

APPENDIX: Some Helpful Formulae and Equations

Classical Model

$$\frac{\epsilon_i}{\epsilon_o} = 2n\kappa = \frac{Ne^2}{m\epsilon_o} \left(\frac{\gamma\omega}{(\omega^2 - \omega_o^2)^2 + \gamma^2\omega^2} \right) \quad \frac{\epsilon_R}{\epsilon_o} = n^2 - \kappa^2 = 1 + \frac{Ne^2}{2m\epsilon_o\omega_o} \left(\frac{\omega_o - \omega}{(\omega - \omega_o)^2 + \frac{\gamma^2}{4}} \right) \quad \text{for } \omega \approx \omega_o$$

Diffraction Optics

$$U_P = -\frac{ik}{4\pi} e^{-i\omega t} \iint_A U_A \frac{e^{ikr}}{r} (\cos(\hat{n}, \hat{r}) + 1) dS \quad U_P = \frac{1}{4\pi} \iint_A \left[U \frac{\partial}{\partial n} \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r} \frac{\partial U}{\partial n} \right] dS$$

$$U_P(u, v) = -\frac{i}{\lambda z} e^{ik|PS|} \exp \left\{ \frac{ik}{2z_a} [x_m^2 + y_m^2] \right\} \iint_A U_o(x_o, y_o) \exp \{-i[ux_o + vy_o]\} dx_o dy_o$$

$$U_P(x, y, z) = -\frac{iU_o}{\lambda z z'} e^{ik|PS|} \iint_A \exp \left\{ \frac{ik}{2z_a} [(x_o - x_m)^2 + (y_o - y_m)^2] \right\} dx_o dy_o \quad x_m = \frac{zx' + z'x}{z + z'}; \quad y_m = \frac{zy' + z'y}{z + z'}; \quad z_a = \frac{zz'}{z + z'}$$

Fresnel, sufficient: $z^3 \gg \frac{\pi}{4\lambda} ((x - x_o)^2 + (y - y_o)^2)^2$

Fresnel, rectangular apertures: $U_P = \frac{U_{P_o}}{(1+i)^2} [C(s) + iS(s)]_u^2 \times [C(s) + iS(s)]_v^2$

Fresnel, circular apertures: $U_P = -iU_o \int_0^{\psi_o} e^{i\psi} Q(\psi) d\psi$ Fresnel zone radii: $r_n = \sqrt{n\lambda z_a}$

Fraunhofer, sufficient: $z_a \gg \frac{\pi(x_o^2 + y_o^2)}{\lambda}$ $u = \frac{kx_m}{z_a}$ and $v = \frac{ky_m}{z_a}$

Special Functions

$$\text{sinc}(\alpha) \equiv \frac{\sin(\pi\alpha)}{\pi\alpha}$$

$$\text{comb}\left(\frac{x_o}{L}\right) \equiv |L| \sum_{-\infty}^{\infty} \delta(x_o - nL)$$

$$\text{Gaus}\left(\frac{x_o}{L}\right) \equiv e^{-\pi(x_o/L)^2}$$

$$\text{rect}\left(\frac{x_o}{L}\right) = \begin{cases} 1 & |x_o| \leq \frac{L}{2} \\ 0 & \text{else} \end{cases}$$

$$\text{cyl}\left(\frac{r}{R}\right) = \begin{cases} 1 & \frac{r}{R} \leq 1 \\ 0 & \text{else} \end{cases}$$

$$\text{step}\left(\frac{x_o}{L}\right) = \begin{cases} 0 & \frac{x_o}{L} \leq 0 \\ 1 & \frac{x_o}{L} > 0 \end{cases}$$

Function

Fourier Transform

rect(x)

$$\text{sinc}\left(\frac{u}{2\pi}\right)$$

$\delta(x)$

1

comb(x)

$$\text{comb}\left(\frac{u}{2\pi}\right)$$

Gaus(x)

$$\text{Gaus}\left(\frac{u}{2\pi}\right)$$

step(x)

$$\frac{1}{2} \delta\left(\frac{u}{2\pi}\right) + \frac{1}{i u}$$

cyl(r)

$$\left(\frac{J_1(u')}{u'}\right)$$

where $u \equiv \frac{kx}{z}$; $u' \equiv \frac{kr}{z}$, the radial-only analogue

Fourier Transform Relations

If $\mathcal{F}\{g(x, y)\} = G(u, v)$ and $\mathcal{F}\{h(x, y)\} = H(u, v)$ then

$$\mathcal{F}\{g(ax, by)\} = \frac{1}{|ab|} G\left(\frac{u}{a}, \frac{v}{b}\right) \quad \mathcal{F}\{g(x-a, y-b)\} = G(u, v) e^{-i(ua+vb)} \quad \mathcal{F}\{g * h\} = G(u, v) H(u, v)$$

$$T(x_o, y_o) = e^{ikn\Delta_o} \exp\left\{-\frac{ik}{2f}(x_o^2 + y_o^2)\right\} \quad \text{transmission (aperture) function for a lens}$$

$$\int_0^a x \exp(-iux) dx = -\frac{a \exp(-iua)}{iu} + \frac{\exp(-iua) - 1}{u^2} \quad \frac{I(x, y)}{I_0} = \frac{\ell_x^2 \ell_y^2}{\lambda^2 z^2} \text{sinc}^2\left[\ell_x \frac{x}{\lambda z}\right] \text{sinc}^2\left[\ell_y \frac{y}{\lambda z}\right]$$

Gaussian Beams etc.

$$\Psi_{l,m} = \frac{C_{l,m}}{w(z)} H_l\left[\frac{\sqrt{2}x}{w(z)}\right] H_m\left[\frac{\sqrt{2}y}{w(z)}\right] \exp\left[-\frac{x^2 + y^2}{w^2(z)}\right] \exp\left[-\frac{ik(x^2 + y^2)}{2R(z)}\right] e^{-i(l+m+1)\phi(z)} e^{ikz}$$

$$w^2(z) = \frac{\lambda z_o}{\pi} \left(1 + \frac{z^2}{z_o^2}\right); \quad R(z) = \frac{z^2 + z_o^2}{z}; \quad \tan \phi = \frac{z}{z_o} \quad \frac{r_1}{L} \gg \frac{\lambda}{r_2}$$

$$\tan \frac{\theta}{2} = \frac{w(z)}{z} \cong \frac{w_o \cdot \frac{z}{z_o}}{z} = \frac{\lambda}{\pi w_o} \cong \frac{\theta}{2} \quad \text{for small } \frac{\lambda}{\pi w_o}$$

$$g_1 = \left(1 - \frac{L}{R_1}\right) \quad g_2 = \left(1 - \frac{L}{R_2}\right) \quad z_o^2 = \frac{L^2 g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2g_1 g_2)^2} \quad 0 < g_1 g_2 < 1$$

$$w_1^2 = \frac{L\lambda}{\pi} \left(\frac{g_2}{g_1(1 - g_1 g_2)}\right)^{1/2} \quad \text{and} \quad w_2^2 = \frac{L\lambda}{\pi} \left(\frac{g_1}{g_2(1 - g_1 g_2)}\right)^{1/2}$$

$$\int_{z_1}^{z_2} k_{\text{eff}} dz = k(z_2 - z_1) - (l + m + 1) \left[\tan^{-1}\left(\frac{z_2}{z_0}\right) - \tan^{-1}\left(\frac{z_1}{z_0}\right) \right] + \frac{\phi_2 + \phi_1}{2} = q\pi$$

$$\omega_q^{l,m} = \frac{q\pi c}{nL} + \frac{c}{nL} \left[(l + m + 1) \left(\tan^{-1}\left(\frac{z_2}{z_0}\right) - \tan^{-1}\left(\frac{z_1}{z_0}\right) - \frac{\phi_2 + \phi_1}{2} \right) \right] = \left[q + \frac{(l + m + 1) \cos^{-1}(\sqrt{g_1 g_2})}{\pi} \right] \frac{c\pi}{nL}$$

$$\text{Airy pattern: } U(r, \theta) = -i U_A \frac{\exp(ik|PS|)}{z} \exp\left(\frac{ik}{2z_a} [x_m^2 + y_m^2]\right) \frac{J_1\left(\frac{kRr}{z}\right)}{\frac{kRr}{z}} \propto \frac{J_1(kR\sin(\theta))}{kR\sin(\theta)}$$

$$\text{Airy disk diameter: } \Delta r = 1.22 \lambda z / R; \quad \Delta \theta = 1.22 \lambda / R \quad f = \frac{\pi\sqrt{R}}{1 - R}$$

$$\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{i\lambda}{\pi n w^2(z)} \quad q' = \frac{Aq + B}{Cq + D}$$

Free space :	$\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$	Thin lens, focal length f :	$\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$
Re fraction at plane :	$\begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$	Curved dielectric :	$\begin{bmatrix} 1 & 0 \\ \frac{(n_1 - n_2)}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}$

$$\text{lensmaker's equation: } \frac{1}{f} = \frac{n_2 - n_1}{n_1} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$\frac{P(E_1)}{P(E_2)} = \exp \left\{ -(E_1 - E_2) / k_B T \right\}$$

$$\rho(\omega) = \sigma_{BB} \langle E(\omega) \rangle = \frac{n^3 \omega^3}{\pi^2 c^3} \frac{\hbar \omega}{\exp\left(\frac{\hbar \omega}{k_B T}\right) - 1} = \frac{4h}{\lambda^3} \frac{1}{\exp\left(\frac{\hbar \omega}{k_B T}\right) - 1}$$

$$I = c \int_0^\infty \rho_{BB}(\omega) d(\omega) = Z_{SB} T^4 \quad Z_{SB} = 5.67 \times 10^{-8} \text{ W/m}^2 \text{K}^4 \quad h = 6.6 \times 10^{-34} \text{ J}\cdot\text{s} \quad k_B = 1.34 \times 10^{-23} \text{ J}\cdot\text{K}^{-1}$$

$$\gamma(\omega) = \frac{\left(N_j - \frac{g_j}{g_i} N_i \right) 3\pi^2 c^2}{n^2 \omega^2 t_{sp}} g(\omega) \quad A_{j \rightarrow i} = \frac{e^2 |x_{ij}|^2}{3\pi \epsilon_0 c^3} \hbar^{-1} n \omega_o^3 g_i \quad B_{i \rightarrow j} = \frac{\pi e^2 |x_{ij}|^2}{3n^2 \epsilon_0} \hbar^{-2} g_j$$

$$\gamma(\omega) = -\alpha(\omega) = -\frac{4\pi\kappa}{\lambda} = -\frac{k}{n_B} \chi_I \quad g_i B_{i \rightarrow j} = g_j B_{j \rightarrow i}$$

$$\chi_I(\omega) = -\frac{\left(N_j - \frac{g_j}{g_i} N_i \right) 3\pi^2 c^3 g(\omega)}{(n_B \omega^3 t_{sp})} \quad \chi_R(\omega) = -\frac{2 \left(N_j - \frac{g_j}{g_i} N_i \right) 3\pi^2 c^3 g(\omega)}{n_B \omega^3 t_{sp}} \frac{\omega_o - \omega}{\Delta\omega} = \frac{2(\omega_o - \omega)}{\Delta\omega} \chi_I$$

$$t_c = n_B \frac{L}{c \mathfrak{L}} \quad t_c = \frac{n_B}{c \left[\zeta - L^{-1} \ln \sqrt{R_1 R_2} \right]} \quad \gamma_t = \zeta - \frac{1}{L} \ell n \sqrt{R_1 R_2} = \frac{n_B}{c t_c}$$

$$\Delta N_t = \left(N_j - \frac{g_j}{g_i} N_i \right)_t = \frac{n^2 \omega^2 t_{sp}}{g(\omega_o) 3\pi^2 c^2} \left(\zeta - \frac{1}{L} \ell n \sqrt{R_1 R_2} \right) \quad \int_{-\infty}^{\infty} g(\omega) d\omega = 1, \text{ thus } g(\omega_o) \Delta\omega \cong 1$$

Doppler: $g(\omega) = g_o \exp\left(-4 \ln 2 \frac{(\omega - \omega_o)^2}{(\Delta\omega_D)^2}\right)$ where $g_o = \frac{c}{\omega_o} \left(\frac{M}{2\pi k_B T} \right)^{1/2}$ and $FWHM \Delta\omega_D = 2\omega_o \sqrt{2k_B T \frac{\ln 2}{M c^2}}$

$$P = \Delta N_t V W_{21} \hbar \omega \quad P = P_S \left(\frac{R}{R_t} - 1 \right) \quad P_S = \frac{\Delta N_t}{t_{sp}} \hbar \omega V \quad P' = \frac{N_o}{2} \hbar \omega \frac{V}{t_2}$$