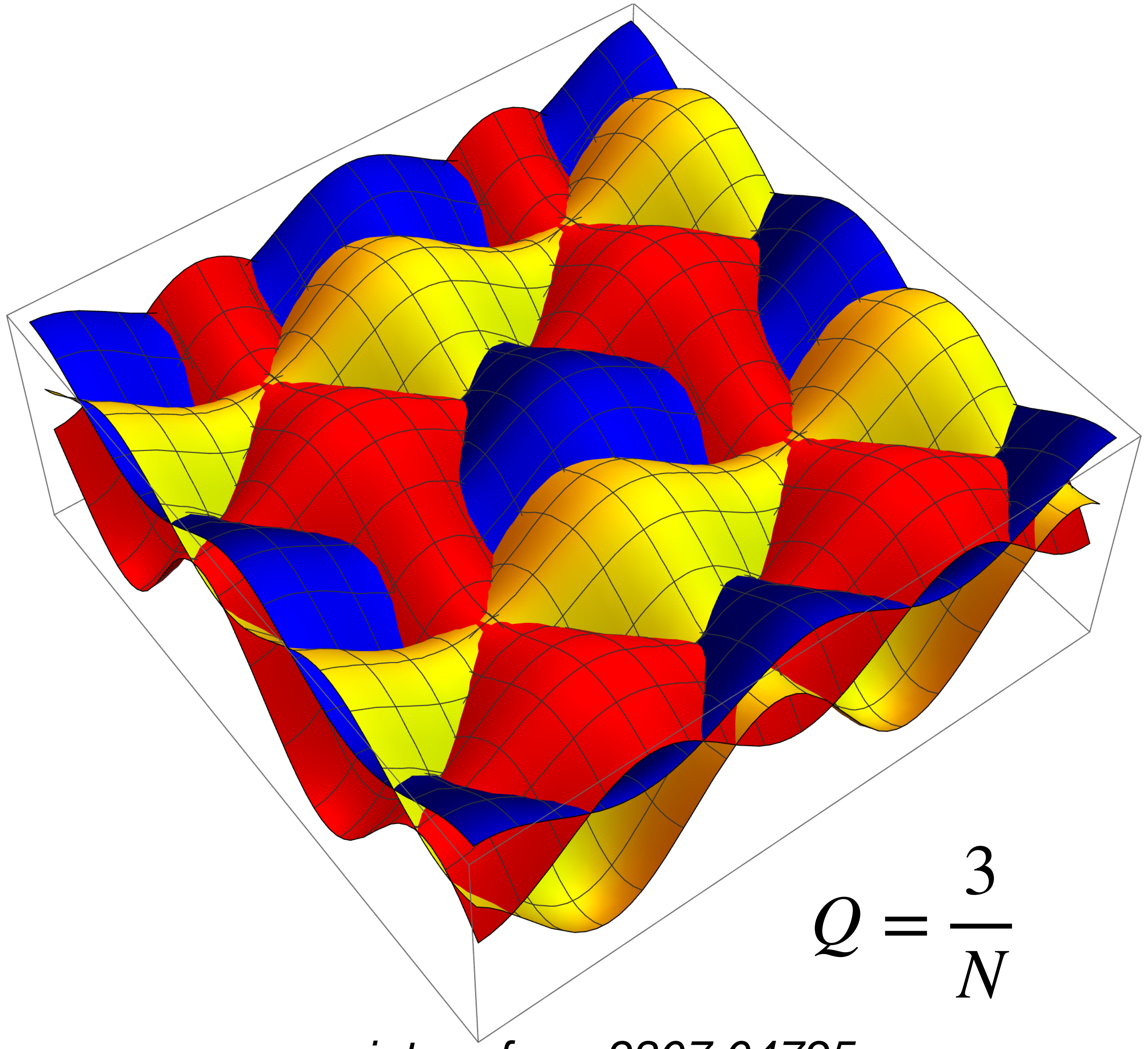
D-branes and the moduli space of fractional instantons on the four-torus

Erich Poppitz

University of Toronto

2604.21980

+ many other works of many authors



$$Q = \frac{3}{N}$$

picture from 2307.04795,
Mohamed Anber (Durham) & EP

talk is, ultimately, about nonperturbative effects in YM on compactified $\mathbb{R}^4$:

... $\mathbb{T}^4$, or its various limits: $\mathbb{R} \times \mathbb{T}^3$, $\mathbb{R}^2 \times \mathbb{T}^2$, $\mathbb{R}^3 \times \mathbb{S}^1$

one may ask *-even in a "Physical Math" seminar-* why, as *not the real world?!*

- don't know that real world does not have a very large $\mathbb{T}^3$, *so long as $\gg$ Hubble*
- stat mech: spontaneous symmetry breaking - TD limit, start w/ finite V, e.g. $\mathbb{T}^3$
- lattice is usually $\mathbb{T}^4$
- generalized anomalies involving, e.g. 1-form center symmetry, revealed on space with two-cycles, e.g. $\mathbb{T}^4$

provided compact space << Λ^{-1}

- various YM theories: semiclassically calculable on $\mathbb{T}^4$, $\mathbb{T}^3 \times \mathbb{R}$, $\mathbb{T}^2 \times \mathbb{R}^2$, $\mathbb{S}^1 \times \mathbb{R}^3$

reveal unusual fractionally charged objects which disorder Wilson loops
interesting insight (at a price - not enough to get Clay Prize)... but best there is, so far

1. Motivation for $Q = \frac{r}{N}, \forall r \in \mathbb{N}$, instantons in SU(N)

brief review of their physics role

2. QFT: constant-F solutions and “missing moduli”

3. D-branes and the “missing moduli”

4. Summary and wishlist

1. Motivation for $Q = \frac{r}{N}, \forall r \in \mathbb{N}$, instantons in SU(N)

brief review of physics role: confinement and chiral symmetry breaking

't Hooft ~'80:

twisted b.c.

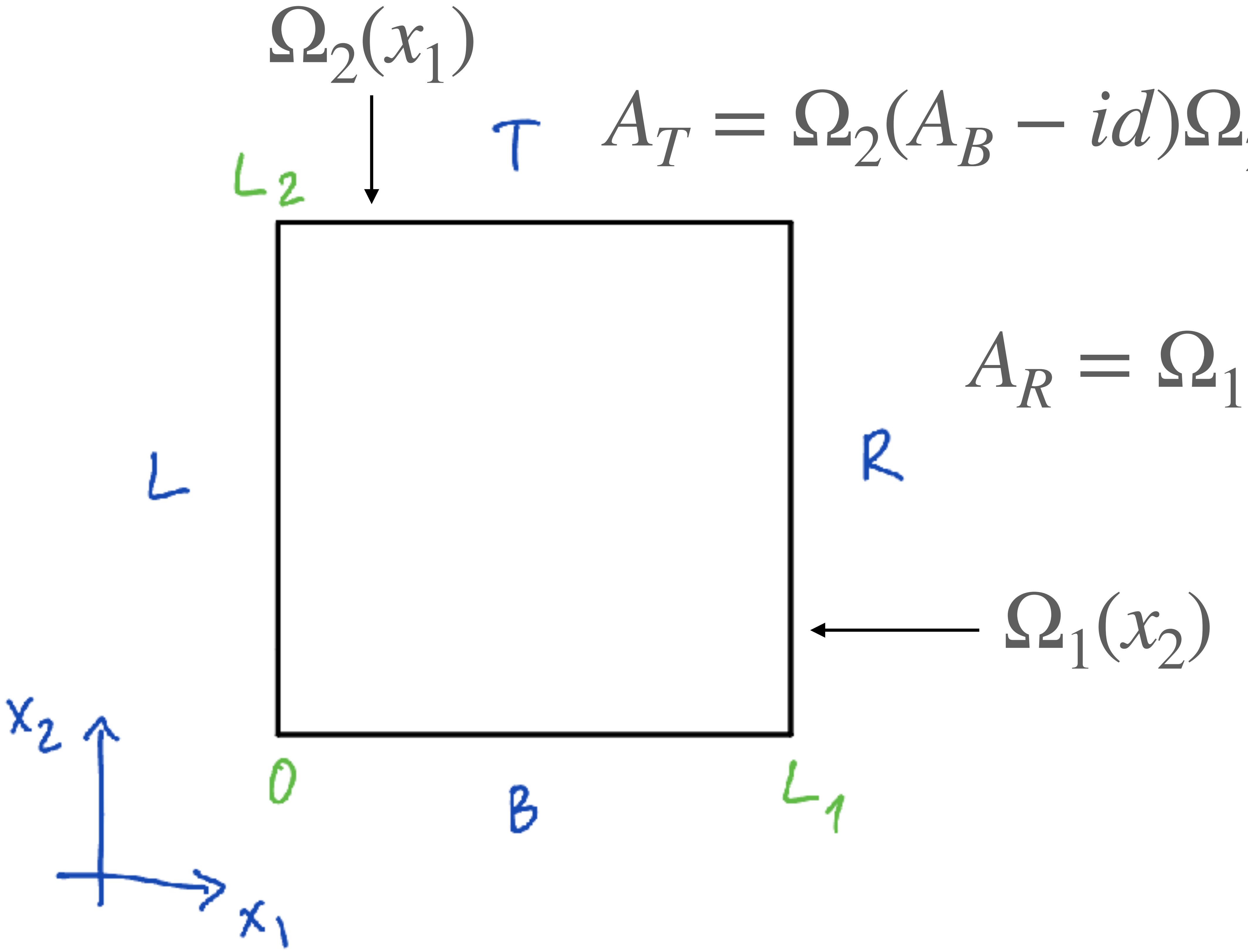
modern:

topological $\mathbb{Z}_N^{(2)}$-form gauge field background,

for $\mathbb{Z}_N^{(1)}$-form ("center") symmetry

a quick review of poor man's twisted bundle

$$\left(\oint_{\mathbb{T}^2_{\mu\nu}} B^{(2)} = \frac{2\pi}{N} n_{\mu\nu} \ (\mathrm{mod}\ 2\pi) \right)$$



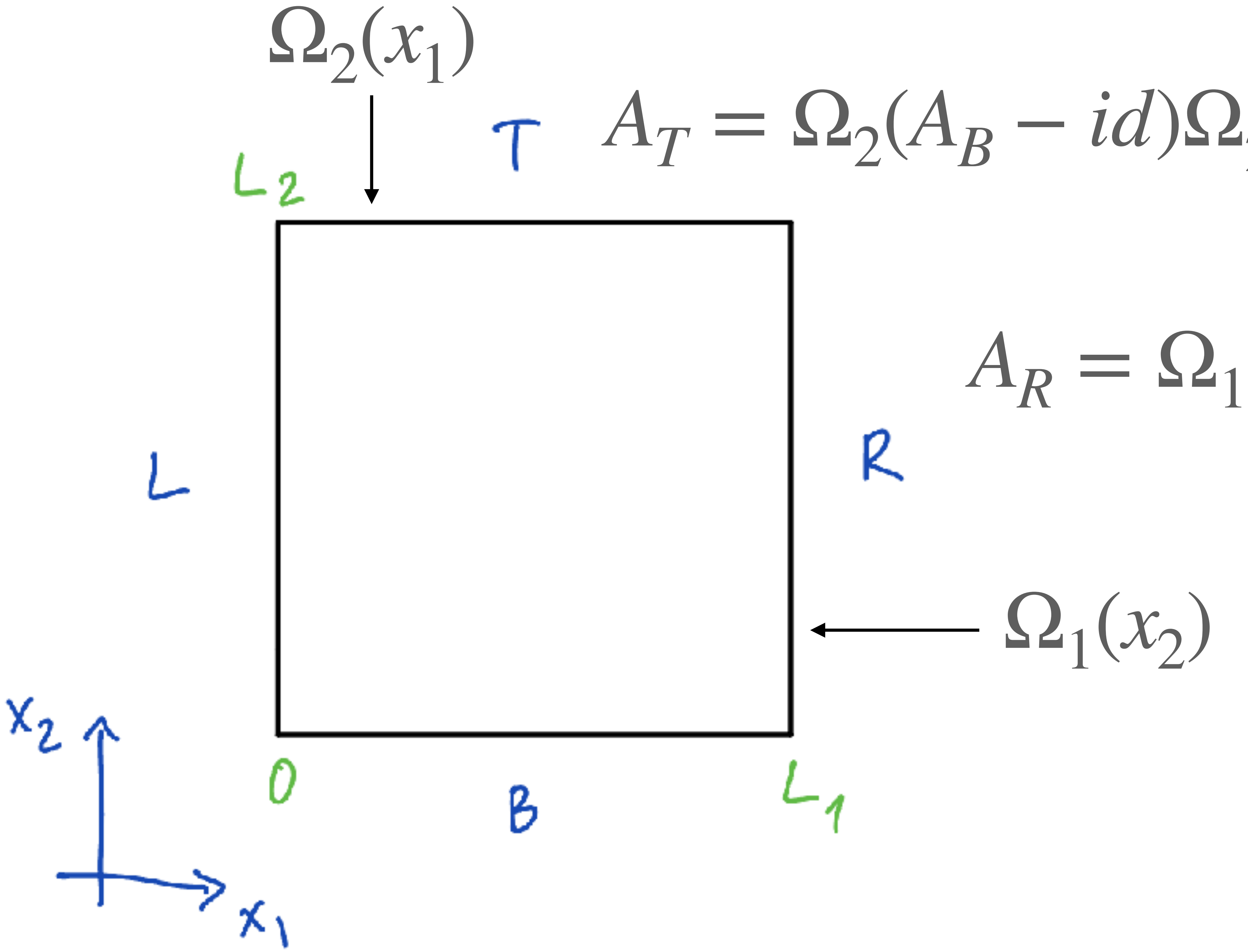
$$A_T = \Omega_2 (A_B - id) \Omega_2^{-1}$$

$$A_R = \Omega_1 (A_L - id) \Omega_1^{-1}$$

$$\Longrightarrow \quad \Omega_1(L_2)\Omega_2(0) = e^{i\frac{2\pi}{N} n_{12}}\, \Omega_2(L_1)\Omega_1(0)$$

a quick review of poor man's twisted bundle

$$\left(\oint_{\mathbb{T}^2_{\mu\nu}} B^{(2)} = \frac{2\pi}{N} n_{\mu\nu} \ (\text{mod } 2\pi) \right)$$



$$A_T = \Omega_2 (A_B - id) \Omega_2^{-1}$$

$$A_R = \Omega_1 (A_L - id) \Omega_1^{-1}$$

$$\Longrightarrow \ \Omega_1(L_2)\Omega_2(0) = e^{i\frac{2\pi}{N} n_{12}} \, \Omega_2(L_1)\Omega_1(0)$$

$$\Big\downarrow \ \mathbb{T}^4$$

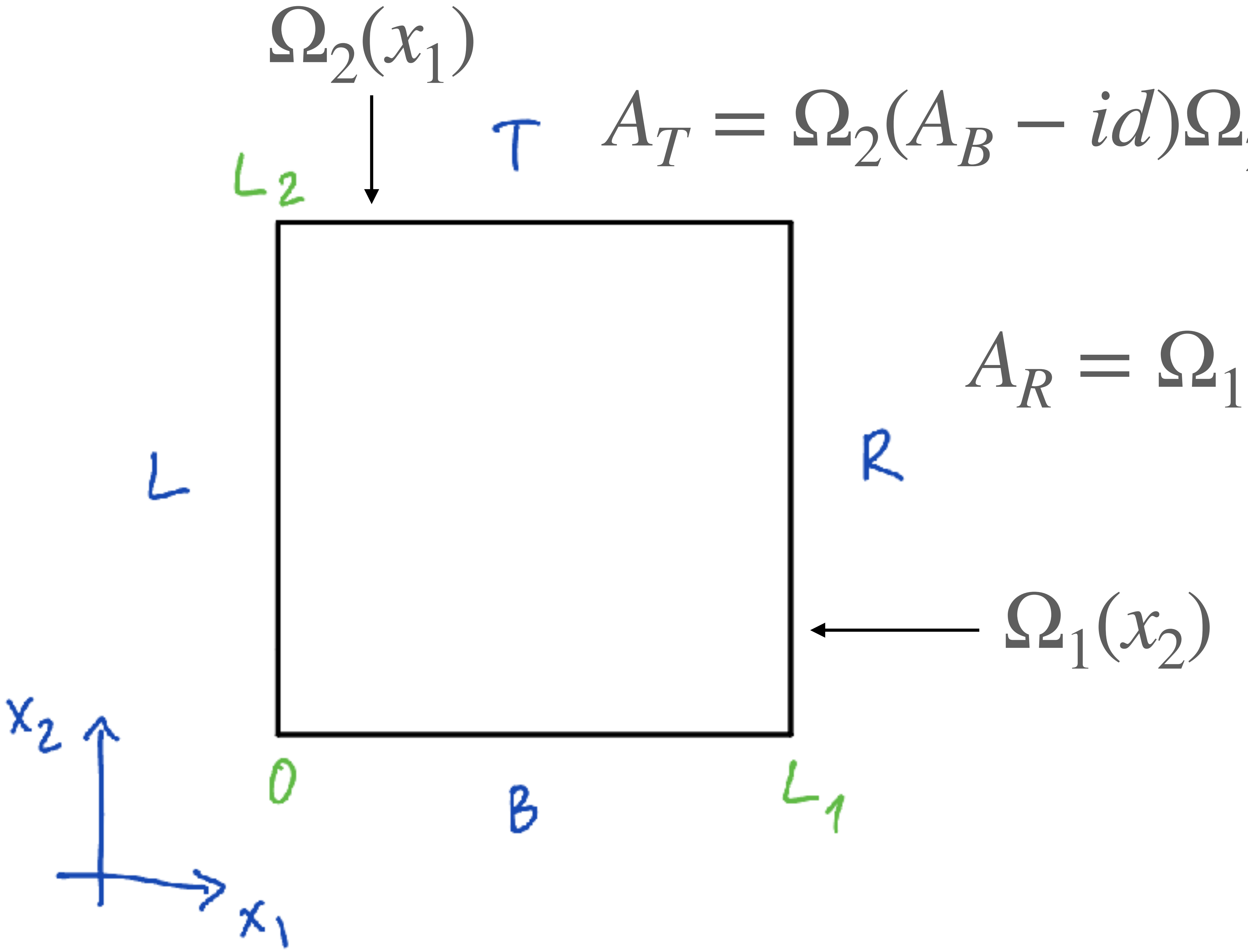
$$\Omega_\mu(L_\nu)\Omega_\nu(0) = e^{i\frac{2\pi}{N} n_{\mu\nu}} \, \Omega_\nu(L_\mu)\Omega_\mu(0)$$

$$Q = \frac{1}{8\pi^2} \int_{\mathbb{T}^4} \text{tr} \, F \wedge F = - \frac{\text{Pf}(n)}{N} \ (\text{mod } 1)$$

't Hooft '81; van Baal '82

a quick review of poor man's twisted bundle

$$\left(\oint_{\mathbb{T}^2_{\mu\nu}} B^{(2)} = \frac{2\pi}{N} n_{\mu\nu} \ (\text{mod } 2\pi) \right)$$



$$A_T = \Omega_2 (A_B - id) \Omega_2^{-1}$$

$$A_R = \Omega_1 (A_L - id) \Omega_1^{-1}$$

$$\Longrightarrow \quad \Omega_1(L_2)\Omega_2(0) = e^{i\frac{2\pi}{N}n_{12}}\, \Omega_2(L_1)\Omega_1(0)$$

$$\Big\downarrow \ \mathbb{T}^4$$

$$\Omega_\mu(L_\nu)\Omega_\nu(0) = e^{i\frac{2\pi}{N}n_{\mu\nu}}\, \Omega_\nu(L_\mu)\Omega_\mu(0)$$

$$Q = \frac{1}{8\pi^2} \int_{\mathbb{T}^4} \text{tr}\, F \wedge F = -\frac{\text{Pf}(n)}{N} \ (\text{mod } 1)$$

't Hooft '81; van Baal '82

if only $n_{12}, n_{34} \neq 0 \ (\text{mod } N)$: $Q = -\dfrac{n_{12}n_{34}}{N} \ (\text{mod } 1)$

't Hooft also found the only analytic, **constant-F, solutions** '81 - return to later …

a quick review of poor man's twisted bundle

$$\Omega_\mu(L_\nu)\Omega_\nu(0) = e^{i\frac{2\pi}{N}n_{\mu\nu}}\,\Omega_\nu(L_\mu)\Omega_\mu(0)$$

$$Q = \frac{1}{8\pi^2}\int_{\mathbb{T}^4} \mathrm{tr}\, F \wedge F = -\frac{\mathrm{Pf}(n)}{N} \pmod 1$$

't Hooft '81; van Baal '82

if only $n_{12}, n_{34} \neq 0 \pmod N$: $Q = -\dfrac{n_{12}n_{34}}{N} \pmod 1$

first some applications:

- burried in *van Baal's* 1984 (!) PhD thesis, unpublished chapter III:
  - multi-branched structure of YM θ-vacuum on small spatial $\mathbb{T}^3$
  - in effect, discovered θ-periodicity/center-symmetry anomaly ($\mathbb{Z}_2^{(0)}$-$\mathbb{Z}_{\mathbb{N}}{}^{(1)}$)

    - *see Cox, Wandler, EP '2106*

- Madrid group 1980's-1993-… *González-Arroyo, García Perez…-* $\mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}^4$

  1st example of "adiabatic continuity"

small-volume -> large-volume

- *González-Arroyo, García Perez,… '93-…* $\mathbb{R} \times \mathbb{T}^3$
- *Ünsal, Yaffe, Shifman,…'07-…* ← $\mathbb{R}^3 \times \mathbb{S}^1$
- *Ünsal, Tanizaki,…'20-… also González-Arroyo, Montero… '98-'00* ← $\mathbb{R}^2 \times \mathbb{T}^2$

small-volume -> large-volume continuity:

- *González-Arroyo, García Perez,… ’93-…* $\mathbb{R} \times \mathbb{T}^3$

$\leftarrow \mathbb{R}^3 \times \mathbb{S}^1$

- *Ünsal, Yaffe, Shifman,…’07-…*

$\leftarrow \mathbb{R}^2 \times \mathbb{T}^2$

- *Ünsal, Tanizaki,…’20-… also González-Arroyo, Montero… ’98-‘00*

- when $\Lambda N L_{\mathbb{T}^k} \ll 1$: weak-coupling $SU(N) \to \mathbb{Z}_N$ or $U(1)$, semiclassics

  dilute gas of fractional instantons, Q=1/N, disorders Wilson loop

  ($\mathbb{Z}_N^{(1)}$-restoration, confinement)

small-volume -> large-volume continuity:

fractional instantons

- *González-Arroyo, García Perez,… ’93-…* $\mathbb{R} \times \mathbb{T}^3$

monopole-instantons

$\mathbb{R}^3 \times \mathbb{S}^1$

- *Ünsal, Yaffe, Shifman,…’07-…*

center vortices

$\mathbb{R}^2 \times \mathbb{T}^2$

- *Ünsal, Tanizaki,…’20-… also González-Arroyo, Montero… ’98-‘00*

- when $\Lambda N L_{\mathbb{T}^k} \ll 1$: weak-coupling $SU(N) \to \mathbb{Z}_N$ or $U(1)$, semiclassics

  dilute gas of fractional instantons, Q=1/N, disorders Wilson loop

  ($\mathbb{Z}_N^{(1)}$-restoration, confinement)

small-volume -> large-volume continuity:

fractional instantons

- *González-Arroyo, García Perez,… '93-…* $\mathbb{R}\times\mathbb{T}^3$

monopole-instantons

$\mathbb{R}^3\times\mathbb{S}^1$

- *Ünsal, Yaffe, Shifman,…'07-…*

$\mathbb{R}^2\times\mathbb{T}^2$ center vortices

- *Ünsal, Tanizaki,…'20-… also González-Arroyo, Montero… '98-'00*

- when $\Lambda N L_{\mathbb{T}^k} \ll 1$: weak-coupling $SU(N) \to \mathbb{Z}_N$ or $U(1)$, semiclassics

  dilute gas of fractional instantons, Q=1/N, disorders Wilson loop

  ($\mathbb{Z}_N^{(1)}$-restoration, confinement)

- for $\Lambda N L_{\mathbb{T}^k} \to \infty$: strong-coupling $\mathbb{R}^4$-limit, numerically show continuity

small-volume -> large-volume continuity:

- *González-Arroyo, García Perez,… '93-…* $\mathbb{R} \times \mathbb{T}^3$ ← fractional instantons
- $\mathbb{R}^3 \times \mathbb{S}^1$ ← monopole-instantons
- *Ünsal, Yaffe, Shifman,…'07-…*
- $\mathbb{R}^2 \times \mathbb{T}^2$ ← center vortices
- *Ünsal, Tanizaki,…'20-…* *also González-Arroyo, Montero… '98-'00*

all related!

- *1998-2024-*

- when $\Lambda N L_{\mathbb{T}^k} \ll 1$: weak-coupling $SU(N) \to \mathbb{Z}_N$ or $U(1)$, semiclassics

  dilute gas of fractional instantons, Q=1/N, disorders Wilson loop

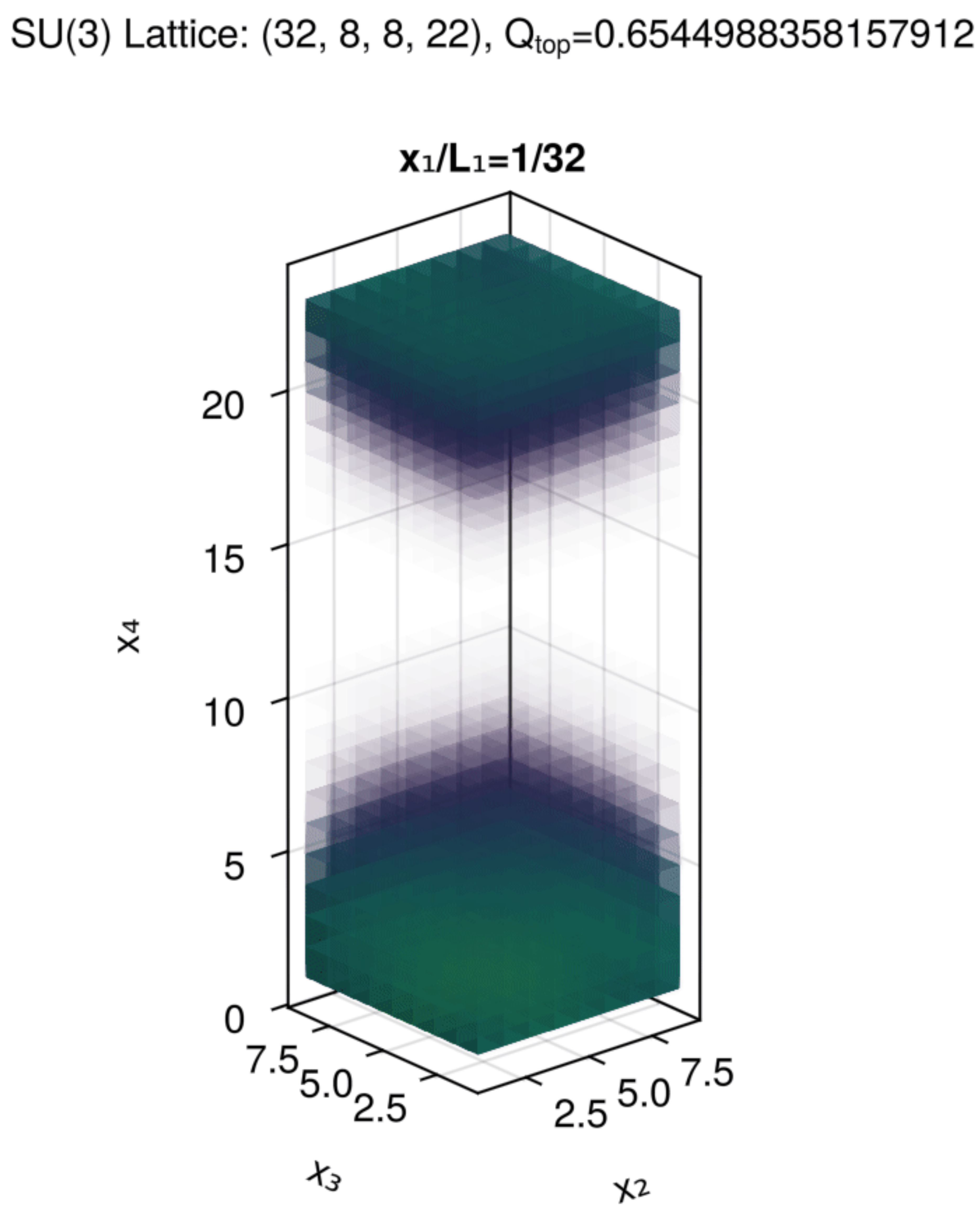
  ($\mathbb{Z}_N^{(1)}$-restoration, confinement)

- all related to fractional instantons on $\mathbb{T}^4$ by changing shape/twists…
  - *González-Arroyo, García Perez '98; Ünsal et al; Tanizaki et al '24; Wandler '24;….* some analytic some numeric

- when $\Lambda N L_{\mathbb{T}^k} \ll 1$: weak-coupling $SU(N) \to \mathbb{Z}_N$ or $U(1)$, semiclassics

  there are remarkably few [exactly two: **a.)**, **b.)** below] analytic solutions available;
  semiclassics uses solutions found by minimizing lattice actions (give idea of size, etc.)

- all semiclassics (string tensions etc.) only up to $\mathcal{O}(1)$ numbers ("semiclassical yet not analytically calculable")



SU(3) charge 2/3 on a (very!) detuned torus - courtesy of A. Cox 2025

- when $\Lambda N L_{\mathbb{T}^k} \ll 1$: weak-coupling $SU(N) \to \mathbb{Z}_N$ or $U(1)$, semiclassics

  there are remarkably few [exactly two: **a.)**, **b.)** below] analytic solutions available;
  semiclassics uses solutions found by minimizing lattice actions (give idea of size, etc.)

- all semiclassics (string tensions etc.) only up to $\mathcal{O}(1)$ numbers ("semiclassical yet not analytically calculable")
- except two:

1. superpotential in SYM on $\mathbb{R}^3 \times \mathbb{S}^1$ due to BPS monopole-instantons **a.)**

   Hollowood, Khoze et al, 2000 (noncancelling determinants (!) corrected only in 2014 *Anber, Teeple, EP)*

2. order-r gaugino condensate in SYM on $\mathbb{T}^4 \; \equiv \; \mathbb{R}^4$ result via ADHM *Dorey et al 2001*

   used some of 't Hooft's $\mathbb{T}^4$ solutions **b.)** + our understanding of their moduli *Anber, EP 2022-2024* summary of

1. Motivation for $Q = \frac{r}{N}, \forall r \in \mathbb{N}$, instantons in SU(N)

brief review of their physics role

$\mathbb{R}^{4-k} \times \mathbb{T}^k$, for $\Lambda N L_{\mathbb{T}^k} \ll 1$: weak-coupling, semiclassics

fractional instantons argued to disorder Wilson loops ($\mathbb{Z}_N^{(1)}$ restoration, confinement)
& responsible for chiral symmetry breaking
- e.g. SYM + many nonSUSY examples, including QCD-like

demonstrating continuity of the $\mathbb{R}^{4-k} \times \mathbb{T}^k \to \mathbb{R}^4$ limit: via lattice simulations or SUSY nonrenormalization

I think that, to a theorist, the semiclassical regime gives enough motivation to desire a better understanding of fractional instantons: their structure, moduli…

end of motivation.

2. QFT: constant-F solutions and "missing moduli"

$$\Omega_\mu(L_\nu)\Omega_\nu(0) = e^{i\frac{2\pi}{N}n_{\mu\nu}}\,\Omega_\nu(L_\mu)\Omega_\mu(0)$$

$$Q = \frac{1}{8\pi^2}\int_{\mathbb{T}^4} \mathrm{tr}\, F \wedge F = -\frac{\mathrm{Pf}(n)}{N} \pmod 1$$

't Hooft '81; van Baal '82

if only $n_{12}, n_{34} \neq 0 \pmod N$: $Q = -\frac{n_{12}n_{34}}{N} \pmod 1$

't Hooft also found the only analytic, **constant-F, solutions** '81

SU(N) solutions we study depend on $k, r \in \mathbb{N}$; where $N = k + \ell$, $\ell \in \mathbb{N}$

$n_{12} = -r, \; n_{34} = 1$ $\qquad Q_{SU(N)} = \frac{r}{N}, \; r \in \mathbb{N}$ $\qquad$ r is arbitary, Q can be also an integer, including 1

$$F_{12} = \begin{pmatrix} -\frac{2\pi\ell r}{NkL_1L_2} I_k & 0 \\ 0 & \frac{2\pi r}{NL_1L_2} I_\ell \end{pmatrix}, \quad F_{34} = \begin{pmatrix} -\frac{2\pi}{NL_3L_4} I_k & 0 \\ 0 & \frac{2\pi k}{N\ell L_3L_4} I_\ell \end{pmatrix}$$

$$\Omega_\mu(L_\nu)\Omega_\nu(0) = e^{i\frac{2\pi}{N}n_{\mu\nu}}\, \Omega_\nu(L_\mu)\Omega_\mu(0)$$

$$Q = \frac{1}{8\pi^2}\int_{\mathbb{T}^4} \mathrm{tr}\, F \wedge F = -\frac{\mathrm{Pf}(n)}{N} \pmod 1$$

't Hooft '81; van Baal '82

if only $n_{12}, n_{34} \neq 0 \pmod N$: $Q = -\frac{n_{12}n_{34}}{N} \pmod 1$

't Hooft also found the only analytic, **constant-F, solutions** '81

SU(N) solutions we study depend on $k, r \in \mathbb{N}$; where $N = k + \ell$, $\ell \in \mathbb{N}$

$n_{12} = -r, \;\; n_{34} = 1$ $\qquad Q_{SU(N)} = \frac{r}{N}, \;\; r \in \mathbb{N}$ $\qquad$ r is arbitary, Q can be also an integer, including 1

$$F_{12} = \begin{pmatrix} -\frac{2\pi\ell r}{NkL_1L_2} I_k & 0 \\ 0 & \frac{2\pi r}{NL_1L_2} I_\ell \end{pmatrix}, \quad F_{34} = \begin{pmatrix} -\frac{2\pi}{NL_3L_4} I_k & 0 \\ 0 & \frac{2\pi k}{N\ell L_3L_4} I_\ell \end{pmatrix}$$

$$F_{12} = F_{34} \implies \frac{L_1L_2}{L_3L_4} = \frac{\ell r}{k}$$

assume tuned shape of $\mathbb{T}^4$ so BPS condition holds

SU(N) solutions we study depend on $k, r \in \mathbb{N}$; where $N = k + \ell$, $\ell \in \mathbb{N}$

$n_{12} = -r, \;\; n_{34} = 1$ $\qquad Q_{SU(N)} = \frac{r}{N}, \;\; r \in \mathbb{N}$; $\qquad$ r is arbitary, Q can be also an integer, including 1

$$F_{12} = \begin{pmatrix} -\frac{2\pi\ell r}{NkL_1L_2} I_k & 0 \\ 0 & \frac{2\pi r}{NL_1L_2} I_\ell \end{pmatrix}, \quad F_{34} = \begin{pmatrix} -\frac{2\pi}{NL_3L_4} I_k & 0 \\ 0 & \frac{2\pi k}{N\ell L_3L_4} I_\ell \end{pmatrix}$$

$$A_2 = -\omega \left(\frac{r x_1}{NkL_1L_2} \right), \quad A_4 = -\omega \left(\frac{x_3}{N\ell L_3L_4} \right), \quad A_1 = A_3 = 0\,;$$

$$\omega = 2\pi \mathrm{diag}\left[\underbrace{\ell, \ell, \ldots, \ell}_{k\,\text{times}}, \underbrace{-k, -k, \ldots, -k}_{\ell\,\text{times}} \right]$$

$$A(x + \hat{e}_\nu L_\nu) = \Omega_\mu(x)(A(x) - id)\Omega_\mu^{-1}(x)$$

SU(N) solutions we study depend on $k, r \in \mathbb{N}$; where $N = k + \ell$, $\ell \in \mathbb{N}$

$n_{12} = -r, \; n_{34} = 1$

$$Q_{SU(N)} = \frac{r}{N}, \; r \in \mathbb{N}.$$

r is arbitary, Q can be also an integer, including 1

deceptively simple!

field strength abelian
transition functions NOT

$$F_{12} = \begin{pmatrix} -\frac{2\pi\ell r}{NkL_1L_2} I_k & 0 \\ 0 & \frac{2\pi r}{NL_1L_2} I_\ell \end{pmatrix}, \quad F_{34} = \begin{pmatrix} -\frac{2\pi}{NL_3L_4} I_k & 0 \\ 0 & \frac{2\pi k}{N\ell L_3L_4} I_\ell \end{pmatrix}$$

$$A_2 = -\omega \left(\frac{r x_1}{NkL_1L_2} \right), \quad A_4 = -\omega \left(\frac{x_3}{N\ell L_3L_4} \right), \quad A_1 = A_3 = 0.$$

Some Twisted Self-Dual Solutions for the Yang-Mills Equations on a Hypertorus*

Gerard 't Hooft**

California Institute of Technology, Pasadena, CA 91125, USA

such an action. All our solutions will be represented in a suitably chosen gauge that makes them look essentially translationally invariant and Abelian. However, considering the difficulty we had in finding them it looked worth-while to publish the result.

It is important that we have solutions with total action decreasing as $1/g^2N$ for $N \to \infty$, so that they will certainly survive in the usual $N \to \infty$ limit [4]. The implications of that for the $N \to \infty$ theory are however not clear to the author.

SU(N) solutions we study depend on $k, r \in \mathbb{N}$; where $N = k + \ell$, $\ell \in \mathbb{N}$

$n_{12} = -r, \; n_{34} = 1$ $\qquad Q_{SU(N)} = \frac{r}{N}, \; r \in \mathbb{N}$

r is arbitary, Q can be also an integer, including 1

deceptively simple!

field strength abelian
transition functions NOT

$$F_{12} = \begin{pmatrix} -\frac{2\pi\ell r}{NkL_1L_2} I_k & 0 \\ 0 & \frac{2\pi r}{NL_1L_2} I_\ell \end{pmatrix}, \quad F_{34} = \begin{pmatrix} -\frac{2\pi}{NL_3L_4} I_k & 0 \\ 0 & \frac{2\pi k}{N\ell L_3L_4} I_\ell \end{pmatrix}$$

$$P_\ell = \gamma_\ell \begin{bmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & & & \\ \dots & & 0 & 1 \\ 1 & 0 & \dots & 0 \end{bmatrix}, \quad Q_\ell = \gamma_\ell \, \mathrm{diag}\left[1, e^{\frac{i2\pi}{\ell}}, e^{2\frac{i2\pi}{\ell}}, \dots\right]$$

$$A_2 = -\omega\left(\frac{r x_1}{NkL_1L_2}\right), \quad A_4 = -\omega\left(\frac{x_3}{N\ell L_3L_4}\right), \quad A_1 = A_3 = 0\,;$$

$$P_\ell Q_\ell = e^{i\frac{2\pi}{\ell}} Q_\ell P_\ell.$$

$$\Omega_1 = \begin{bmatrix} P_k^{-r} e^{i2\pi\ell r \frac{x_2}{NkL_2}} & 0 \\ 0 & e^{-i2\pi r \frac{x_2}{NL_2}} I_\ell \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} Q_k & 0 \\ 0 & I_\ell \end{bmatrix},$$

$$\Omega_3 = \begin{bmatrix} e^{i2\pi \frac{x_4}{NL_4}} I_k & 0 \\ 0 & e^{-i2\pi k \frac{x_4}{N\ell L_4}} P_\ell \end{bmatrix}, \quad \Omega_4 = \begin{bmatrix} I_k & 0 \\ 0 & Q_\ell \end{bmatrix}.$$

$$\omega = 2\pi \mathrm{diag}\left[\underbrace{\ell, \ell, \dots, \ell}_{k\,\text{times}}, \underbrace{-k, -k, \dots, -k}_{\ell\,\text{times}}\right]$$

$$A(x + \hat{e}_\nu L_\nu) = \Omega_\mu(x)(A(x) - id)\Omega_\mu^{-1}(x)$$

SU(N) solutions we study depend on $k, r \in \mathbb{N}$; where $N = k + \ell$, $\ell \in \mathbb{N}$

$n_{12} = -r, \; n_{34} = 1$ $\qquad Q_{SU(N)} = \frac{r}{N}, \; r \in \mathbb{N}$

r is arbitary, Q can be also an integer, including 1

$$F_{12} = \begin{pmatrix} -\frac{2\pi\ell r}{NkL_1L_2} I_k & 0 \\ 0 & \frac{2\pi r}{NL_1L_2} I_\ell \end{pmatrix}, \quad F_{34} = \begin{pmatrix} -\frac{2\pi}{NL_3L_4} I_k & 0 \\ 0 & \frac{2\pi k}{N\ell L_3L_4} I_\ell \end{pmatrix}$$

deceptively simple!

field strength abelian
transition functions NOT

moduli?

$$A_2 = -\omega \left(\frac{r x_1}{NkL_1L_2} \right), \quad A_4 = -\omega \left(\frac{x_3}{N\ell L_3L_4} \right), \quad A_1 = A_3 = 0\,;$$

- constant connections must commute with Ω_μ

$$\Omega_1 = \begin{bmatrix} P_k^{-r} e^{i2\pi\ell r \frac{x_2}{NkL_2}} & 0 \\ 0 & e^{-i2\pi r \frac{x_2}{NL_2}} I_\ell \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} Q_k & 0 \\ 0 & I_\ell \end{bmatrix},$$

$$\Omega_3 = \begin{bmatrix} e^{i2\pi \frac{x_4}{NL_4}} I_k & 0 \\ 0 & e^{-i2\pi k \frac{x_4}{N\ell L_4}} P_\ell \end{bmatrix}, \quad \Omega_4 = \begin{bmatrix} I_k & 0 \\ 0 & Q_\ell \end{bmatrix}.$$

$$\omega = 2\pi \mathrm{diag}\left[\underbrace{\ell, \ell, \ldots, \ell}_{k\,\text{times}}, \underbrace{-k, -k, \ldots, -k}_{\ell\,\text{times}} \right]$$

$$A(x + \hat{e}_\nu L_\nu) = \Omega_\mu(x)(A(x) - id)\Omega_\mu^{-1}(x)$$

SU(N) solutions we study depend on $k, r \in \mathbb{N}$; where $N = k + \ell$, $\ell \in \mathbb{N}$

$n_{12} = -r, \;\; n_{34} = 1$ $\qquad Q_{SU(N)} = \frac{r}{N}, \; r \in \mathbb{N}$ $\qquad$ r is arbitary, Q can be also an integer, including 1

$$F_{12} = \begin{pmatrix} -\frac{2\pi\ell r}{NkL_1L_2} I_k & 0 \\ 0 & \frac{2\pi r}{NL_1L_2} I_\ell \end{pmatrix}, \quad F_{34} = \begin{pmatrix} -\frac{2\pi}{NL_3L_4} I_k & 0 \\ 0 & \frac{2\pi k}{N\ell L_3L_4} I_\ell \end{pmatrix}$$

$$A_2 = -\omega\left(\frac{r x_1}{NkL_1L_2}\right), \quad A_4 = -\omega\left(\frac{x_3}{N\ell L_3L_4}\right), \quad A_1 = A_3 = 0\,.$$

deceptively simple!

field strength abelian
transition functions NOT

moduli?

- constant connections must commute with Ω_μ
- $\exists 4\gcd(k, r)$ such moduli

Anber, EP 2307

explicitly, $\forall\, \mu$: $\quad \phi = \mathrm{diag}(\underbrace{a^1, a^2, \ldots, a^k}_{k}, \underbrace{a, a, \ldots, a}_{\ell})$

where $\quad a^c = a^{[c-r]_k}, \quad a^1 + a^2 + \ldots + a^k + \ell a = 0$

$[c - r]_k \equiv c - r \;(\mathrm{mod}\; \mathrm{k}),\; [\mathrm{k}]_\mathrm{k} \equiv \mathrm{k}$

SU(N) solutions we study depend on $k, r \in \mathbb{N}$; where $N = k + \ell$, $\ell \in \mathbb{N}$

$n_{12} = -r, \;\; n_{34} = 1$ $\qquad Q_{SU(N)} = \frac{r}{N}, \;\; r \in \mathbb{N},$ $\qquad r$ is arbitary, Q can be also an integer, including 1

$$F_{12} = \begin{pmatrix} -\frac{2\pi\ell r}{NkL_1L_2} I_k & 0 \\ 0 & \frac{2\pi r}{NL_1L_2} I_\ell \end{pmatrix}, \quad F_{34} = \begin{pmatrix} -\frac{2\pi}{NL_3L_4} I_k & 0 \\ 0 & \frac{2\pi k}{N\ell L_3L_4} I_\ell \end{pmatrix}$$

$$A_2 = -\omega \left(\frac{r x_1}{NkL_1L_2} \right), \quad A_4 = -\omega \left(\frac{x_3}{N\ell L_3L_4} \right), \quad A_1 = A_3 = 0\,;$$

deceptively simple!

field strength abelian
transition functions NOT

moduli?

- constant connections must commute with Ω_μ
- $\exists 4\gcd(k, r)$ such moduli

Anber, EP 2307

"missing moduli":
unless $\gcd(k, r) = r$, not enough to satisfy index theorem

Schwarz; E. Weinberg; Taubes '77-'82 $\qquad$ $\dim(\text{moduli}) = 4NQ = 4r$

$\gcd(k, r) = r$ is only possible for $r < N$, hence, e.g. all $Q \geq 1$ have this issue for gcd$(k, r) = r$: flat connections are all the moduli; described moduli space globally

Anber, EP 2307, 2408

used integral over this moduli space to calculate r-point ($r < N$) gaugino condensate, agrees with *Dorey et al* ADHM calculation on $\mathbb{R}^4$

<u>for gcd$(k, r) = r$:</u> flat connections are all the moduli; described moduli space globally

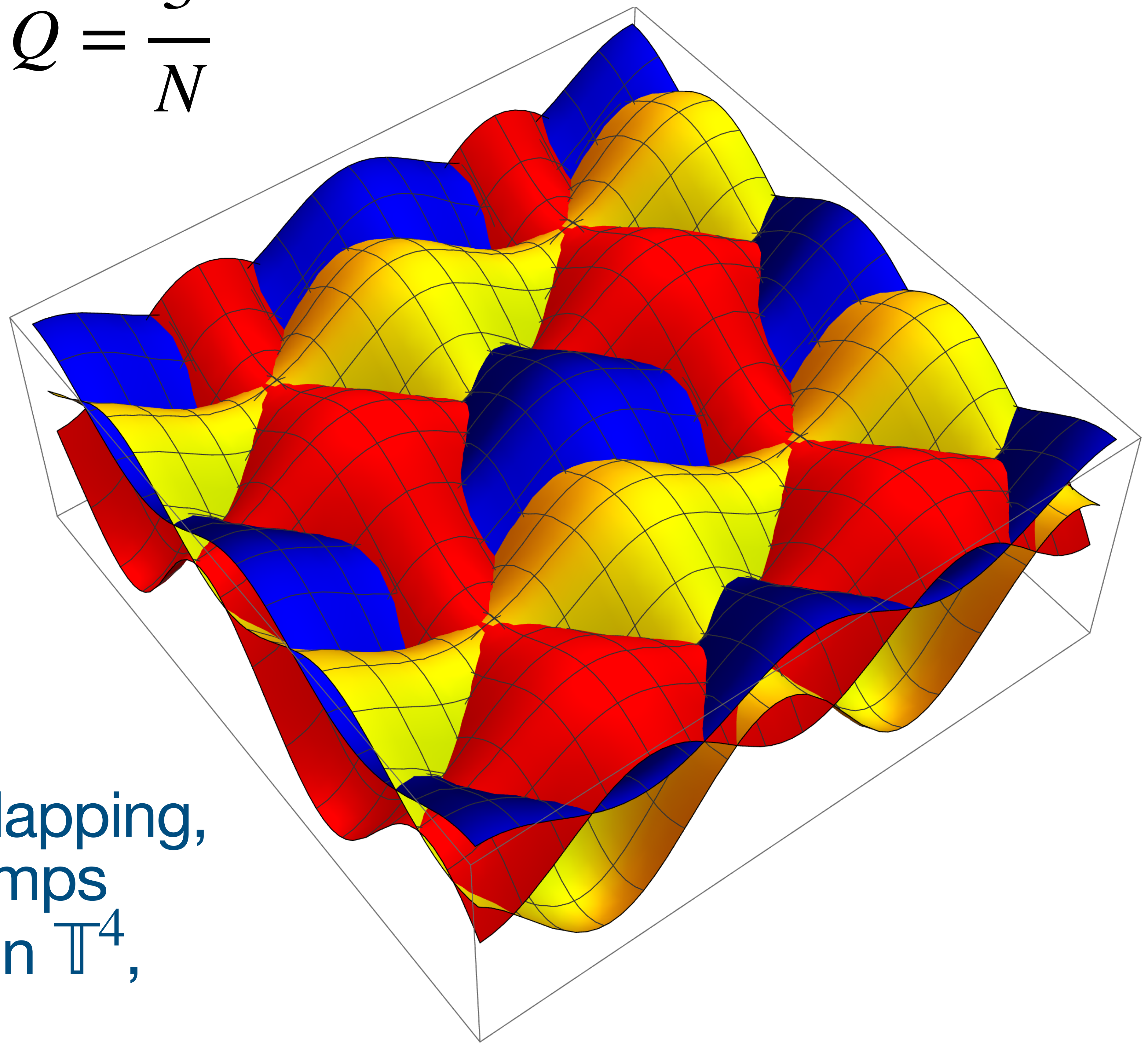
Anber, EP 2307, 2408

used integral over this moduli space to calculate r-point ($r < N$) gaugino condensate, agrees with *Dorey et al* ADHM calculation on $\mathbb{R}^4$

... showing this for visualization/advertisement ...

picture from 2307.04795,
Mohamed Anber (Durham) & EP

$Q = \frac{3}{N}$



this is a Q=3/N example, F_{12} shown in one 2-plane (double size shown)
obviously not constant solution, but an approximate one, related to it analytically via the "Δ-expansion" (*González-Arroyo et al, 2000*)

$$\Delta = \frac{r\ell L_3 L_4 - k L_1 L_2}{\sqrt{L_1 L_2 L_3 L_4}}$$

$$\Delta = 0 \implies \frac{L_1 L_2}{L_3 L_4} = \frac{\ell r}{k}$$

F_{12} = sum of 3 overlapping, instanton-liquid-like lumps
each has 4 positions on $\mathbb{T}^4$,
so $4r = 12$ moduli

for $\gcd(k, r) = r$: flat connections are all the moduli; described moduli space globally

Anber, EP 2307, 2408

used integral over this moduli space to calculate r-point ($r < N$) gaugino condensate, agrees with *Dorey et al* ADHM calculation on $\mathbb{R}^4$

for $\gcd(k, r) \neq r$: which is the case for all $Q \geq 1$ and some $\frac{1}{N} < Q < 1$

need to account for $4r - 4\gcd(k, r)$ "missing" moduli

in *Anber, Cox, EP 2504* we understood how these $4r - 4\gcd(k, r)$ moduli appear, using a QFT analysis of the self-dual small perturbations

(very technical story…)

main moral:

when the "missing" moduli are turned on, solutions become non-constant, i.e. constant-F are measure zero!

- our results from 2504 much easier to obtain via D-branes, hence I now describe this
- my hope, as I discuss at the end, is that via D-branes, one can go further…

end of: 2. QFT: constant-F solutions and "missing moduli"

3. D-branes and the "missing moduli"

- our results from 2504 much easier to obtain via D-branes, hence I now describe this

  ...D-branes in type II are the natural place for self-dual configurations: ADHM via branes

Witten; Douglas '90s

steps:

1. embed the constant-F $SU(N)$ solutions (N, k, r) into $U(N)$ bundle on $\mathbb{T}^4$ ... QFT

2. constant-$\mathscr{F}$ $U(N)$ flux -> worldvolume of N D_{p+4} wrapped on $\mathbb{T}^4$ (8 supercharges)

   T-duality in two (choice of transition functions) $\mathbb{T}^4$ directions removes flux, turns into stacks of D_{p+2} wrapped on intersecting 2-cycles on the dual $\tilde{\mathbb{T}}^4$, preserving 8 supercharges

3. moduli space locally = Higgs branch supersymmetric vacua of $p+1$-dim 8 supercharge worldvolume theory

-> exactly our results from 2504, essentially with no calculation (except of intricate brane wrappings)

- *my hope, as I discuss at the end, is that via D-branes, one can go further… here I need help!*

1. embed the constant-F $SU(N)$ solutions (N, k, r) into $U(N)$ bundle on $\mathbb{T}^4$ … QFT

$$\mathcal{A}_\mu = A_\mu + \mathbf{1}_N \, a_\mu$$

($U(N) \to \mathcal{A}_\mu$, $SU(N) \to A_\mu$, $U(1) \to a_\mu$)

$$\Omega \in SU(N): \; \Omega_\mu(x+\hat{e}_\nu L_\nu)\Omega_\nu(x) = e^{i\frac{2\pi n_{\mu\nu}}{N}}\Omega_\nu(x+\hat{e}_\mu L_\mu)\Omega_\mu(x),$$

$$\omega \in U(1): \; \omega_\mu(x+\hat{e}_\nu L_\nu)\omega_\nu(x) = e^{-i\frac{2\pi n_{\mu\nu}}{N}}\omega_\nu(x+\hat{e}_\mu L_\mu)\omega_\mu(x)$$

add (minimal) fractional U(1) fluxes $f = da$ in 12 and 34 planes: $n_{12} = -r, n_{34} = 1$

$$\int_{\mathbb{T}^4} \mathrm{ch}_2(\mathcal{F}) = \frac{1}{8\pi^2}\int_{\mathbb{T}_4} \mathrm{tr}\mathcal{F}\wedge\mathcal{F} = \frac{1}{8\pi^2}\int_{\mathbb{T}_4} \mathrm{tr}F\wedge F + \frac{N}{8\pi^2}\int_{\mathbb{T}^4} f\wedge f$$

$$Q_{SU(N)} = \frac{r}{N}, \;\; r \in \mathbb{N}$$

$$U(N) \quad SU(N) \quad U(1)$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$\mathcal{A}_\mu = A_\mu + \mathbf{1}_N \, a_\mu$$

$$\Omega \in SU(N): \; \Omega_\mu(x+\hat{e}_\nu L_\nu)\Omega_\nu(x) = e^{i\frac{2\pi n_{\mu\nu}}{N}}\Omega_\nu(x+\hat{e}_\mu L_\mu)\Omega_\mu(x)\,,$$
$$\omega \in U(1): \; \omega_\mu(x+\hat{e}_\nu L_\nu)\omega_\nu(x) = e^{-i\frac{2\pi n_{\mu\nu}}{N}}\omega_\nu(x+\hat{e}_\mu L_\mu)\omega_\mu(x)$$

add (minimal) fractional U(1) fluxes $f = da$
in 12 and 34 planes: $n_{12} = -r, n_{34} = 1$

$$\int_{\mathbb{T}^4} \mathrm{ch}_2(\mathcal{F}) = \frac{1}{8\pi^2}\int_{\mathbb{T}_4} \mathrm{tr}\mathcal{F}\wedge\mathcal{F} = \frac{1}{8\pi^2}\int_{\mathbb{T}_4} \mathrm{tr}F\wedge F + \frac{N}{8\pi^2}\int_{\mathbb{T}^4} f\wedge f$$ =0, for $\mathcal{F}$ below

$$Q_{SU(N)} = \frac{r}{N}, \; r\in\mathbb{N}$$

$$\int_{\mathbb{T}^2_{(x_1,x_2)}} \frac{\mathrm{ch}_1(\mathcal{F})}{N} = -\frac{r}{N}$$
$$\int_{\mathbb{T}^2_{(x_3,x_4)}} \frac{\mathrm{ch}_1(\mathcal{F})}{N} = \frac{1}{N}$$

$SU(N)$:
$$F_{12} = \begin{pmatrix} -\frac{2\pi\ell r}{NkL_1L_2}I_k & 0 \\ 0 & \frac{2\pi r}{NL_1L_2}I_\ell \end{pmatrix}, \quad F_{34} = \begin{pmatrix} -\frac{2\pi}{NL_3L_4}I_k & 0 \\ 0 & \frac{2\pi k}{N\ell L_3L_4}I_\ell \end{pmatrix}$$

$U(N)$:
$$\mathcal{F}_{12} = \begin{pmatrix} -\frac{2\pi r}{kL_1L_2}I_k & 0 \\ 0 & 0\times I_\ell \end{pmatrix}, \quad \mathcal{F}_{34} = \begin{pmatrix} 0\times I_k & 0 \\ 0 & \frac{2\pi}{\ell L_3L_4}I_\ell \end{pmatrix}$$

$$F = \mathcal{F} - \frac{I_N}{N}\mathrm{tr}\,\mathcal{F}$$

$U(N)$:

$$\mathcal{F}_{12} = \begin{pmatrix} -\frac{2\pi r}{kL_1L_2} I_k & 0 \\ 0 & 0 \times I_\ell \end{pmatrix}, \quad \mathcal{F}_{34} = \begin{pmatrix} 0 \times I_k & 0 \\ 0 & \frac{2\pi}{\ell L_3 L_4} I_\ell \end{pmatrix}.$$

2. constant-$\mathscr{F}$ $U(N)$ flux -> worldvolume of $N\, D_{p+4}$ wrapped on $\mathbb{T}^4$ (8 supercharges)

T-duality in two (choice of transition functions) $\mathbb{T}^4$ directions removes flux, turns into stacks of D_{p+2} wrapped on intersecting 2-cycles on the dual $\tilde{\mathbb{T}}^4$, preserve 8 supercharges

$U(N)$:

$$\mathcal{F}_{12} = \begin{pmatrix} -\frac{2\pi r}{k L_1 L_2} I_k & 0 \\ 0 & 0 \times I_\ell \end{pmatrix}, \quad \mathcal{F}_{34} = \begin{pmatrix} 0 \times I_k & 0 \\ 0 & \frac{2\pi}{\ell L_3 L_4} I_\ell \end{pmatrix}.$$

T-duality in two (choice of transition functions) $\mathbb{T}^4$ directions removes flux, turns into stacks of D_{p+2} wrapped on intersecting 2-cycles on the dual $\tilde{\mathbb{T}}^4$, preserve 8 supercharges

skip details of: choice of gauge for $\Sigma_\mu = \omega_\mu \Omega_\mu$ (make $\Sigma_2 = \Sigma_4 = 1$)

T-duality in x_2, x_4, the ones with trivial Σ_μ: D_{p+4} on $\mathbb{T}^4 \to D_{p+2}$ on $\tilde{\mathbb{T}}^4$

BPS condition for intersecting stacks of D_{p+2} on $\tilde{\mathbb{T}}^4$

$\Sigma_1, \Sigma_3 \neq 1\ldots$ make for intricate reconnections of D_{p+2} on $\tilde{\mathbb{T}}^4$

result: find two D_{p+2} stacks wrapped on 2-cycles on $\tilde{\mathbb{T}}^4$ (y_1, y_2, y_3, y_4)

each 2-cycle projects onto 1-d line in $\tilde{\mathbb{T}}^2_{12}$ and 1-d line in $\tilde{\mathbb{T}}^2_{34}$

$g \equiv \gcd(k, r)$ plays again an important role

$$w_{\mu\nu} = \int_{\mathbb{T}^2} d\sigma d\tau \left(\frac{\partial t_\mu}{\partial \sigma}\frac{\partial t_\nu}{\partial \tau} - \frac{\partial t_\nu}{\partial \sigma}\frac{\partial t_\mu}{\partial \tau} \right)$$

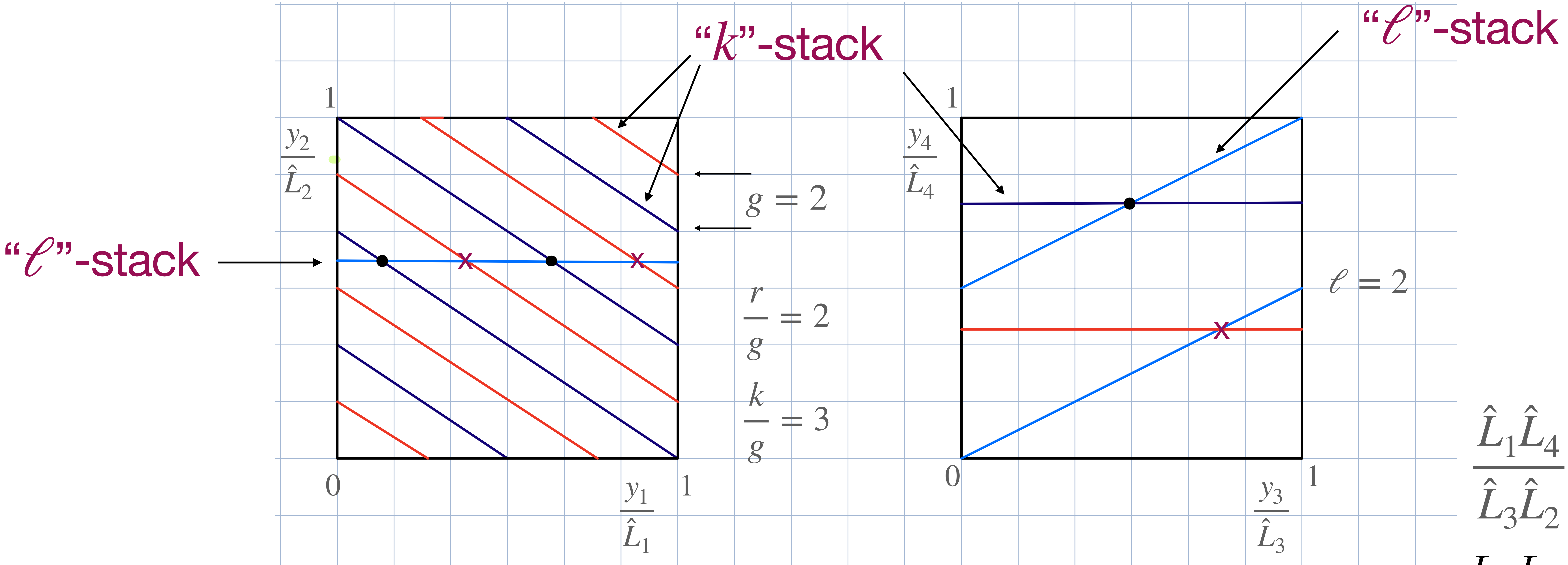
describe windings as 1dim lines in $\tilde{\mathbb{T}}^2_{12}, \tilde{\mathbb{T}}^2_{34}$ or as $\mathbb{T}^2_{\text{worldvolume}\ (\sigma,\tau)} \to \mathbb{T}^4_{(t_\mu)}$ (RR charges)

$g \equiv \gcd(k, r)$, $N = k + \ell$, $Q_{SU(N)} = r/N$

"k"-stack has g parallel branes: wrap $\frac{k}{g}$ times in y_1, $\frac{r}{g}$ times in y_2, once in y_3, at fixed y_4

"ℓ"-stack of a single brane: wraps once in y_1, at fixed y_2, ℓ times in y_3, once in y_4

each of the g branes of the "k"-stack has $\frac{r}{g}$ intersections with the "ℓ"-stack brane



$$\frac{\hat{L}_1 \hat{L}_4}{\hat{L}_3 \hat{L}_2} =$$

$k = 6$, $\ell = 2$, $r = 4$, $g = 2 \implies Q = 1/2\ SU(8)$

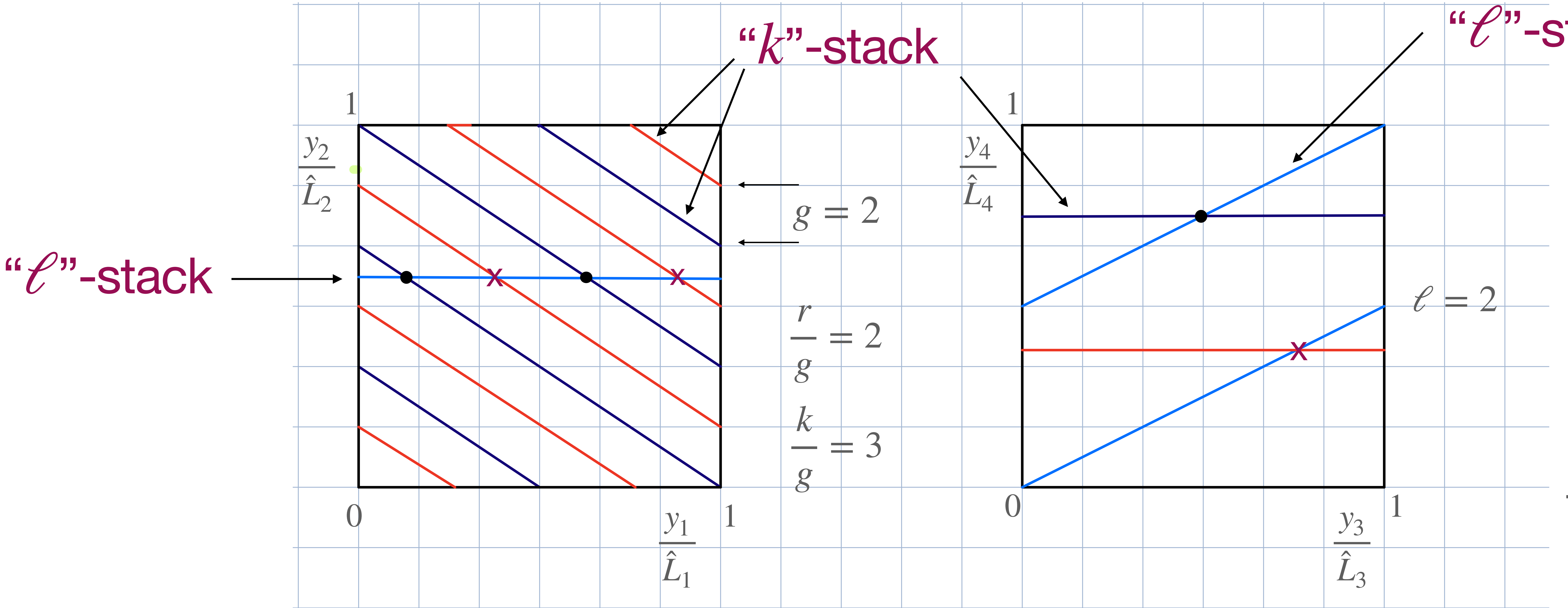
$$\frac{L_1 L_2}{L_3 L_4} = \frac{\ell r}{k}$$

RR charges, from $w_{\mu\nu}$, just the right ones: $N\ D_{p+4}$ branes + correct $U(1)$ fluxes

$$n_{12} = -r \qquad n_{34} = 1$$

$$\tilde{Q}^{RR}_{13} = \underbrace{k/g}_{w^{(k)}_{13}} \times g + \underbrace{\ell}_{w^{(\ell)}_{13}} \times 1 = k + \ell = N$$

$$\tilde{Q}^{RR}_{14} = -\underbrace{0}_{w^{(k)}_{14}} \times g + \underbrace{1}_{w^{(\ell)}_{14}} \times 1 = 1 \qquad \tilde{Q}^{RR}_{23} = -\underbrace{r/g}_{w^{(k)}_{23}} \times g + \underbrace{0}_{w^{(\ell)}_{23}} \times 1 = -r$$



$$\frac{\hat{L}_1 \hat{L}_4}{\hat{L}_3 \hat{L}_2} =$$

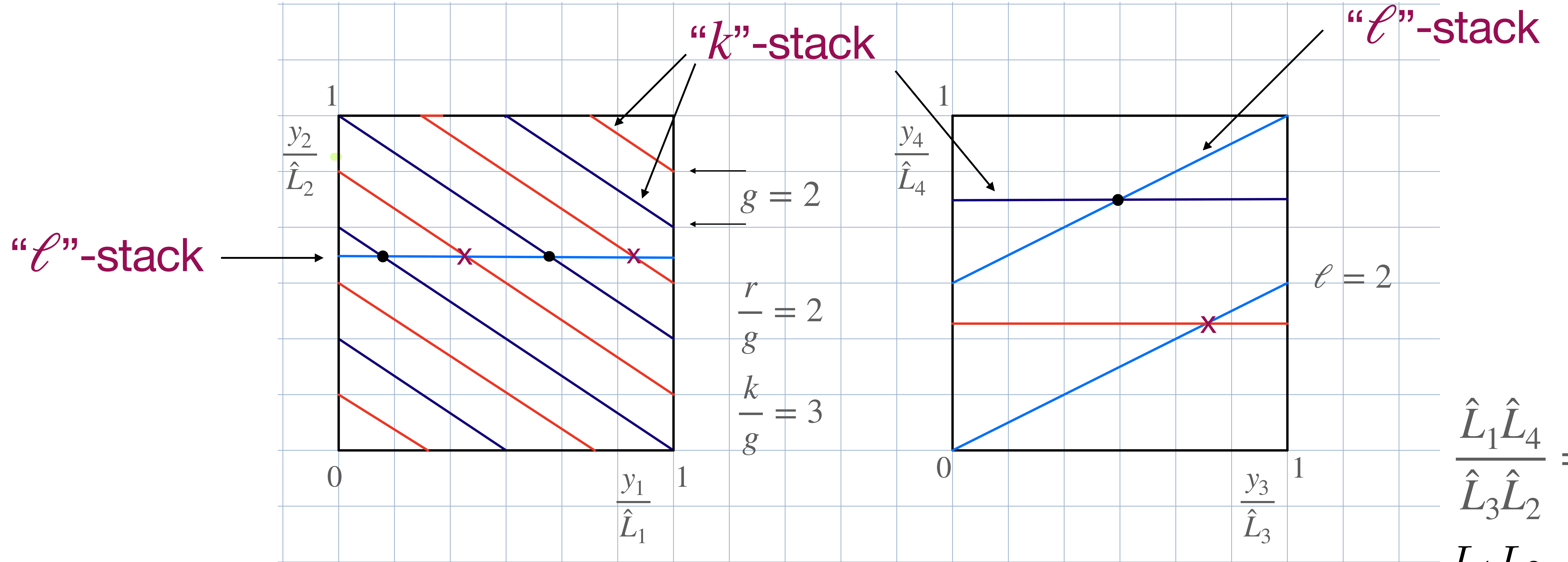
$$\frac{L_1 L_2}{L_3 L_4} = \frac{\ell r}{k}$$

$k = 6,\ \ell = 2,\ r = 4,\ g = 2 \implies Q = 1/2\ SU(8)$

each D_{p+2} has two positions + two worldvolume Wilson lines on $\tilde{\mathbb{T}}^4$, 4 moduli

there are $g+1$ branes in the two stacks, so $4g+4$ moduli: $4g$ of $SU(N)$ + 4 of $U(1)$

+ massless hyper (1,-1) under $U(1)\times U(1)$ lives at each intersection



$$\frac{\hat{L}_1\hat{L}_4}{\hat{L}_3\hat{L}_2}=$$

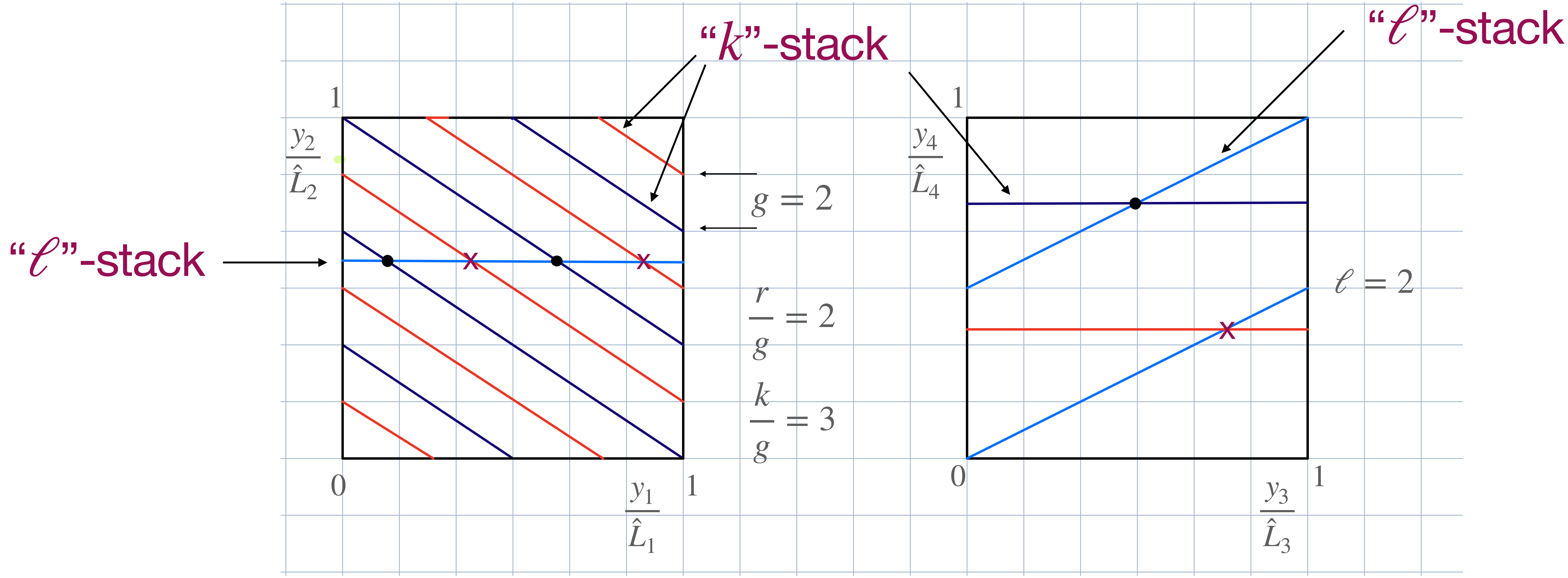
$$k=6,\ \ell=2,\ r=4,\ g=2 \implies Q=1/2\ SU(8)$$

$$\frac{L_1L_2}{L_3L_4}=\frac{\ell r}{k}$$

3. to study moduli space locally - Higgs branch supersymmetric vacua of $p+1$-dim 8 supercharge worldvolume theory

ignoring $\tilde{\mathbb{T}}^4$-variations [discuss later] study $p+1=3+1$-dim $\mathcal{N}=2$ theory vacua

$U(1)_1 \times U(1)_2 \times \ldots \times U(1)_g \times U(1)_\ell$ $\quad D_5 : \; y_0, \underbrace{y_1, y_2, y_3, y_4}_{2 \text{ cycle } \in \tilde{\mathbb{T}}^4}, y_5, y_6, y_7$

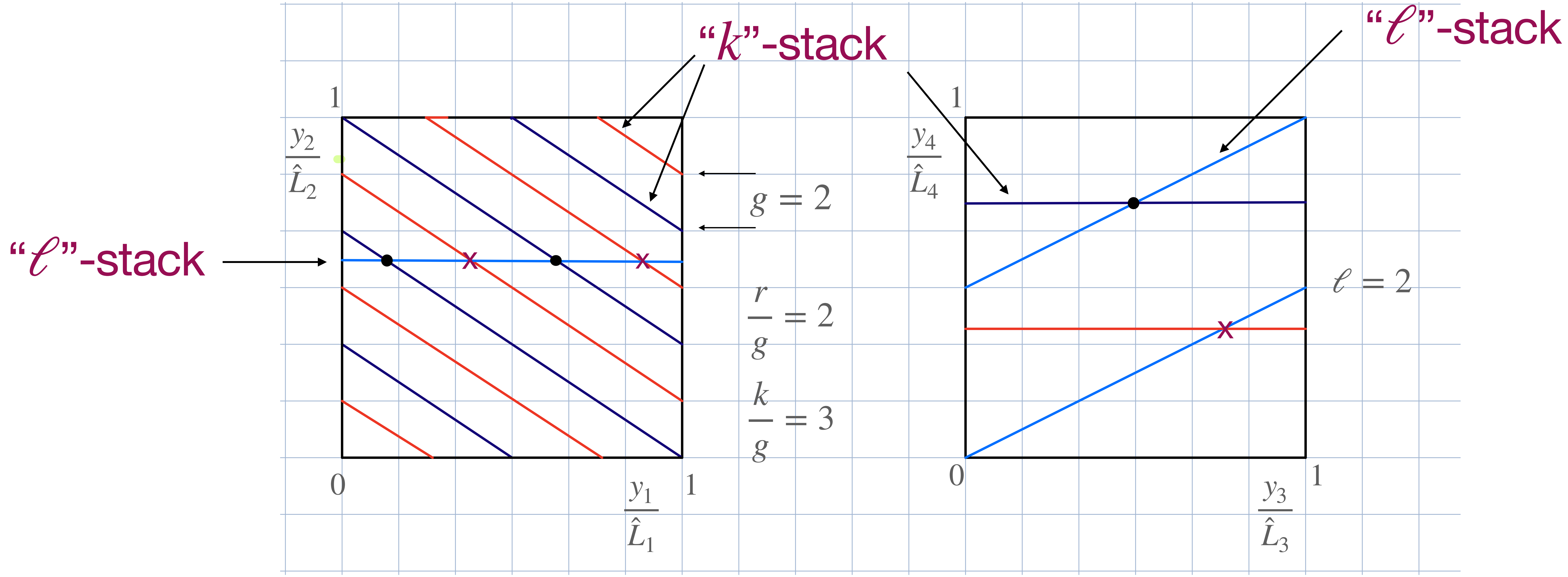


Coulomb branch ϕ_3 ϕ_1, ϕ_2

in $\mathcal{N} = 1$ terms, each $U(1)$ has adjoint chiral: D_5 position in y_8, y_9 + 2 adjoint chiral: D_5 position/Wilson line in $\tilde{\mathbb{T}}^4$

ignoring $\tilde{\mathbb{T}}^4$-variations [discuss later] study $p + 1 = 3 + 1$-dim $\mathcal{N} = 2$ theory vacua

$U(1)_1 \times U(1)_2 \times \ldots \times U(1)_g \times U(1)_\ell$ $D_5: \; y_0, \underbrace{y_1, y_2, y_3, y_4}_{2 \text{ cycle } \in \tilde{\mathbb{T}}^4}, y_5, y_6, y_7$

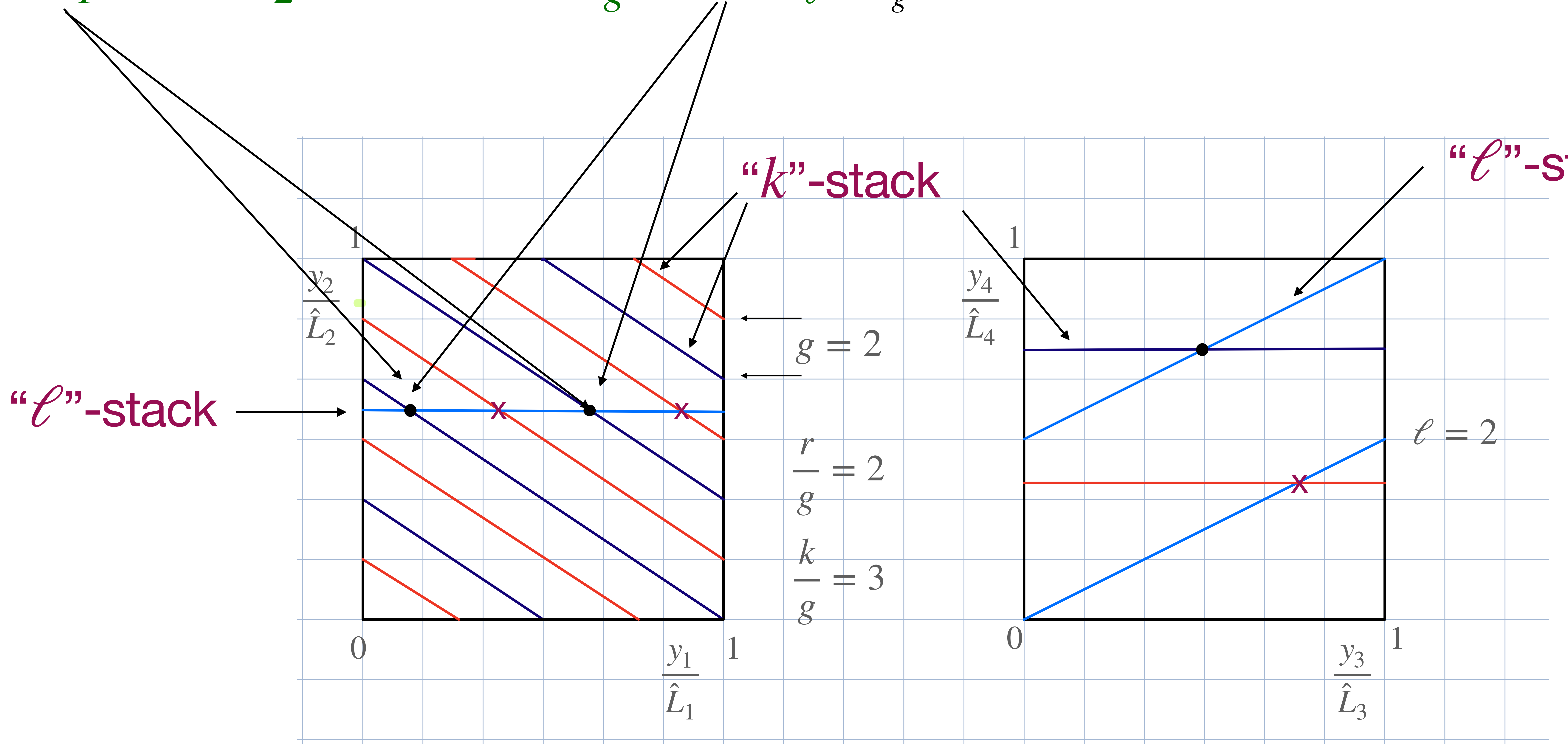


Coulomb branch ϕ_3 ϕ_1, ϕ_2

in $\mathcal{N} = 1$ terms, each $U(1)$ has adjoint chiral: D_5 position in y_8, y_9 + 2 adjoint chiral: D_5 position/Wilson line in $\tilde{\mathbb{T}}^4$

ignoring $\tilde{\mathbb{T}}^4$-variations [discuss later] study $p + 1 = 3 + 1$-dim $\mathcal{N} = 2$ theory vacua

$U(1)_1 \times U(1)_2 \times \ldots \times U(1)_g \times U(1)_\ell$ $\frac{r}{g}$ hypers (2 chiral $\mathcal{N} = 1$) under each $U(1)_i \times U(1)_\ell$: $q, \tilde{q}$

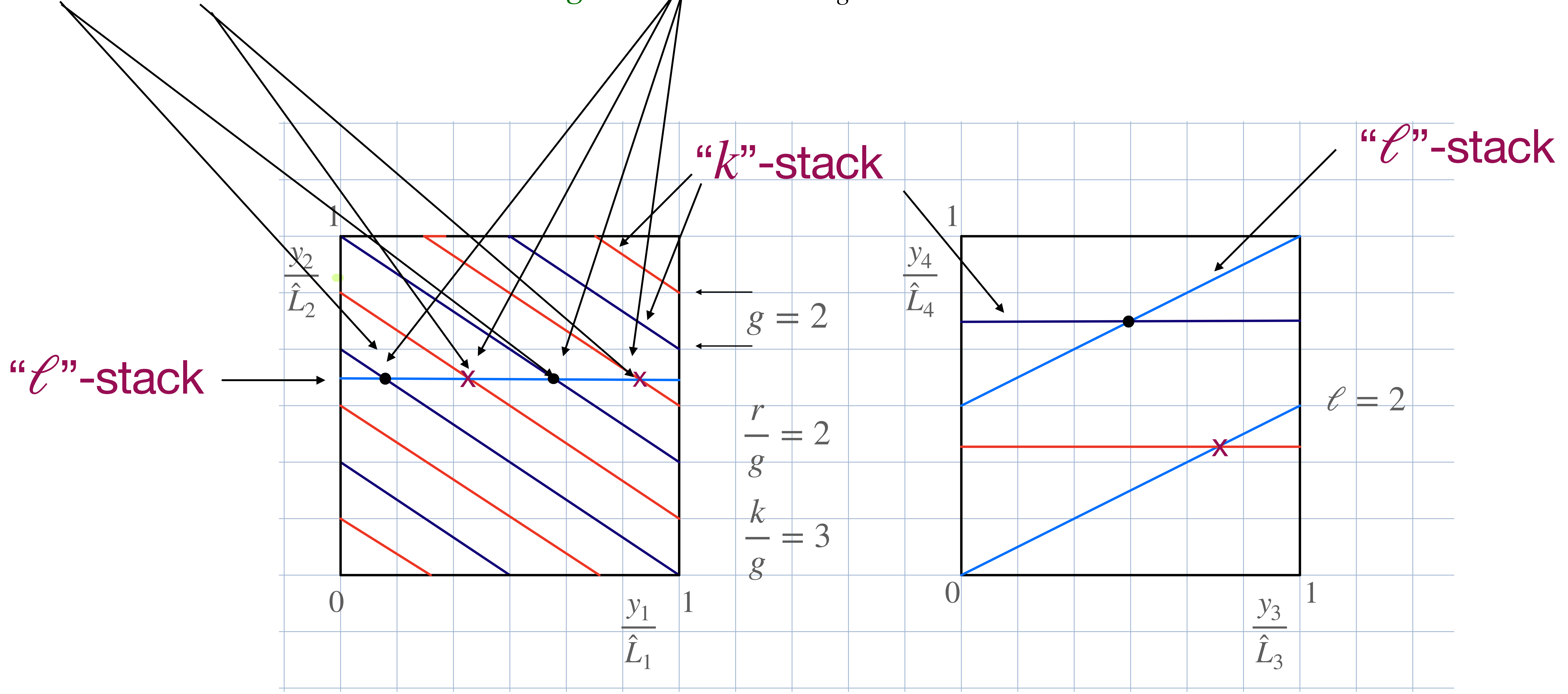


Coulomb branch ϕ_3 ϕ_1, ϕ_2

in $\mathcal{N} = 1$ terms, each $U(1)$ has adjoint chiral: D_5 position in y_8, y_9 + 2 adjoint chiral: D_5 position/Wilson line in $\tilde{\mathbb{T}}^4$

ignoring $\tilde{\mathbb{T}}^4$-variations [discuss later] study $p + 1 = 3 + 1$-dim $\mathcal{N} = 2$ theory vacua

$U(1)_1 \times U(1)_2 \times \ldots \times U(1)_g \times U(1)_\ell$ $\frac{r}{g}$ hypers (2 chiral $\mathcal{N} = 1$) under each $U(1)_i \times U(1)_\ell$: $q, \tilde{q}$



Coulomb branch ϕ_3 ϕ_1, ϕ_2

in $\mathcal{N} = 1$ terms, each $U(1)$ has adjoint chiral: D_5 position in y_8, y_9 + 2 adjoint chiral: D_5 position/Wilson line in $\tilde{\mathbb{T}}^4$

ignoring $\tilde{\mathbb{T}}^4$-variations [discuss later] study $p + 1 = 3 + 1$-dim $\mathcal{N} = 2$ theory vacua

$\frac{r}{g}$ hypers (2 chiral $\mathcal{N} = 1$) under each $U(1)_i \times U(1)_\ell$: $q, \tilde{q}$

field	$U(1)_\ell$ charge	$U(1)_1$ charge	...	$U(1)_i$ charge	...	$U(1)_g$ charge	multiplicity
q_1^a	1	-1					$a = 1, ..., r/g$
$\tilde{q}_a^1$	-1	1					$a = 1, ..., r/g$
$\dots$							$a = 1, ..., r/g$
q_i^a	1			-1			$a = 1, ..., r/g$
$\tilde{q}_a^i$	-1			1			$a = 1, ..., r/g$
$\dots$							$a = 1, ..., r/g$
q_g^a	1					-1	$a = 1, ..., r/g$
$\tilde{q}_a^g$	-1					1	$a = 1, ..., r/g$
ϕ_A							$A = 1, 2, 3$
ϕ_A^i							$A = 1, 2, 3; i = 1, ..., g$

bifundamental hypers /2 chiral/ $q_i^a, \tilde{q}_a^i$

A=1,2 adjoint hyper

A=3 adjoint vector Coulomb branch

$U(1)_1 \times U(1)_2 \times \ldots \times U(1)_g \times U(1)_\ell$

$$W = \sum_{i=1}^{g} \phi_{3i} \sum_{a=1}^{\frac{r}{g}} \tilde{q}^i_a q^a_i + \phi_3 \sum_{i=1}^{g} \sum_{a=1}^{\frac{r}{g}} \tilde{q}^i_a q^a_i$$

=0 on Higgs branch

$$\overline{(\phi_3 + \phi_{3i})} \tilde{q}^a_i = 0, \quad \overline{(\phi_3 + \phi_{3i})} q^a_i = 0$$

$$\sum_{a=1}^{\frac{r}{g}} \tilde{q}^i_a q^a_i = 0, \; i = 1, \ldots g$$ g complex constraints

$$\sum_{a=1}^{\frac{r}{g}} |\tilde{q}^i_a|^2 - |q^a_i|^2 = 0$$ g real constraints

F- and D-term conditions, Higgs branch

$U(1)$ holonomies

$$\text{dim of Higgs branch} = \underbrace{4g+4}_{\text{adjoint hypers}} + \underbrace{4r}_{\text{bifundamental hypers}} - \underbrace{4g}_{D-, F-, \text{gauge constraints}} = 4r + 4$$

geometric moduli ϕ_1, ϕ_2 (→ adjoint hypers)

missing moduli (→ bifundamental hypers)

$4NQ = 4r$, **as per index theorem**

identical equations and counting (minus the $4\ U(1)$ holonomies) obtained in *Anber, Cox, EP 2504* paper

in local analysis of self-dual perturbations of constant-F instantons

... using the tens of pages of Appendices of *Anber, EP 2307*

... vs virtually no calculation here!

$$\sum_{a=1}^{\frac{r}{g}} \tilde{q}_a^i q_i^a = 0, \;\; i = 1, ...g \qquad g \text{ complex constraints}$$

$$\sum_{a=1}^{\frac{r}{g}} |\tilde{q}_a^i|^2 - |q_i^a|^2 = 0 \qquad g \text{ real constraints}$$

F- and D-term conditions, Higgs branch

$U(1)$ holonomies

$$\text{dim of Higgs branch} = \underbrace{4g+4}_{\text{adjoint hypers}} + \underbrace{4r}_{\text{bifundamental hypers}} - \underbrace{4g}_{D-,F-,\text{gauge constraints}} = 4r+4$$

geometric moduli ϕ_1, ϕ_2

missing moduli

$4NQ = 4r$, **as per index theorem**

for $r = g$ - a single intersection only - $q = \tilde{q} = 0$, only geometric moduli identical equations and counting (minus the $4\ U(1)$ holonomies) obtained in *Anber, Cox, EP 2504* paper

in local analysis of self-dual perturbations of constant-F instantons

... using the tens of pages of Appendices of *Anber, EP 2307*

... vs virtually no calculation here!

$$\sum_{a=1}^{\frac{r}{g}} \tilde{q}^i_a q^a_i = 0, \ \ i = 1, ...g \qquad g \text{ complex constraints}$$

$$\sum_{a=1}^{\frac{r}{g}} |\tilde{q}^i_a|^2 - |q^a_i|^2 = 0 \qquad g \text{ real constraints}$$

F- and D-term conditions, Higgs branch

$U(1)$ holonomies

$$\text{dim of Higgs branch} = \underbrace{4g+4}_{\text{adjoint hypers}} + \underbrace{4r}_{\text{bifundamental hypers}} - \underbrace{4g}_{D-,F-,\text{gauge constraints}} = 4r + 4$$

geometric moduli ϕ_1, ϕ_2

missing moduli

$4NQ = 4r$, **as per index theorem**

While this makes me happy, I would like to learn more... where does this leave us, re. $Q = \frac{r}{N}$ on $\mathbb{T}^4$?

4. Summary and wishlist

- ***I told you that $Q = \frac{r}{N}$ instantons on $\mathbb{T}^4$ are responsible, in various calculable settings, for confinement, chiral symmetry breaking: so physically of interest.***

- ***I considered the embedding of constant-F, $Q = \frac{r}{N}, \forall r \in \mathbb{N}$, BPS instantons in string theory and used this picture for a local study of the moduli space.***

- ***The results are satisfying … agree with index theorem, QFT study, easy to obtain…***

- ***… but ignore $\tilde{\mathbb{T}}^4$ variations where the most interesting properties are hiding!***

"ignoring $\tilde{\mathbb{T}}^4$-variations [discuss ~~later~~ now] study $p + 1 = 3 + 1$-dim $\mathcal{N} = 2$ theory vacua"

4. wishlist *… where help is needed*

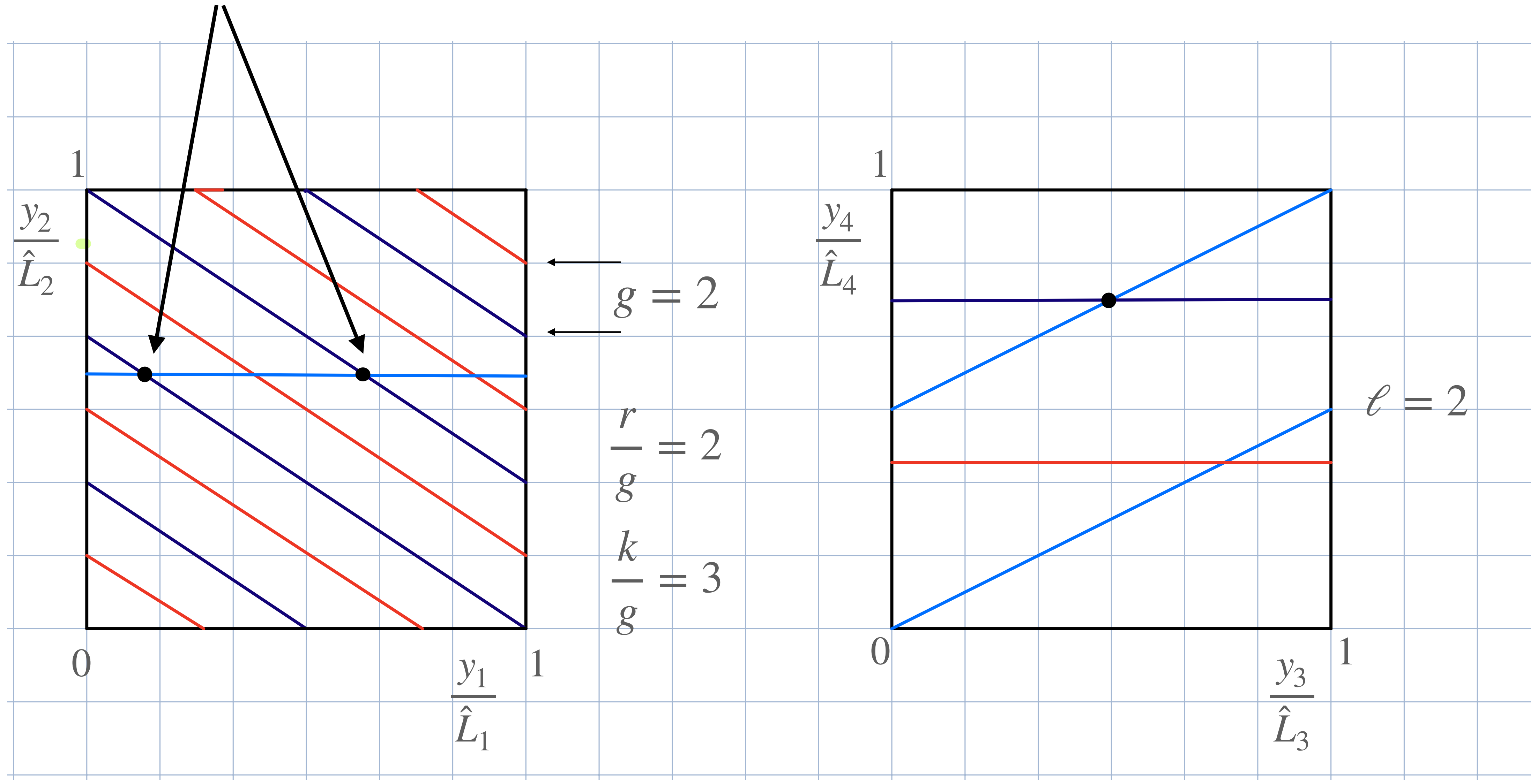
- ***Giving the bifundamental hypers ("missing moduli") vevs makes solutions space-time dependent. Not much is known of these, apart from very limited numerics!*** - *Anber, Cox, EP 2504*

For $\mathbb{R}^4$ instantons of charge-$p \in \mathbb{Z}$, D_5/D_9 (say) the ADHM moduli space is also a Higgs branch; ADHM background "lives" on a probe D_1, can be found explicitly.

On $\mathbb{T}^4$ both geometric and bifundamental hyper moduli are expected to be compact… can one study via branes? **Global structure of moduli space?**

It is natural to expect that $Q = r/N$ with $r = pN$ reproduces ADHM in infinite volume limit, for fixed overall size. Details? There may be some interesting math…

bifundamental hypers: vevs?



I also told you early on that "monopole-instantons" on $\mathbb{R}^3 \times \mathbb{S}^1$, "center-vortices" on $\mathbb{R}^2 \times \mathbb{T}^2$ and "fractional instantons" on $\mathbb{R} \times \mathbb{T}^3$ are all related (numerics/analytics). Can this be seen/better understood using the brane picture?

Finally, detuning the $\mathbb{T}^4$ shape away from the BPS limit leads to an open string tachyon at intersections… end point of tachyon condensation vs. Δ-expansion in QFT?

Thank you!