Fractional instantons and nonperturbative gauge theory

Erich Poppitz, U. of Toronto

mostly work with **Mohamed Anber** (Durham U.)
and some with **F. David Wandler** (was at Toronto, now in neuroscience)

+ the work of many other people…

the big picture:

determining the vacuum structure, showing confinement and dynamical mass generation are difficult, strong coupling problems in nonperturbative gauge theory

attitudes

after all, problem is solved on the lattice (confinement "proven")

but, lattice can't deal with many theories:
$\theta \neq 0, \chi\text{GT},$
(most of) SUSY

use the magic of SUSY (Seiberg-Witten theory)

not 'real world'… but some hints

having analytical control, even at a price, gives useful insight

this talk!

roughly, about the role of nonperturbative fluctuations in the YM vacuum

this talk: ***about the role of nonperturbative fluctuations in the vacuum in 4d SU(N) YM***

instantons (1970's: BPST, ADHM… integer Q) - oldest known nonperturbative objects in YM; do not cause confinement (no disordering of Wilson loops)

fractional instanton solutions found 1979: 't Hooft, Q = $\frac{r}{N}, r \in \mathbb{Z}$

dynamics? → hints for fractional Q: $\theta \to \frac{\theta}{N}$ at large-N, Witten 1980

$$E_{vac} = N^2 \min_k F(\frac{\theta - 2\pi k}{N})$$

hints taken seriously by some (few) … Zhitnitsky ~ '91 +

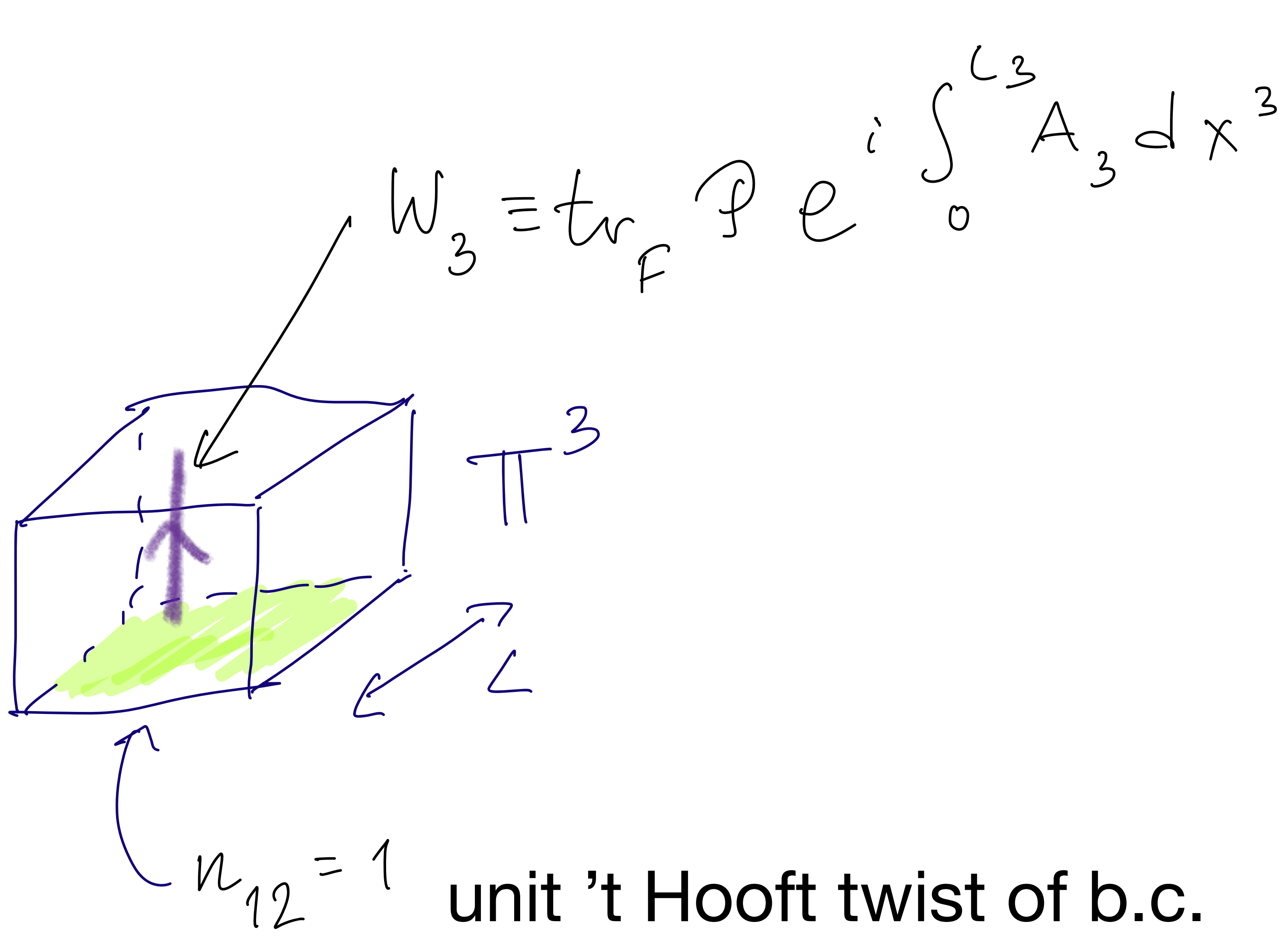
dynamics? → semiclassical dynamics on small $\mathbb{T}^3$ (weak coupling)

dilute gas of fractional instantons (van Baal, González-Arroyo et al, 1980s —>1990s)

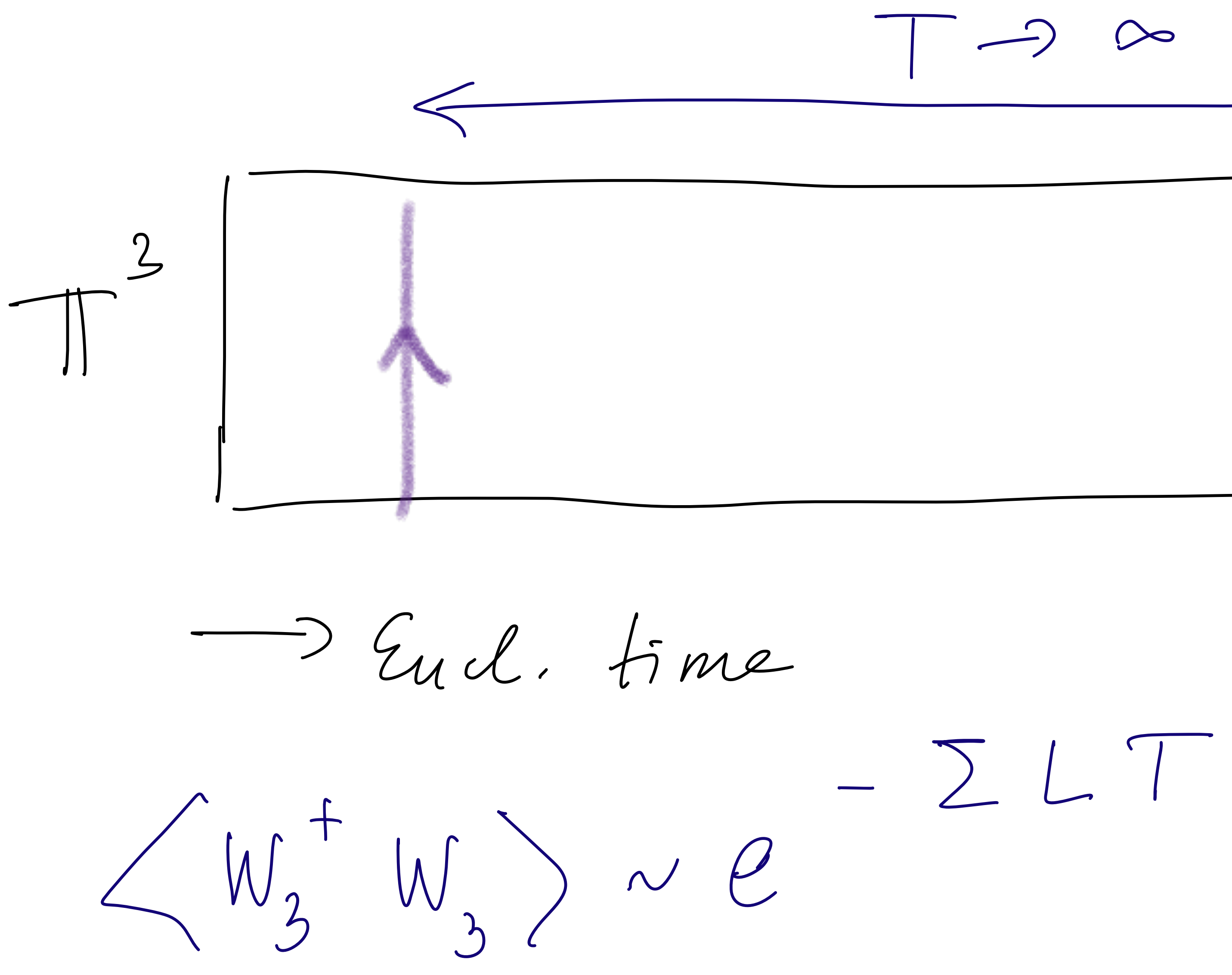
review a little of the history:

semiclassical dynamics on small $\mathbb{T}^3$ (weak coupling)

P. van Baal, González-Arroyo, Martínez, García Pérez et al 1980s —>1990s “Madrid group”



unit ’t Hooft twist of b.c.



($\langle W_1^\dagger W_1 \rangle$, etc.-ask!)

small-L ($L\Lambda \ll 1$): dilute gas of fractional instantons give area law $\Sigma = \frac{2c}{L^2}\ e^{-\frac{8\pi^2}{Ng^2}}$

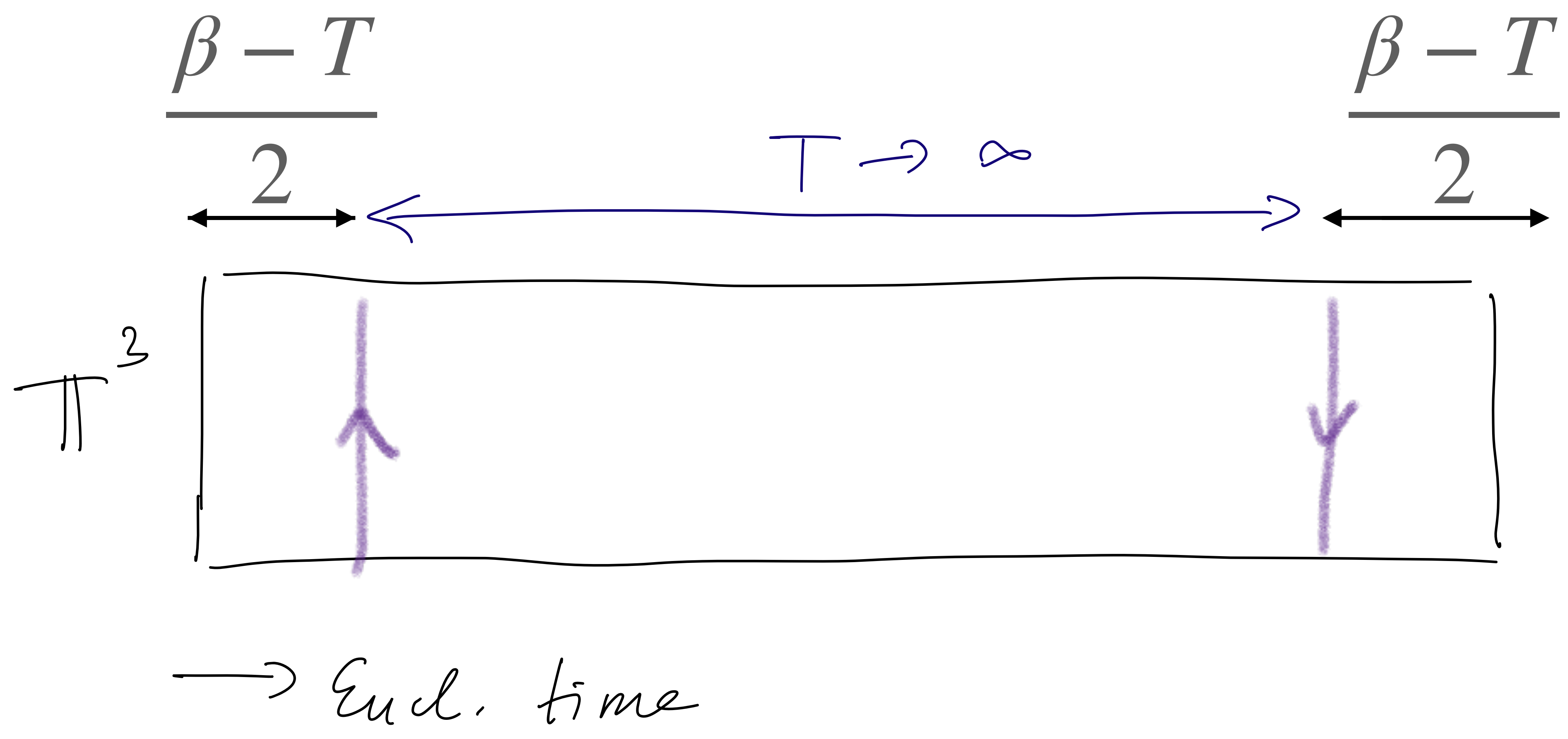
mix analytic/numerical: no analytic solutions, but dilute gas of $Q = 1/2$ seen in lattice configs.; SU(2)

large-L ($L\Lambda \gg 1$): continuous transition of Σ to infinite volume limit, no phase transition

analytic control lost; conjectured “**F**ractional **I**nstanton **L**iquid **M**odel” (González-Arroyo, review 2302.12356)

use last week's talk stuff, SU(2):

$\beta - T \to \infty \qquad \beta \to \infty \qquad T \to \infty$



$$\langle W_3^\dagger W_3 \rangle = \mathrm{Tr}\left(e^{-\frac{\beta - T}{2}H}\, W_3^\dagger\, e^{-TH}\, W_3\, e^{-\frac{\beta - T}{2}H} \right)$$

W_3 changes flux by 1

$$= e^{-E_0 \frac{\beta - T}{2}} e^{-E_1 T} \,|\langle e_3, E_0 \,|\, W_3^\dagger \,|\, e_3 + 1, E_1 \rangle|^2$$

$$= e^{-E_0 \beta} e^{-(E_1 - E_0)T} \,|\langle e_3, E_0 \,|\, W_3^\dagger \,|\, e_3 + 1, E_1 \rangle|^2$$

normalization

difference of min energies in $e_3 = 0, e_3 = 1$ sectors

$\langle W_3^+ W_3 \rangle \sim e^{-\Sigma L T}$

claim

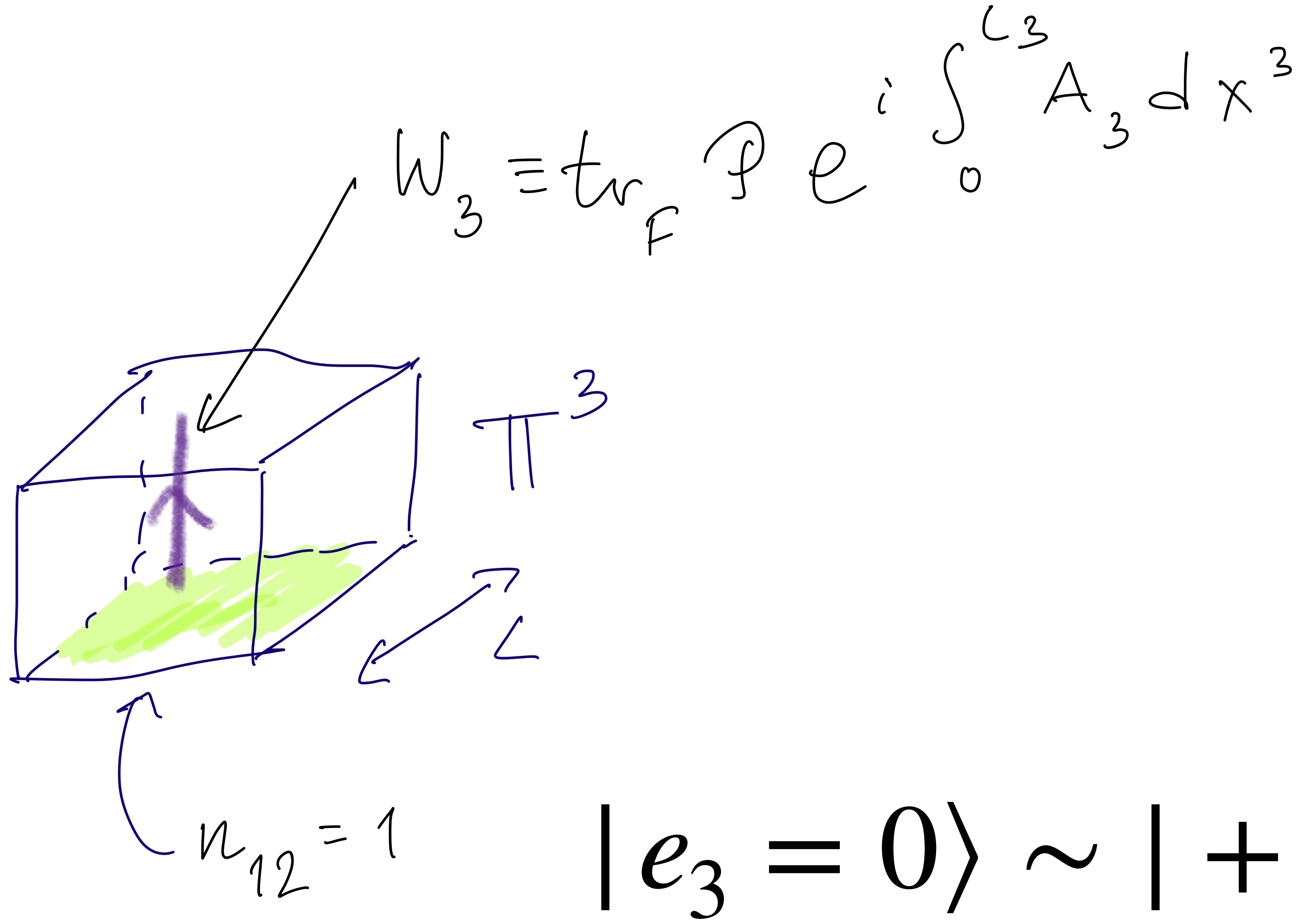
hence: $\langle W_3^\dagger W_3 \rangle \sim e^{-T(E_1 - E_0)}$

next small-$\mathbb{T}^3$ show:

$$E_1 - E_0 = L\Sigma, \;\; \Sigma \sim (g^2 L)^{-2} e^{-\frac{4\pi^2}{g^2}}$$

hence, small-$\mathbb{T}^3$ "area" law: $\langle W_3^\dagger W_3 \rangle \sim e^{-TL\Sigma}$

hence: $\langle W_3^\dagger W_3 \rangle \sim e^{-T(E_1 - E_0)}$ ➡ will argue $E_1 - E_0 = L\Sigma, \;\; \Sigma \sim (g^2 L)^{-2} e^{-\frac{4\pi^2}{g^2}}$



twist -> no continuous zero modes

small torus, semiclassical quantization, $A_0 = 0$

classical zero-energy: $A^+ = 0, \, A^- = iT_3 dT_3^{-1}$

$\langle W_3 \rangle_+ = 1, \; \langle W_3 \rangle_- = -1$

$\hat{T}_3$

related by center symmetry

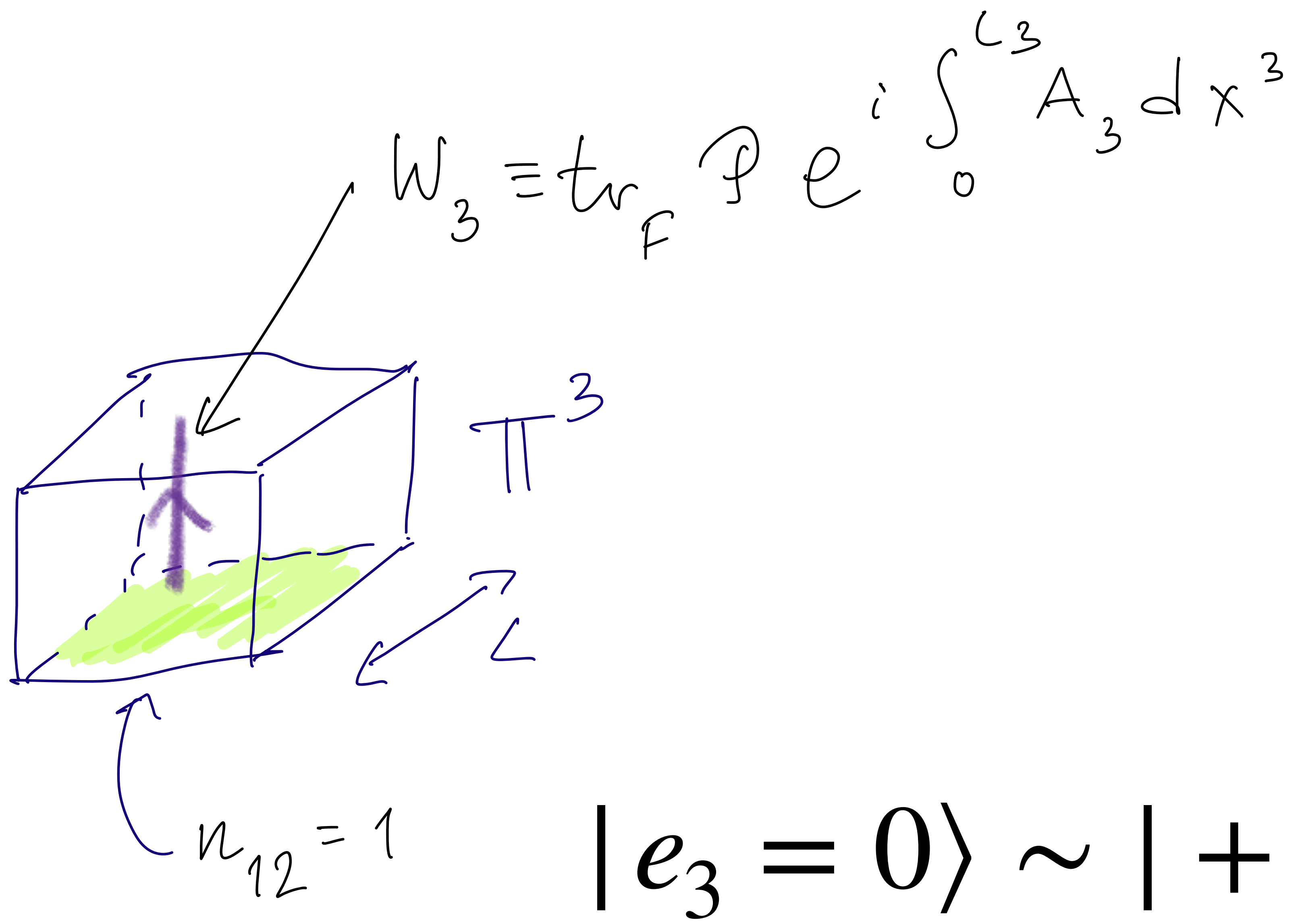
perturbatively degenerate vacua

$$|e_3 = 0\rangle \sim |+\rangle + |-\rangle, \; |-\rangle = \hat{T}_3 |+\rangle$$

$$|e_3 = 1\rangle \sim |+\rangle - |-\rangle$$

(aside: these two 0-energy vacua saturate Witten index in SYM)

hence: $\langle W_3^\dagger W_3 \rangle \sim e^{-T(E_1 - E_0)}$ ➡ will argue $E_1 - E_0 = L\Sigma,\ \Sigma \sim (g^2 L)^{-2} e^{-\frac{4\pi^2}{g^2}}$



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$\beta \to \infty$

$$e^{-\beta E_0} \sim 2\langle + | e^{-\beta H} | + \rangle + \langle + | e^{-\beta H} T_3 | + \rangle + \langle + | e^{-\beta H} T_3^\dagger | + \rangle$$

$$e^{-\beta E_1} \sim 2\langle + | e^{-\beta H} | + \rangle - \langle + | e^{-\beta H} T_3 | + \rangle - \langle + | e^{-\beta H} T_3^\dagger | + \rangle$$

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Q= 0 + Z **Q=1/2 + Z** **Q=-1/2 + Z**

$\beta \to \infty$

$$2Z_0 \qquad\qquad 2Z_{1/2}$$

$$e^{-\beta E_0} \sim 2\langle + | e^{-\beta H} | + \rangle + \overline{\langle + | e^{-\beta H} T_3 | + \rangle + \langle + | e^{-\beta H} T_3^\dagger | + \rangle}$$

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Q= 0 + Z **Q=1/2 + Z** **Q=-1/2 + Z**

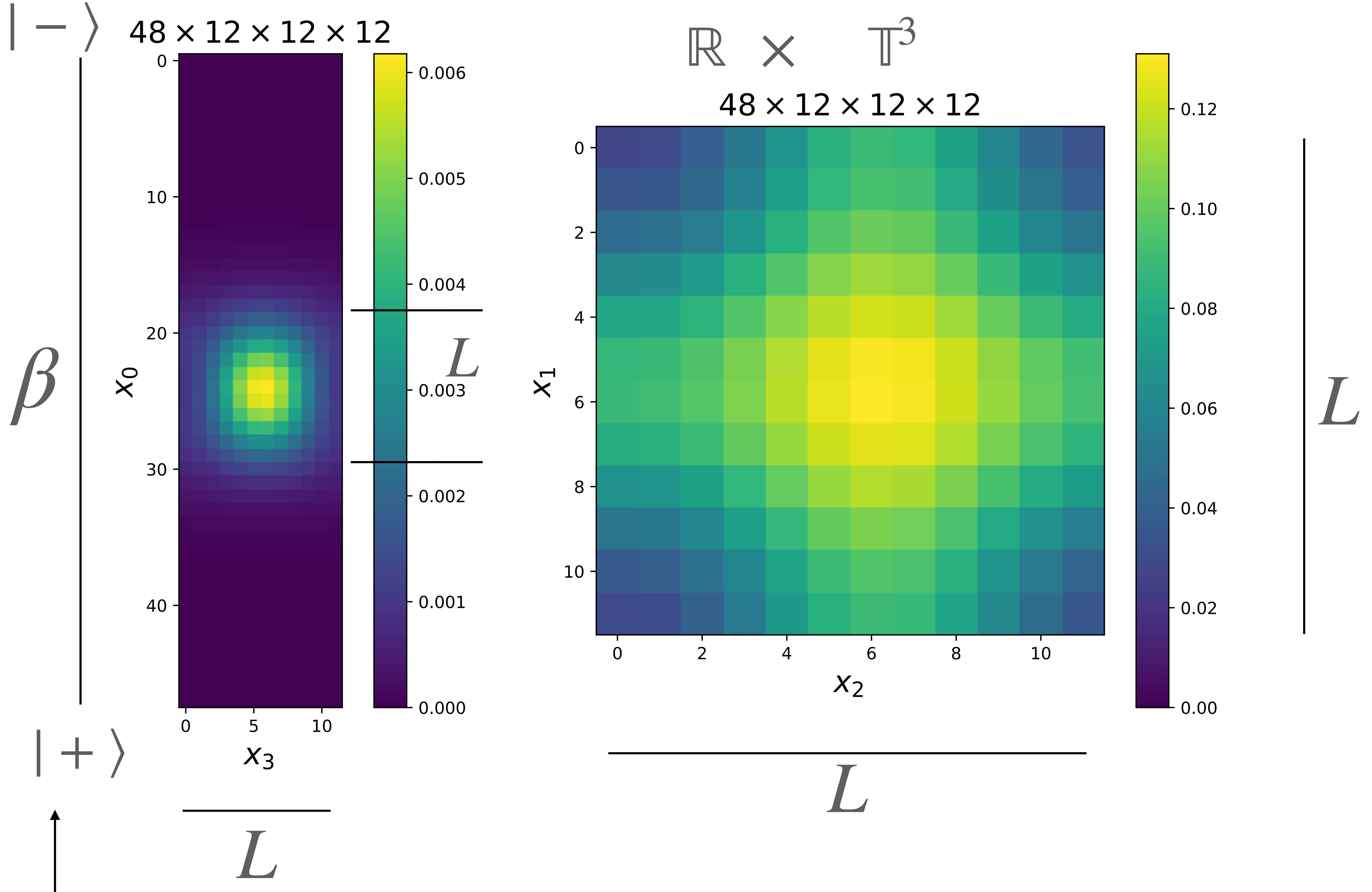
$$E_1 - E_0 = -\frac{1}{\beta} \ln \frac{Z_0 - Z_{1/2}}{Z_0 + Z_{1/2}} \simeq \frac{2Z_{1/2}}{\beta Z_0}$$

$$Z_{1/2} \sim e^{-\frac{4\pi^2}{g^2(L)}} \ll 1$$

at small β, single Q=1/2 in numerator, perturbative denominator, but for $E_1 - E_0$ need large β:

assumption: dilute gas, Q=1/2 instantons (size L, volume L^4) w/ semiclassical density

1990s, “Madrid group”



$\langle W_3 \rangle_+ = 1, \langle W_3 \rangle_- = -1$

pic from Wandler 2406.07636

(1990s, “Madrid group”)

$$\beta \to \infty \qquad \overset{2Z_0}{|||} \qquad \overset{2Z_{1/2}}{|||}$$

$$e^{-\beta E_0} \sim 2\langle + | e^{-\beta H} | + \rangle + \overline{\langle + | e^{-\beta H} T_3 | + \rangle + \langle + | e^{-\beta H} T_3^\dagger | + \rangle}$$

$$e^{-\beta E_1} \sim 2\langle + | e^{-\beta H} | + \rangle - \langle + | e^{-\beta H} T_3 | + \rangle - \langle + | e^{-\beta H} T_3^\dagger | + \rangle$$

Q= 0 + Z **Q=1/2 + Z** **Q=-1/2 + Z**

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$$Z_0 = \sum_{\bar{n},n=0\ (n+\bar{n}\ \text{even})}^{\infty} \frac{e^{-\frac{4\pi^2}{g^2}(n+\bar{n}) + i\frac{\theta}{2}n - i\frac{\theta}{2}\bar{n}}}{n!\bar{n}!} \left(\frac{c\beta L^3}{L^4}\right)^{n+\bar{n}}$$

$$Z_{1/2} = \frac{1}{2} \sum_{n,\bar{n}=0\ (n+\bar{n}\ \text{odd})}^{\infty} \frac{e^{-\frac{4\pi^2}{g^2}(n+\bar{n}) + i\frac{\theta}{2}n - i\frac{\theta}{2}\bar{n}}}{n!\bar{n}!} \left(\frac{c\beta L^3}{L^4}\right)^{n+\bar{n}} = \ldots \simeq e^{-\frac{4\pi^2}{g^2}} \cos\frac{\theta}{2} \times \frac{c\beta}{L} \times Z_0$$

$$E_1 - E_0 = -\frac{1}{\beta} \ln \frac{Z_0 - Z_{1/2}}{Z_0 + Z_{1/2}} \simeq \frac{2Z_{1/2}}{\beta Z_0} \simeq \frac{2c}{L} \cos \frac{\theta}{2} e^{-\frac{4\pi^2}{g^2}}$$

$$\underline{E_1 - E_0 = \frac{2c}{L} e^{-\frac{4\pi^2}{g^2}} \cos \frac{\theta}{2}}$$

at small β, single Q=1/2 in numerator, perturbative denominator, but for $E_1 - E_0$ need large β:

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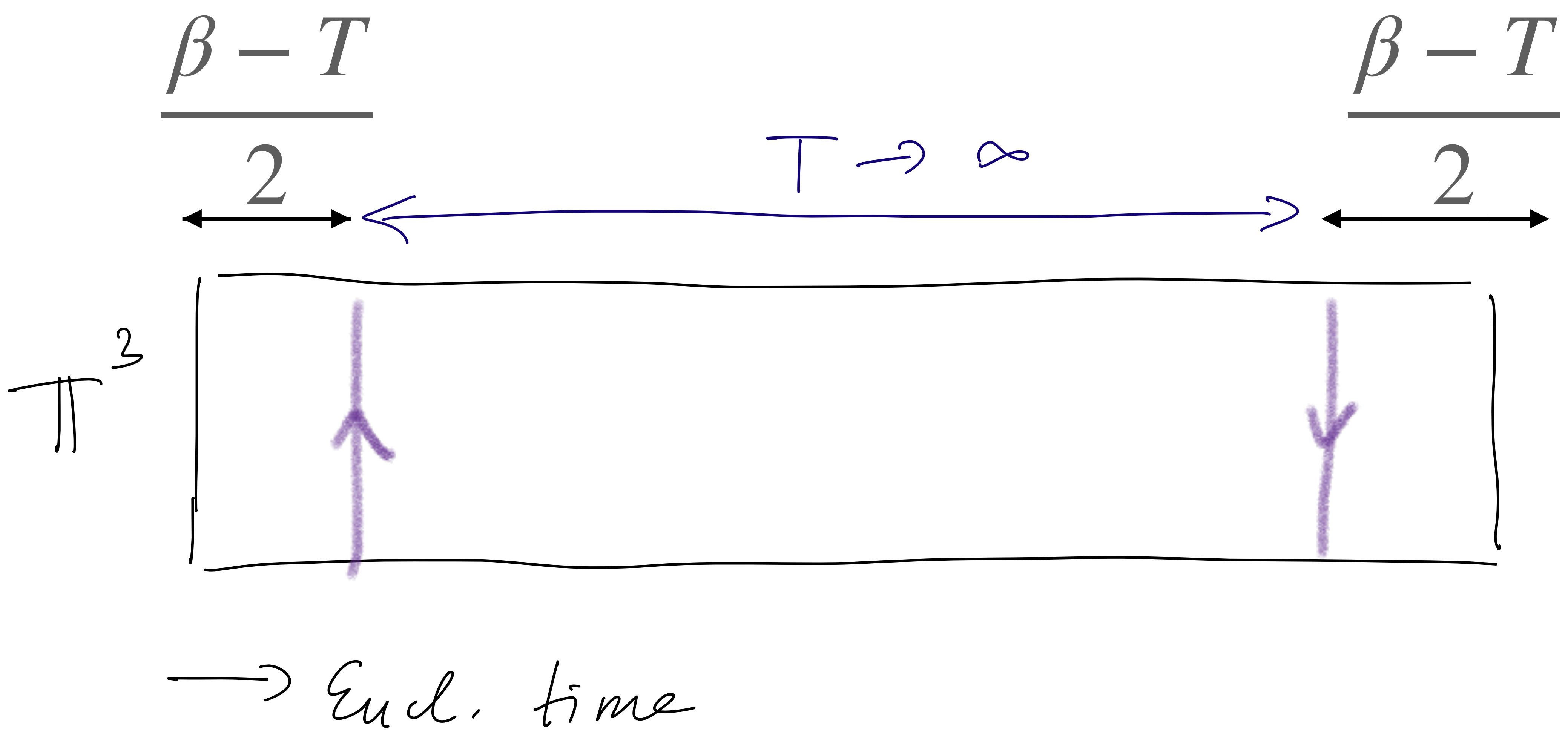
González-Arroyo, Martínez, García Pérez et al 1980s —>1990s



$$\langle W_3^\dagger W_3 \rangle \sim e^{-\Sigma L T}$$

established claim!

semiclassical small-L string tension

Σ = 2 x fugacity of Q=1/2 objects

$$\langle W_3^\dagger W_3 \rangle \sim e^{-TL\,\overbrace{\left(\frac{2c}{L^2} e^{-\frac{4\pi^2}{g^2}}\right)}^{\Sigma}}$$

$$\langle W_3^\dagger W_3 \rangle \sim e^{-T(E_1 - E_0)} \longrightarrow E_1 - E_0 = \frac{2c}{L} e^{-\frac{4\pi^2}{g^2}} \cos\frac{\theta}{2}$$

González-Arroyo, Martínez, García Pérez et al 1980s —>1990s

$$\Sigma\Big|_{\mathrm{small-L}} \simeq \frac{2c}{L^2}(\Lambda L)^{\frac{11}{3}} = 2c\Lambda^{\frac{22}{6}}L^{\frac{5}{3}}$$

$$\langle W_3^\dagger W_3 \rangle \sim e^{-TL\,\overbrace{\left(\frac{2c}{L^2}e^{-\frac{4\pi^2}{g^2}}\right)}^{\Sigma}}$$

P. van Baal, 1984, PhD thesis, unpublished Ch.3 ->? 't Hooft

$$\langle W_3^\dagger W_3 \rangle \sim e^{-T(E_1-E_0)}$$

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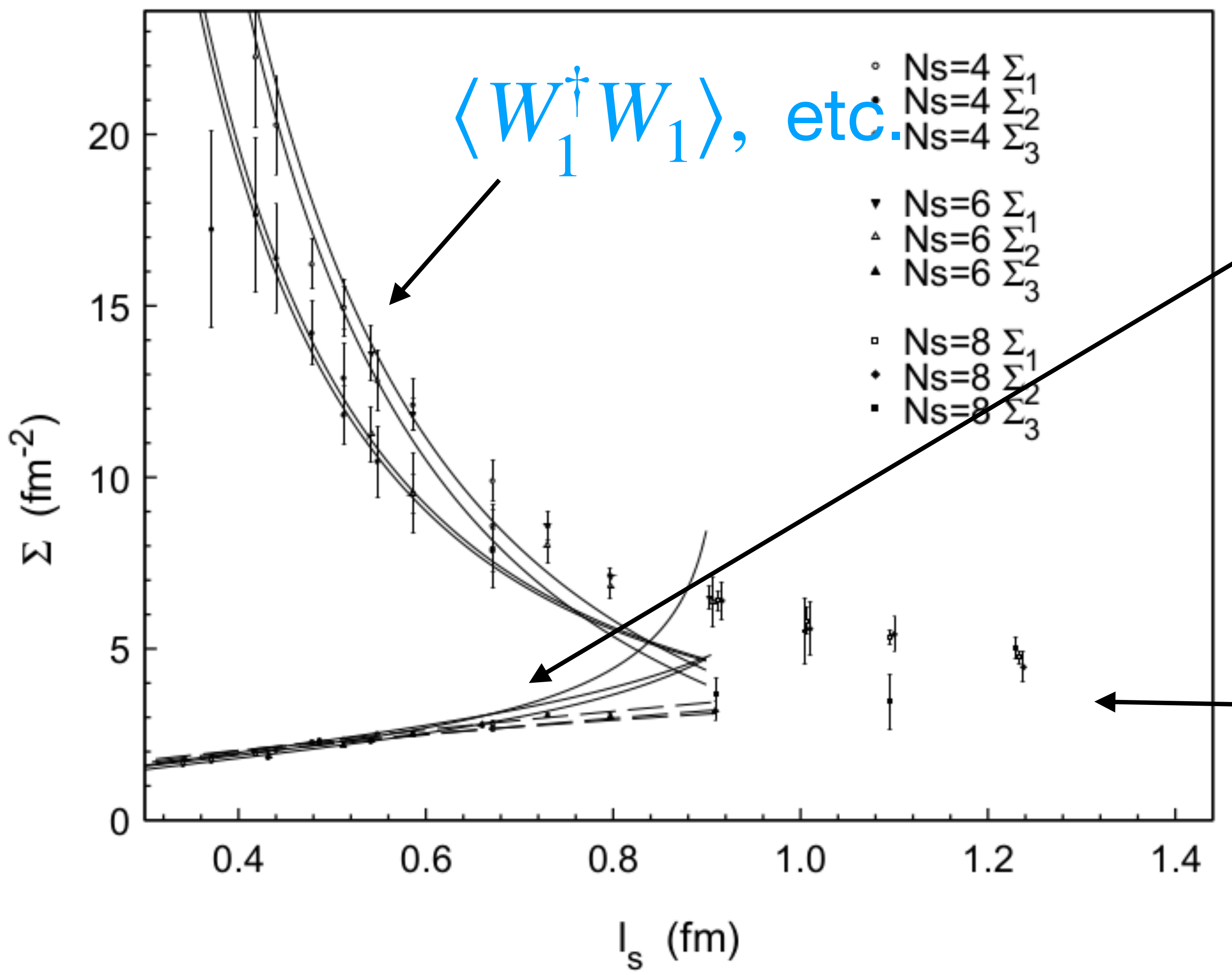
González-Arroyo, Martínez, García Pérez et al 1980s —>1990s

$$\Sigma\Big|_{\rm small-L} \simeq \frac{2c}{L^2}(\Lambda L)^{\frac{11}{3}} = 2c\Lambda^{\frac{22}{6}}L^{\frac{5}{3}}$$

fit semiclassical **ansatz** to lattice data…*1993-95 papers*

split of pertubatively degenerate e-fluxes grows with L



$$\langle W_3^\dagger W_3\rangle \sim e^{-TL\,(\frac{2c}{L^2}e^{-\frac{4\pi^2}{g^2}})}$$

Σ

approach ∞ volume limit
string tension above “1 fm”

P. van Baal, 1984, PhD thesis, unpublished Ch.3 ->? ’t Hooft

$$\langle W_3^\dagger W_3\rangle \sim e^{-T(E_1-E_0)}$$

$$E_1 - E_0 = \frac{2c}{L}e^{-\frac{4\pi^2}{g^2}}\cos\frac{\theta}{2}$$

+ **aside:** *vacuum degeneracy at* $\theta = \pi$ (exact!)

+ *center-parity anomaly: perimeter law* $\langle W_3 W_3^\dagger\rangle \to$ const *($\mathbb{Z}_2$ TQFT)*

end history fun…

fractional instanton solutions found 1979: 't Hooft, Q = $\frac{r}{N}, r \in \mathbb{Z}$

dynamics?

hints for fractional Q: $\theta \to \frac{\theta}{N}$ at large-N, Witten 1980

$$E_{vac} = N^2 \min_k F(\frac{\theta - 2\pi k}{N})$$

hints taken seriously by some (few) … Zhitnitsky ~ '91 +

semiclassical dynamics on small $\mathbb{T}^3$ (weak coupling)

dilute gas of fractional instantons

P. van Baal, González-Arroyo et al, 1980s —>1990s

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this talk is about the recent "pick-up" in this 30+ years old activity and the role of fractional instantons - *forgotten by many and unknown to the youngest…*

- the reason why just reviewed!

recent “pick-up” in this 30+ years old activity for various reasons…

“Quanta”
Spring ’23 —>

MATHEMATICAL PHYSICS

A New Kind of Symmetry Shakes Up Physics

23

So-called “higher symmetries” are illuminating everything from particle decays to the behavior of complex quantum systems.

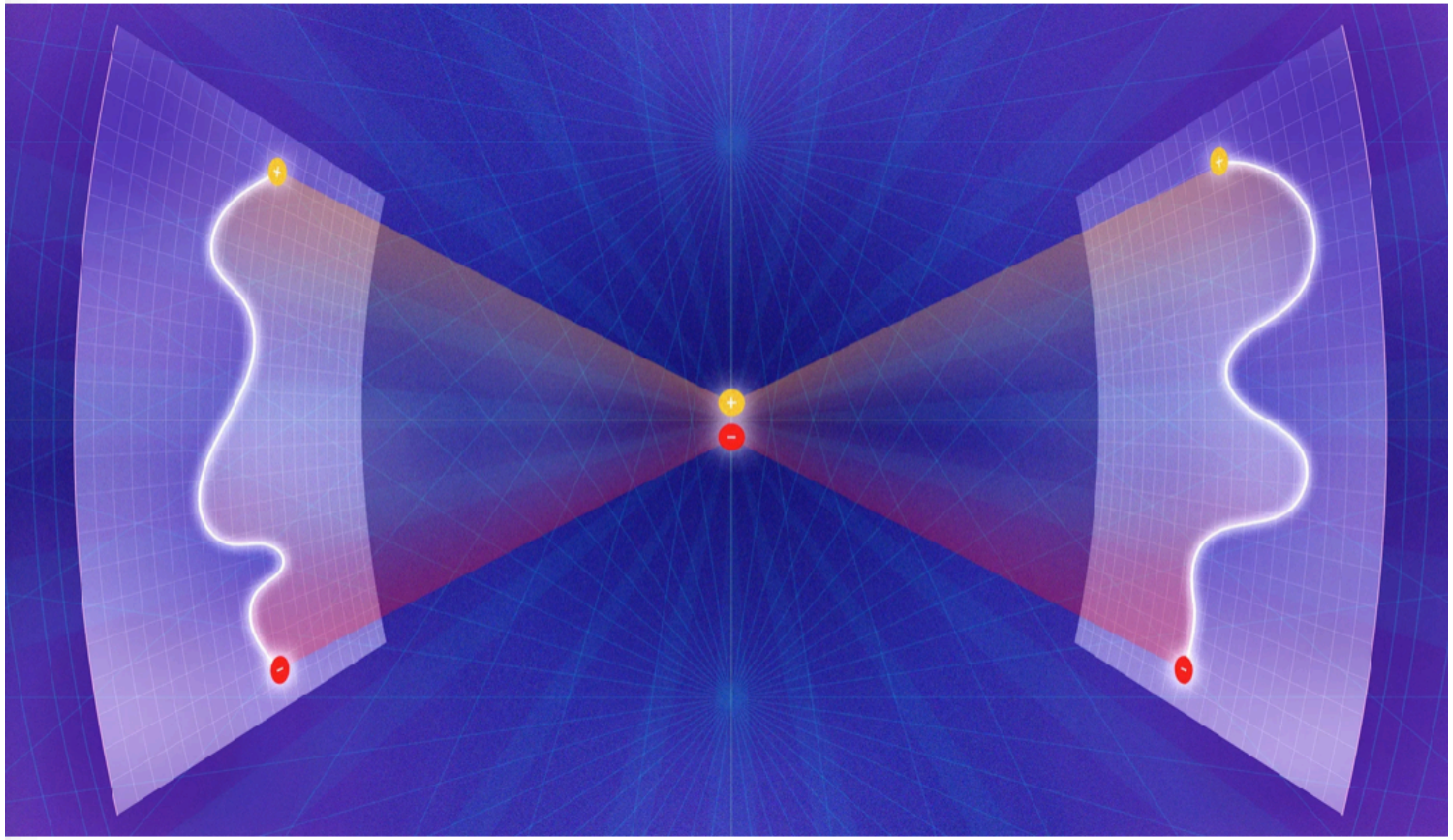
1.

renewed interest in $\mathbb{T}^4$
due to generalized anomalies

missed in the 1980s

to see need spacetime with noncontractible 2-cycles

Gaiotto,Kapustin,Komargodski,Seiberg
2014-



The symmetries of 20th-century physics were built on points. Higher symmetries are based on one dimensional lines.

Samuel Velasco/Quanta Magazine

recent "pick-up" in this 30+ years old activity for various reasons…

$\mathbb{R}^3 \times \mathbb{S}^1$ D-brane work of K. Lee and P. Yi from 1990s!

↑

2. **recent interest in $\mathbb{R}^3 \times \mathbb{S}^1$ compactifications of 4d gauge theories** Ünsal…2007+;

explicit (as opposed to $\mathbb{R} \times \mathbb{T}^3$) monopole-instanton solutions of topological charge $Q = \frac{1}{N}$

allow analytic semiclassical studies of nonperturbative physics on $\mathbb{R}^3 \times \mathbb{S}^1$

argue for /shown/ continuous connection to $\mathbb{R}^4$

also, on $\mathbb{R}^2 \times \mathbb{T}^2$ Tanizaki-Ünsal…2020+ (but also García Pérez, González-Arroyo…1990s)

and the relation between the seemingly different fractional instantons on $\mathbb{R}^2 \times \mathbb{T}^2, \mathbb{R}^1 \times \mathbb{T}^3, \mathbb{R}^3 \times \mathbb{S}^1, \mathbb{T}^4$

(García Pérez, González-Arroyo…1990s; Wandler EP 2022; Wandler 2024; Ünsal et al 2024; Tanizaki et al 2024)

recent “pick-up” in this 30+ years old activity for various reasons…

1. renewed interest in $\mathbb{T}^4$ due to generalized anomalies involving 1-form center symmetry

Gaiotto,Kapustin,Komargodski,Seiberg 2014-

2. recent interest in $\mathbb{R}^3 \times \mathbb{S}^1$ compactifications of 4d gauge theories Ünsal…2007+;

… and relations between fractional instantons on $\mathbb{R}^2 \times \mathbb{T}^2, \mathbb{R}^1 \times \mathbb{T}^3, \mathbb{R}^3 \times \mathbb{S}^1, \mathbb{T}^4$

3. progress in analytically constructing solutions with $Q = \frac{k}{N}$

García Pérez, González-Arroyo…2000, González-Arroyo 2018; Anber, EP 2022, 2023, 2024

will tell you about these

+ role in chiral symmetry breaking - via the calculation the gaugino condensate in SYM

rest of talk about:

SYM in 4d: SU(N) + 1 massless adjoint Weyl fermion λ_α^a (SUSY emergent when $m_\lambda = 0$)

chiral U(1) broken to $\mathbb{Z}_{2N}$ by anomaly

$\mathbb{Z}_{2N}$ spontaneously broken to $\mathbb{Z}_2$ by bilinear gaugino condensate ($\lambda^2(x) \equiv \mathrm{tr}\, \lambda^\alpha(x)\lambda_\alpha(x)$)

$$\langle \lambda^2 \rangle = e^{i\frac{2\pi k}{N}}\, c\, \Lambda^3,\ k = 1,\ldots,N, \quad c = 16\pi^2$$

the "mother" of all exact results in SUSY

1983-1999: Novikov, Shifman, Vainshtein, Zakharov; Amati, Konishi, Rossi, Veneziano; Affleck, Dine, Seiberg; Cordes; Finnell, Pouliot (SQCD —> SYM on R^4); Davies, Hollowood, Khoze, Mattis;… 2014 Anber, Teeple, EP (SYM on $R^3 \times S^1$ —> SYM on R^4)

↑

semiclassical weakly-coupled instanton calculations + power of SUSY

recent independent large-N lattice determination!

2406.08955
Bonnano, García Pérez,
González-Arroyo, Okawa et al

SYM in 4d: SU(N) + 1 massless adjoint Weyl fermion λ_α^a (SUSY emergent when $m_\lambda = 0$)

chiral U(1) broken to $\mathbb{Z}_{2N}$ by anomaly

$\mathbb{Z}_{2N}$ **spontaneously broken to $\mathbb{Z}_2$ by bilinear gaugino condensate ($\lambda^2(x) \equiv \text{tr}\ \lambda^\alpha(x)\lambda_\alpha(x)$)**

$$\langle \lambda^2 \rangle = e^{i\frac{2\pi k}{N}}\, c\, \Lambda^3,\ k = 1,\ldots,N, \quad c = 16\pi^2$$

here, I will discuss the calculation of the condensate on $\mathbb{T}^4$

it had been noted a long time ago that a nonzero bilinear adjoint fermion condensate requires an instanton with two adjoint zero modes, i.e. topological charge 1/N, since "index(adjoint Dirac) = 2 N Q"… *a.k.a. "instanton quarks"*

$$\langle \lambda^2 \rangle = e^{i\frac{2\pi k}{N}} \, c \, \Lambda^3, \; k = 1,\ldots,N, \;\; c = 16\pi^2$$

1. finish a 40 years old story: new developments allow it! - first attempt in 1984, Cohen and Gomez, ***could not and did not*** compute "c" at the time

2. the semiclassical objects (instantons on twisted torus) are closely related to both center vortices and monopoles, argued to be responsible for confinement/mass gap/chiral symmetry breaking - as opposed to BPST/ADHM instantons used in $\mathbb{R}^4$ calculation

use both new insights: *a.)* generalized anomalies + *b.)* moduli space of $Q = \frac{k}{N}$ on torus

3. the calculation raises interesting questions about semiclassics, boiling down to the basic definition of path integrals … (recent progress…)

(also, SYM is the one theory where one expects small-L and $\mathbb{R}^4$results to match!)

in fact, on $\mathbb{T}^4$ we'll be able to do more than

$$\langle \lambda^2 \rangle = c\,\Lambda^3$$ (taking one particular vacuum)

SUSY Ward identities: $\langle \lambda^2(x_1)\lambda^2(x_2)\ldots\lambda^2(x_k)\ \rangle \equiv \langle \lambda^{2k} \rangle = (c\,\Lambda^3)^k$

=> x-independence / + clustering /

<u>verified</u> in weak-coupling calculation of $\langle \lambda^{2k} \rangle$ in SQCD on $\mathbb{R}^4$

using ADHM+holomorphy Dorey, Hollowood, Khoze, Mattis 2002

we calculate $\langle \lambda^{2k} \rangle$ on small $\mathbb{T}^4$, gcd(N,k)=1; result agrees with $\mathbb{R}^4$

$$\langle (\mathrm{tr}\,\lambda^2)^k \rangle\Big|_{\mathbb{T}^4} = N\,\langle (\lambda^2)^k \rangle_{\text{one vacuum on } \mathbb{R}^4}$$

generalized

anomalies + def. of path $\int$!

use both new insights: *a.)* generalized anomalies + *b.)* moduli space of $Q = \frac{k}{N}$ on torus

a.) generalized anomalies

Hamiltonian: $\mathbb{T}^3$ with 't Hooft twist $m_3 = n_{12} = -k$, $gcd(N, k) = 1$

we calculate: $\langle (\lambda^2)^k \rangle \equiv \mathrm{tr}_{\mathcal{H}_{m_3}} \left(e^{-\beta H} (-1)^F (\lambda^2)^k \, T_3 \right)$

$\mathbb{T}^3$ Hilbert space with above twist "'t Hooft flux in x_3"

center symmetry (1-form) generator along "'t Hooft flux"

strategy: calculate at small L, β (weak coupling) then continue to large volume (SUSY)

salient point: anomaly between (0-form) $\mathbb{Z}_{2N}$ chiral and center symmetry (1-form)

implies exact N-fold degeneracy of all states in $\mathcal{H}_{m_3}$

trace with T_3 sums absolute value of $(\lambda^2)^k$ in N degenerate sectors

Cox, Wandler, EP, 2021 Anber, EP, 2022

use both new insights: *a.)* generalized anomalies + *b.)* moduli space of $Q = \frac{k}{N}$ on torus

a.) generalized anomalies →

$$\langle(\lambda^2)^k\rangle \equiv \mathrm{tr}_{\mathcal{H}_{m_3}} \left(e^{-\beta H} (-1)^F (\lambda^2)^k \, T_3 \right)\Big|_{\beta, L \to \infty} = N \langle (\lambda^2)^k \rangle_{\text{one vacuum on } \mathbb{R}^4}$$

the Hilbert space trace over $\mathcal{H}_{m_3}$ with the T_3 inserted is the path integral over $\mathbb{T}^4$ $(L^3 \times \beta)$ with 't Hooft twists $n_{12} = -k, n_{34} = 1$:

$$\sum_{\nu \in \mathbb{Z}} \int [DA_\mu][D\lambda][D\bar{\lambda}] \left[\prod_{i=1}^{k} \mathrm{tr}(\lambda\lambda)(x_i)\right] e^{-S_{SYM} - i\theta\left(\nu + \frac{k}{N}\right)}\Big|_{n_{12}=-k\,, n_{34}=1}$$

so that leading semiclassical contribution requires $Q = \frac{k}{N}$ instantons.

What are they? What is *b.)* moduli space of $Q = \frac{k}{N}$ on torus

b.) moduli space of $Q = \frac{k}{N}$ on torus

fractional instanton solutions found 1979: ’t Hooft, Q = $\frac{k}{N}, k \in \mathbb{Z}$

issues: i.) instantons self-dual (BPS) only for tuned $\mathbb{T}^4$: $kL_1L_2 = k(N-k)L_3L_4$

ii.) index of adjoint operator is k (as per index theorem),
but have extra antichiral 0-modes

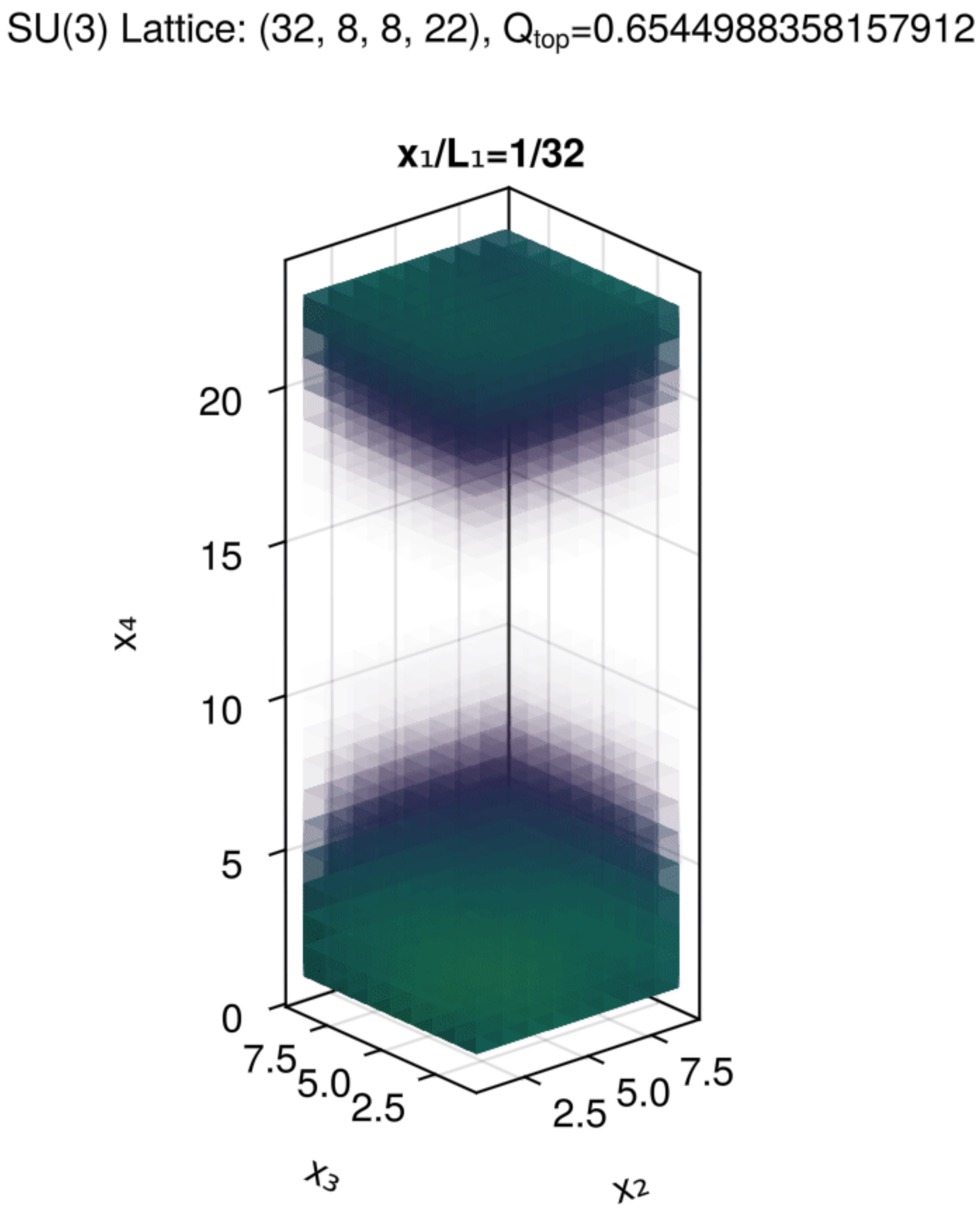
both can be solved by detuning $\mathbb{T}^4$ away from $kL_1L_2 = k(N-k)L_3L_4$

idea: García Pérez, González-Arroyo, Pena 2000, implemented for k=1, N=2, 2000 (& G.-A.: k=1, any N, 2018)

and constructing self-dual instanton as an expansion in small $\Delta = \frac{kL_1L_2 - k(N-k)L_3L_4}{\sqrt{L_1L_2L_3L_4}}$

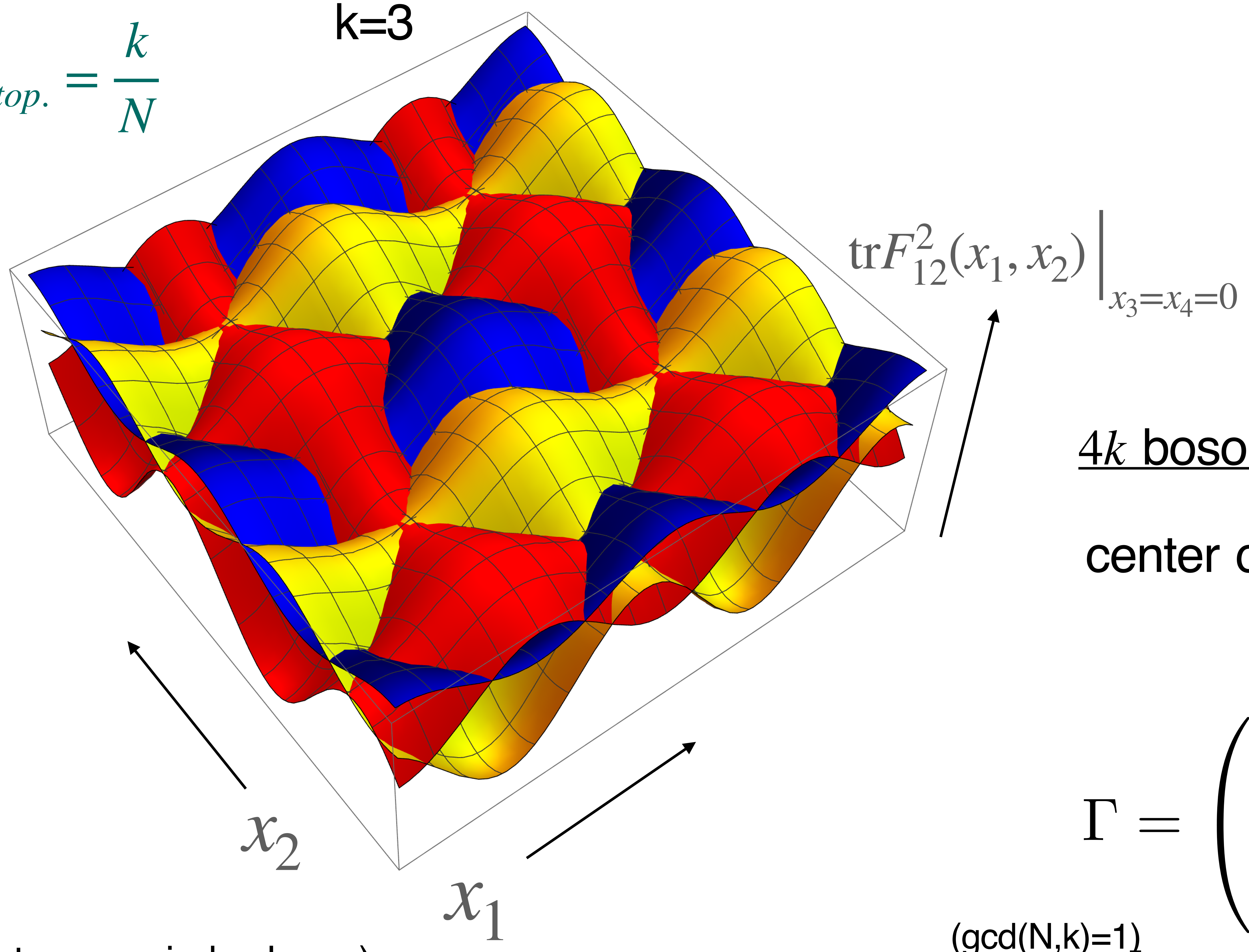
for k>1, we found <u>”multi-fractional instantons”…</u>

b.) *just for fun: an "exact" solution (numerical) Q=2/3 in SU(3),* courtesy A. Cox (Toronto)



b.) moduli space of $Q = \dfrac{k}{N}$ on torus

Anber, EP 2307.07495, 2408.16058



(two torus periods shown)

- k lumps, $Q_{top.} = \dfrac{k}{N}$
- strongly overlapping, liquid-like
- each lump carries 2 gaugino zero modes

(see 2307.07495)

<u>$4k$ bosonic moduli, as per index thm.:</u>

center of mass motion + relative motion ($\Gamma_r^{SU(k)}$ = $SU(k)$ root lattice)

$$\Gamma = \left(\prod_{\mu=1}^{4} \frac{\mathbb{S}^1_\mu \times \Gamma_r^{SU(k)}}{\mathbb{Z}_k} \right) / S_k$$

(gcd(N,k)=1)

(in SUSY: $\int_\Gamma$ only!!!)

combined weight-lattice/c.m. shifts

permutation of lumps = $SU(k)$ Weyl

Γ includes images of instanton under global center symmetry (in $\mathbb{T}^3$ only) + modding by gauge equivalences

combining all… SUSY -> nonzero modes cancel, only $\int_\Gamma$ remains

Γ includes images of instanton under global center symmetry (in $\mathbb{T}^3$ only…!)

(there is a story here related to subtlety of the def. of path integral, only recently understood)

$$\langle (\mathrm{tr}\,\lambda^2)^k \rangle = \sum_{\nu \in \mathbb{Z}} \int [DA_\mu][D\lambda][D\bar\lambda] \left[\prod_{i=1}^{k} \mathrm{tr}(\lambda\lambda)(x_i)\right] e^{-S_{SYM} - i\theta(\nu + \frac{k}{N})} \Big|_{n_{12}=-k\,, n_{34}=1}$$

$$= \mathrm{N} \left(\frac{16\pi^2 M_{\mathrm{PV}}^3}{g^2} e^{-\frac{8\pi^2}{N g^2}} \right)^k \int \prod_{C'=1}^{k} d\zeta_1^{C'} d\zeta_2^{C'} \zeta_2^{C'} \zeta_1^{C'}$$

integral over bosonic and fermionic moduli; $M_{PV}^{n_B - \frac{1}{2} n_F} = M_{PV}^{4k-k} = M_{PV}^{3k}$

path integral on twisted $\mathbb{T}^4$ w/ gcd(N,k)=1: $\langle (\mathrm{tr}\,\lambda^2)^k \rangle = \mathrm{N} \left(16\pi^2 \Lambda^3\right)^k$

$$\langle (\lambda^2)^k \rangle \equiv \mathrm{tr}_{\mathcal{H}_{m_3}} \left(e^{-\beta H} (-1)^F (\lambda^2)^k \, T_3 \right) \Big|_{\text{infinite volume}} = N \langle (\lambda^2)^k \rangle_{\text{one vacuum on } \mathbb{R}^4}$$

thus, the main points of my talk:

1.

semiclassical objects contributing to gaugino condensate on the torus are related to center vortices and monopoles, argued responsible for chiral symmetry breaking and confinement

(just state… won't describe relation…)

2.

multifractional instantons with $Q = k/N$ "fractionalize" into k objects in an instanton liquid like configuration, whose moduli we now understand

3.

calculated $\langle \lambda^{2k} \rangle$ on small $\mathbb{T}^4$, gcd(N,k)=1; agrees with $\mathbb{R}^4$ at weak-coupling and recent lattice!

$$\langle (\mathrm{tr}\,\lambda^2)^k \rangle \Big|_{\mathbb{R}^4} = (16\pi^2\Lambda^3)^k$$

compared to the k>1 ADHM calculation on $\mathbb{R}^4$, this is (to us) infinitely simpler

this completes - and extends! - a calculation started in 1984

none of this would be possible without recent *(2000+)* **progress in:**

a.) **understanding the role of generalized anomalies in twisted Hilbert space on torus,**

b.) **the nature and moduli space of multi-fractional instantons,**

and

c.) **some subtleties of defining the path integral from the Hilbert space trace** *(too technical to discuss, skipped) - THANKS TO AN ANONYMOUS REFEREE!!!*

where does this leave us regarding confinement, $\mathbb{R}^4$, the future, etc.?

the relation between the seemingly different fractional instantons, responsible for nonperturbative phenomena on $\mathbb{R}^2 \times \mathbb{T}^2, \mathbb{R}^1 \times \mathbb{T}^3, \mathbb{R}^3 \times \mathbb{S}^1, \mathbb{T}^4$ is interesting and not totally explored yet *(various people working on various aspects)*

extending our calculation of $\langle (\lambda^2)^k \rangle$ to $\gcd(N,k) \neq 1$ may hold interesting lessons

now, semiclassics holds when some compact direction is small, with details depending on theory and twists (b.c.)$\cdots$but, absent the SUSY magic, extending it to large volume requires lattice studies and/or models

this is my own conservative view; can't offer a 'rose garden' $\cdots$

where does this leave us regarding confinement, $\mathbb{R}^4$, the future, etc.?

fractionalization and moduli space understood for $Q = k/N$ with $k = 1,\ldots,N-1$ only;

how about general-Q fractionalization, including integer Q?

may hold lessons for FILM (or other) models of the vacuum…

another thing of interest *(given the success of ADHM/branes)* **may be the relation between fractional instantons and D-branes -** unexplored since 1997;

’t Hooft’s solutions on tuned $\mathbb{T}^4$ are T-dual to wrapped intersecting (BPS) D2-branes

- tachyon condensation vs the detuned-$\mathbb{T}^4$ Δ-expansion??