast Fourier Transforms (FFTs; an optimized algorithm that approximates Fourier Transforms), so effectively what they do is fold back (or reflect) the images at the walls of the box. So fr instance, the hand-drawn figure for the diamond is better and clearer. (This note is from Prof. R. Marjoribanks)



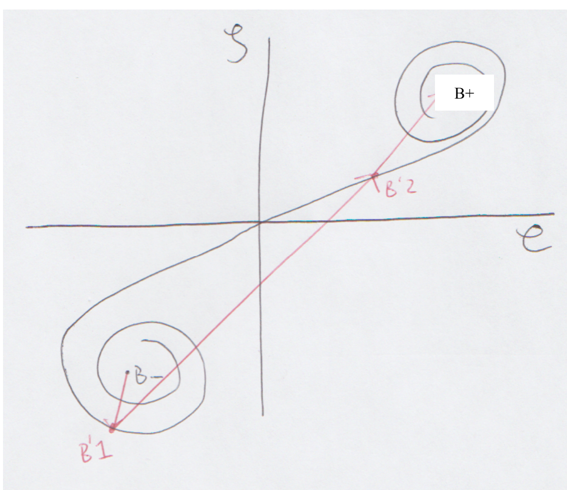
Q4 a) Babinet's principle states that the sum of the optical disturbances (*alias* E-fields) due to two complementary apertures is equal to the undisturbed field.

$$E_1 + E_2 = E_0$$



The integrals of the disturbance for an aperture can be found on the Cornu spiral. To demonstrate Babinet, the phasor representing a narrow slit is shown on the spiral below between points B1 and B2. A half-infinite aperture ending at one edge of the slit is represented by B-B1, another taking up at the other slit-edge by B2B+.

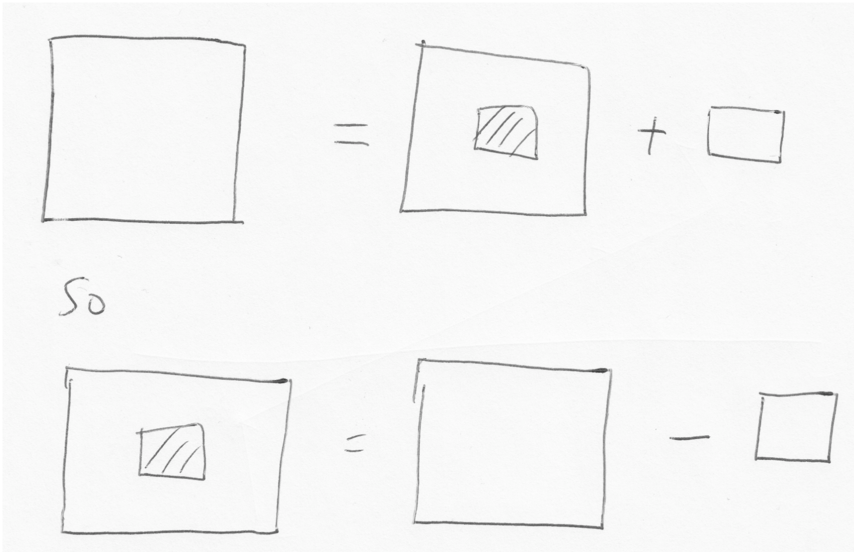
We can see that
B1B2 + B-B1 + B2B+ = B-B+
which is the phasor for the undisturbed field: Babinet's principle is basically the vector sum.



Another drawing shows the same situation from a different observation point – it only changes the location of B`1 and B`2, not the result (or the length along the spiral of B1B2 = B`1B`2)

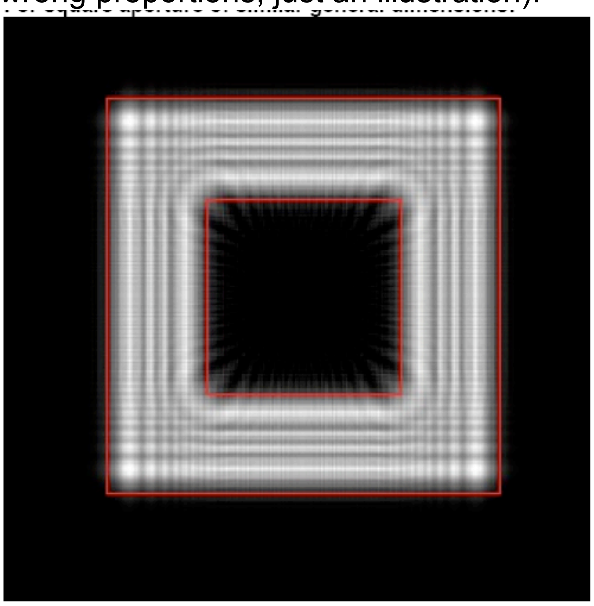
[*RSM NOTE:* B1 and B2 slide together along the Cornu spiral, as we change observation point… we shift only the parameterizations, both at once]

b) Fraunhofer diffraction becomes valid when $R >> a^2 / \lambda$
$>> (50E\text{-}6)^2 / 5E\text{-}7$ = 0.5 mm from screen



The diffracted disturbance (E-field) from the small rectangular aperture, seen at the observation plane, will be subtracted from that of the large rectangular aperture. The intensity at each point can be found by sliding B1, B2, corresponding to edges of slit, together along the Cornu spiral, the intensity given by the square modulus of the difference of the complex numbers between them for each observation point (i.e., the square of the length of the vector between B1 to B2). The disturbance in each axis for each rectangle is simply a near-field slit pattern, and the two patterns interfere, after which the square modulus gives the intensity, for the pattern. So you have a grid-like structure of fringes for the superposition (interference) between the separate solutions for each rectangle, smaller and larger. Remember that first the E-fields add and interfere, then taking the intensity of that gives the pattern.

The patterns can also be plotted numerically, and will resemble this (which has wrong proportions, just an illustration):



Q5

a) For stability, find parameters g_1, g_2:

$$g_1 = 1 - \frac{L}{R_1} = 1 - \frac{1}{\infty} = 1$$

$$g_2 = 1 - \frac{L}{R_2} = 1 - \frac{1}{3} = 0.67$$

The product $g_1 g_2 = 0.67$ satisfies stability for the cavity $0 < g_1 g_2 < 1$.

The Gaussian beam matches the mirror curvature at either end, with formula:

$$R(z) = z + \frac{z_0^2}{z}$$

$R = \infty$ at flat mirror, $z=0$ there; $R(1\text{m})=3\text{m}$, so

$$z_0(oscillator) = \sqrt{3-1} = \sqrt{2}$$

Then the beamwaist on the flat mirror is:

$$w_0 = \sqrt{\frac{z_0 \lambda}{\pi}} = 0.67\,\text{mm}$$

where we've chosen an example wavelength $\lambda = 1$ µm. The far-field divergence angle is:

$$\theta = \frac{\lambda}{\pi w_0} = \frac{w_0}{z_0} = 0.47\,\text{mrad}$$

b) The second cavity has these parameters:

$$5\,\text{m} = R(1.25\,\text{m}) = 1.25\,\text{m} + \frac{z_0^2}{1.25\,\text{m}}$$

$$\Rightarrow z_0 = 2.165\,\text{m}, \quad w_0 = 0.83\,\text{mm}$$

The two flat mirrors with the output of the oscillator and input to the regen are facing each other 4 m apart. We need to find where to put a lens and with what focal length so that we can relay the flat-wavefront beamwaist from the output flat mirror to the *right diameter* flat-wavefront beamwaist at the input to the regen.

From that point, this can be done the hard way or the not-so-hard way: the hard way is to write the general ray-matrix for an unknown lens an unknown distance x along the 4 m separation:

$$M = \begin{bmatrix} 1 & (4-x) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

and then use the new $ABCD$ matrix in a fractional-linear transformation

$$q_{regen} = T(q_{osc}) = \frac{A q_{osc} + B}{C q_{osc} + D}$$

to map the beam-parameter q_{osc} at the first flat mirror onto the required parameter q_{regen} on the second flat mirror, using:

$$q = \frac{i\lambda}{\pi w_0^2} = -iz_0$$

Since both these q's are imaginary-valued, the real part of $T(q_{\text{osc}})$ must vanish and you can get formulae for x and f.

However, it's always easier to split the problem up in a *physical* sense, if it will make two decoupled equations.

Here, we do that by noting that when crossing the thin lens the beam does not change diameter. So the position of the lens must be where the beam of the oscillator is the same size w as the beam of the regen, working backwards from the flat input mirror. That is, for each cavity separately we use:

$$w^2(z) = w_0^2\left(1 + \frac{z}{z_0^2}\right)$$

to get:

$$w_0^2\left(1 + \frac{z^2}{z_0^2}\right) = w^2(z) = w^2(-4+z) = w_0^2\left(1 + \frac{(-4+z)^2}{z_0^2}\right)$$

where blue is for the oscillator parameters, red for the regen. This can be rewritten as the quadratic equation:

$$0 = \left(\frac{w_0^2}{z_0^2} - \frac{w_0^2}{z_0^2}\right)z^2 - 8\frac{w_0^2}{z_0^2}z + \left(w_0^2 - w_0^2 + 16\frac{w_0^2}{z_0^2}\right)$$

which for our parameters becomes

$$0 = 7.8\times10^{-8}z^2 + 1.17\times10^{-6}z - 2.59\times10^{-6}$$

and the lens goes at z=1.95 m, measured from the output flat mirror of the oscillator.

The lens must then change the curvature of the beam from positive (expanding) to negative (converging) to go to the second beamwaist. We find the curvatures at this location:

$$R(1.95\,\text{m}) = z + \frac{z_0^2}{z} = 1.95 + \frac{\left(\sqrt{2}\right)^2}{1.95} = 2.98\,\text{m}$$

$$R(-4\,\text{m} + 1.95\,\text{m}) = (-4+1.95) + \frac{2.17^2}{(-4+1.95)} = -4.34\,\text{m}$$

The relation for the focal length is then

$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}$$

which gives a focal length of 1.77 m.

c) The oscilloscope will show the beat-frequency between the two transverse modes of the regen if the detector sees just a spot in the beam (and doesn't average the whole thing). The frequencies of different transverse modes are given by:

$$\nu_{qmn} = \frac{c}{2L}\left(q + \frac{1}{\pi}(m+n+1)\cos^{-1}\sqrt{g_1 g_2} \right)$$

For the regen, g_2 = 1–1.25/5=0.75, and $\cos^{-1}\sqrt{g_1 g_2}$ =0.72. The frequency difference of transverse modes is smaller than the difference between longitudinal modes, and the net difference between TEM_{00} and TEM_{10} modes is

$$\Delta\nu = \nu_{q10} - \nu_{q00} = \frac{c}{2L}\left(\frac{1}{\pi}\cos^{-1}\sqrt{g_1 g_2} \right) = 27.5\,\text{MHz}$$

This shows up as a modulation in the photodiode signal with a period around 36ns, which is about 4-5 roundtrips.