The curious case of multi-instantons and the necessity of Lefshetz thimbles

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“Thimble and buttons” in “Fashion District,” Toronto

Instanton—anti-instanton thimble in N=2 SUSY QM, 1507.04063
Motivation: ... really, more than half of my slides

Instantons play a role in many physical problems. In QFT, whenever semiclassics “works”, key to understanding important physics, e.g.:

N=1 SUSY theories: nonperturbative superpotentials.
N=2 SUSY theories: Seiberg-Witten curves.
Phenomenological models of chiral symmetry breaking in QCD.

Mass gap, confinement & center stability:
QCD(adj)/SYM & deformed Yang–Mills theory on $\mathbb{R}^{1,2} \times S^1_L$, at small $L$

already at weak coupling, a major difficulty:
“How to define & calculate multi-instanton contributions?”

Motivation: ... no time/need to explain all-but to see QFT origin

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Not merely a question of calculating exponentially suppressed effects.

Instanton—anti-instanton (I-I*), for example, contributions have been found to give the leading effect in many cases.

**Ex. 1:** SYM, mass gap (confinement) and center stability due to such configurations: vacuum is a dilute gas of “magnetic bions” and “neutral bions.” both are different types of I-I* “molecules”

for SU(2)
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“neutral bions” are particularly bizarre: they are MM* “molecules”

\[ 2 \times M \text{ action} \quad \text{deviation of holonomy from center} \]

\[ \mathcal{A} \sim \int_0^\infty dr e^{-\left( \frac{-2 \times 4 \pi L}{g^2} + (4n_f - 2) \log r \right)}, \quad n_f = 1 \]

Coulomb attraction
fermion-zero mode exchange attraction

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Coulomb attraction fermion-zero mode exchange attraction

(neutral bions are responsible for center stability and also cancel magnetic bion vacuum energy in SYM)

Turns out, the MM* amplitude makes sense. Despite the attractive-only interactions, a "stable molecule" exists! We know from:

1. supersymmetry, exact \( W \to V = |W'|^2 \)

2. analytic continuation:
   MM* "live" at complex separation

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MM* in some sense “classical” (live in Euclidean)
- no time and no quantum fluctuations to stabilize, not, e.g. positronium!
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\[ 
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\]

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\[ 
A \sim \int_0^\infty d \epsilon e^{-\left( -2 \times \frac{4\pi L}{g^4 r} + (4n_f - 2) \log r \right)}, \quad n_f = 1 
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Complexification crucial. Hypothesis that MM* lie on a different “Lefshetz thimble” from the perturbative vacuum - distinguished by a phase (“HTA”)…?
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Ex. 1: SYM, mass gap....
Ex. 2: “Resurgent” cancellations: imaginary parts due to Borel resummation of perturbation theory vs imaginary parts of I-I*

high orders of perturbation theory double-well QM, non Borel-summable:

\[ E^0_{pert.} = -\frac{3}{\pi} \sum_{n=1}^{\infty} \frac{(3g)^n}{n!} k! \]

ambiguity of Borel sum of pert. series:

\[ \Delta E[\text{borel sum}] = -\frac{3}{\pi} \left( \pm i\pi \left( \frac{3g}{2} \right) \right) \]

II* contribution:
requires analytic continuation
Bogomolnyi, Zinn-Justin
Motivation:

Complexification seems crucial. **Hypothesis/dream/ is that** MM* **lie on a different “Lefshetz thimble” from the perturbative vacuum and are distinguished from it by a phase associated with the thimble... “like” in 1dim integrals:**

\[
I(\hbar) = \int_{-\infty}^{\infty} dx \, e^{-\frac{1}{\hbar} f(x)} \quad \xrightarrow{\text{steepest descent method}} \quad \sum_{\sigma} n_{\sigma} \int_{\mathcal{J}_\sigma} dz \, e^{-\frac{1}{\hbar} f(z)}.
\]

Decompose the original real path of integration into steepest descent paths, or “thimbles”, going through different saddles (recall phase is constant on each such contour)

(I think) we are far from understanding of what “**Defining the Path Integral on Lefshetz Thimbles**” means.

All I will do is to show you a simple, yet not completely trivial, example supporting the need of complexification...
Subject/summary of talk:

N=2 SUSY QM = 4d WZ model reduced to 2d

\[ g \mathcal{L}_E = \left| \dot{z}(t) \right|^2 + \left| W'(z) \right|^2 + \left( \bar{\chi}_1 \chi_2 \right) \left( -\partial_t + \begin{pmatrix} 0 & W''(z) \\ W'(z) & 0 \end{pmatrix} \right) \begin{pmatrix} \chi_1 \\ \bar{\chi}_2 \end{pmatrix} \]

\[ W(z) = \prod_{i=1}^{k+1} (z - z_i) \quad |I_W| = k \]

Witten index=number of critical points of W(z)

E_vac=0, as opposed to N=1 SUSY QM: well known.
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\( E_{\text{vac}} = 0 \), as opposed to \( N=1 \) SUSY QM: well known.

**Goal:** Understand \( E_{\text{vac}} = 0 \) from next-order semiclassicals.

**Upshot:** It’s not completely trivial. {Relation to motivation: complexification!}
Main part of talk:

N=2 SUSY QM = 4d WZ model reduced to 2d

\[ g \mathcal{L}_E = |\dot{z}(t)|^2 + |W'(z)|^2 + \left( \begin{array}{c} \bar{\chi}_1 \\ \chi_2 \end{array} \right) \left( -\partial_t + \left( \begin{array}{cc} 0 & W''(z) \\ W'(z) & 0 \end{array} \right) \right) \left( \begin{array}{c} \chi_1 \\ \bar{\chi}_2 \end{array} \right) \]

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\[ W(z) = \frac{1}{3} z^3 - z \alpha^2 \]

take “a” real (plot for a=1)

BPS (anti)instantons

\[ \dot{z} = \pm W' \]

| \leftrightarrow \|

|* \leftrightarrow |

l,l*: tunnelling between minima; two fermion zero modes each
(with opposite “chirality” from 4d p.o.v.)
Main part of talk: \( W(z) = \frac{1}{3}z^3 - za^2 \) take “a” real (plot for a=1)

potential w/ two minima

BPS (anti)instantons
\[ \dot{z} = \pm \bar{W}' \]

| \( l \) \( \leftrightarrow \) \( l^* \)

\( l, l^* \): tunnelling between minima; two fermion zero modes each (with opposite “chirality” from 4d p.o.v.)

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To rephrase:

after all, the far away \( l^* \) will lift the zero modes of \( l \) (and v.v.):

fermion exchange

instanton anti-instanton
Main part of talk:  
**Goal:** Understand $E_{\text{vac}} = 0$ from next-order semiclassics.
**To rephrase:** Why I-I* ‘events’ do not contribute to $E_{\text{vac}}$?

after all, the far away I* will lift the zero modes of I (and v.v.):

\[
I_1 = \int_{J_1} d\tau \ e^{\frac{4\omega^3}{g} e^{-\omega \tau} - 2\omega \tau}
\]

Two issues:

- at small I-I* separation - all the above is nonsense
- gives *negative* $E_{\text{vac}}$ if exponentiated
Main part of talk:

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\[
I_1 = \int_{\mathcal{J}_1} d\tau \ e^{\frac{4\omega^3}{g} e^{-\omega \tau} - 2\omega \tau}
\]

\[
I_2 = \int_{\mathcal{J}_2} d\tau \ e^{\frac{4\omega^3}{g} e^{-\omega \tau} - \omega \tau}
\]

Yukawa squared = \(g\)
Main part of talk:

Two issues:
- at small I-I* separation - all the above is nonsense
- gives *negative* E_vac

\[ I_1 = \int d\tau \ e^{\frac{4\omega^3}{g} e^{-\omega\tau} - 2\omega\tau} \]

entire story rests on relative factor - somewhat hard calculation

\[ E_0 \propto -e^{-2S_0} (4\omega^3 I_1 + g I_2) \]

notice: different orders in g!
both come with same sign: how to cancel?

\[ I_2 = \int d\tau \ e^{\frac{4\omega^3}{g} e^{-\omega\tau} - \omega\tau} \]

Yukawa squared = g
\[ E_0 \propto -e^{-2S_0}(4\omega^3 I_1 + gI_2) \]

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THE TWO "LEFSHETZ THIMBLES"

\[ [II]_F \]

\[ [II]_Y \]

\[ \log(2\omega^3/g) \]

Naive cycle
Main part of talk:

\[ E_0 \propto -e^{-2S_0}(4\omega^3 I_1 + gI_2) \]

\[ I_1 = \int_{\mathcal{J}_1} d\tau \ e^{\frac{4\omega^3}{g}e^{-\omega \tau} - 2\omega \tau} \quad I_2 = \int_{\mathcal{J}_2} d\tau \ e^{\frac{4\omega^3}{g}e^{-\omega \tau} - \omega \tau} \]

THE TWO "LEFSHETZ THIMBLES"

\[ e^{-2\omega \tau_{\text{crit.}}} = e^{-i2\pi + 2\log \frac{2\omega^3}{g}} = \left(\frac{2\omega^3}{g}\right)^2 \]

\[ e^{-\omega \tau_{\text{crit.}}} = e^{-i\pi + \log \frac{4\omega^3}{g}} = -\frac{4\omega^3}{g} \]
Main part of talk:

\[ E_0 \propto -e^{-2S_0}(4\omega^3 I_1 + gI_2) \]

\[ I_1 = \int_{\mathcal{J}_1} d\tau \ e^{\frac{4\omega^3}{g}e^{-\omega \tau} - 2\omega \tau} \]

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**THE TWO “LEFSHETZ THIMBLES”**

\[ [II]_F \quad [II]_Y \]

- Naive cycle
- \( e^{-2\omega \tau_{\text{crit.}}} = e^{-i2\pi + 2\log(\frac{2\omega^3}{g})} = \left(\frac{2\omega^3}{g}\right)^2 \)

Imaginary part, \( \text{Im} \omega \tau_{1,2} = i\pi \) of critical separation responsible for change of relative sign (one vs. two “massive propagators”)

+ two vs. one “massive propagators” at saddle compensates for relative \( g \) (nontrivial interplay of complexification and perturbation theory)
Main part of talk:

\[ E_0 \propto -e^{-2S_0}(4\omega^3 I_1 + gI_2) \]

\[ I_1 = \int_{\mathcal{J}_1} d\tau \, e^{\frac{4\omega^3}{g} \tau - 2\omega\tau} \]

\[ I_1 = \frac{g^2}{16\omega^7} \]

\[ 4\omega^3 I_1 + gI_2 = 0 \]

Comments/Results:

1. Imaginary part, \( \text{Im}\ \omega\tau_{1,2} = i\pi \) of critical separation responsible for change of relative sign - one vs. two “massive propagators”; \( g \)-order!

2. Absolute value of separation is large at small \( g \) - self consistent! \( I \) and \( I^* \) are never on top of each other: complex separation

3. Integrating over the *entire* (…?) thimble gives \( E_{\text{vac}} = 0 \)!
Goal: Understand $E_{\text{vac}} = 0$ from plain next-order semiclassics… no localization, no deformation invariance…

Upshot: It’s not completely trivial. {Relation to motivation: complexification!}

Found that complexifying the quasi-zeromode crucial. $I$ and $I^*$ “live” a complex & large separation apart; consistent next-to-leading order semiclassics.
N=2 SUSY QM = 4d WZ model reduced to 2d, Witten index ≠ 0

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Comments:

All was done to $I-I^*$ order… not immediately clear how to proceed to higher orders.

Showed that “quasi-zeromode” complexification crucial; notice that this is just one direction in field space (the most relevant for this case!).

Suggests that complexification of path integral important.

Magnetic and neutral bions in SYM can be seen to emerge in a similar way, at (generally) complex separations. (Recall SYM is only SUSY w/out scalars… YM)

Solving analogous puzzles in SW theory harder… but worthwhile, beyond QM?

status: “theoretical experiment” in search of a theory… finite dimensional thimbles (lattice)? mathematics?