

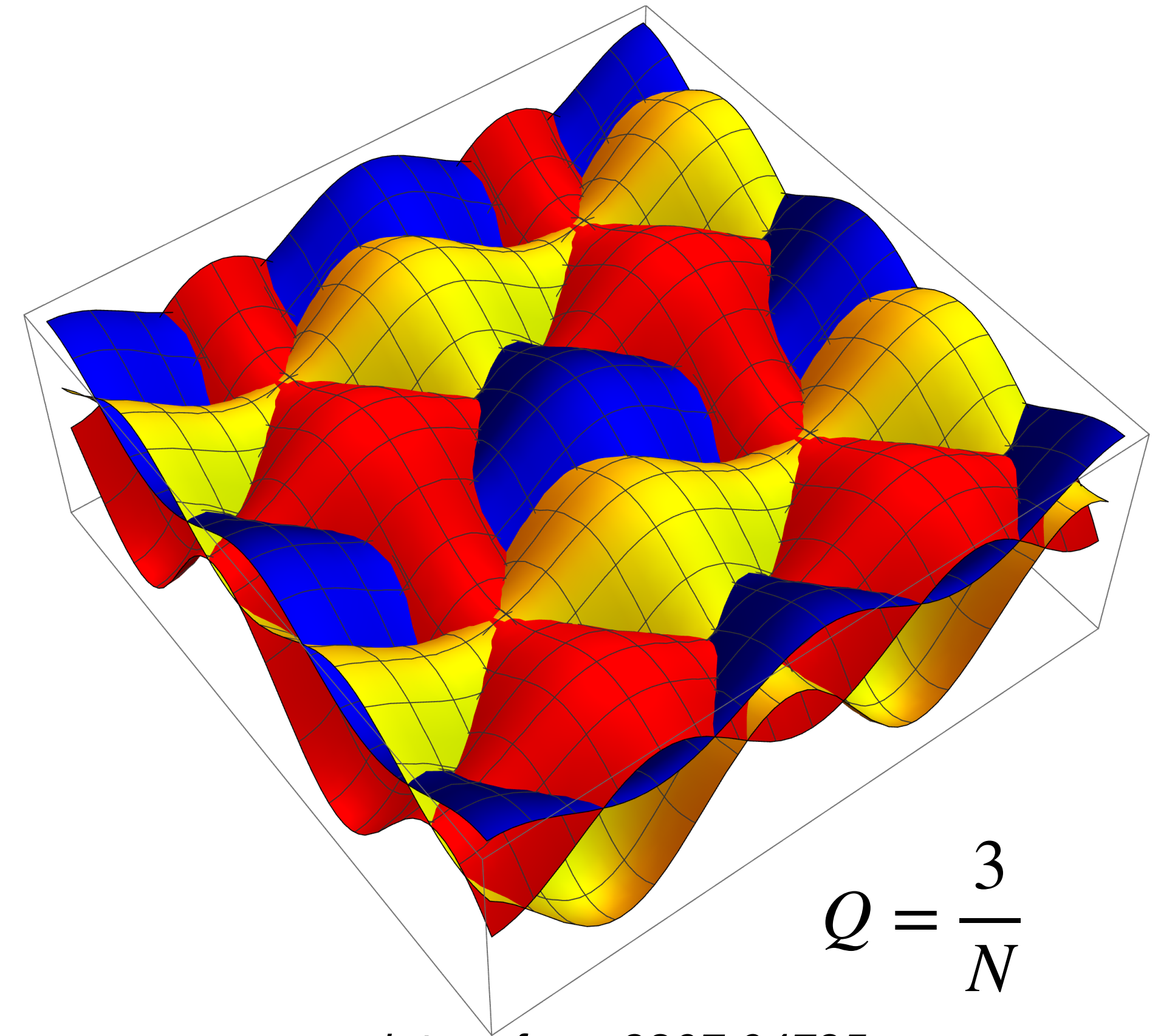
D-branes and the moduli space of fractional instantons on the four-torus

Erich Poppitz

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2604.21980

+ many other works of many authors



*picture from 2307.04795,
Mohamed Anber (Durham) & EP*

fractional instantons on the twisted four-torus \rightarrow

study instantons, self-dual $SU(N)$ gauge field configurations

on $\mathbb{T}_{n_{12}, n_{34}}^4$, the “twisted four-torus,” $Q_{top.} \equiv -\frac{n_{12}n_{34}}{N} \pmod{1}$

't Hooft; van Baal; early 1980s

interesting from physics point of view (gauge theory dynamics: confinement...)

+ interesting math problem (moduli spaces)

+ relations to D-branes (recall ADHM/branes)

minimal background: $T^a = (T^a)^\dagger, \text{tr } T^a = 0, N \times N, SU(N)$

$$x \equiv x + \hat{e}_\mu L_\mu, \forall \mu = 1, 2, 3, 4; x \in \mathbb{T}^4$$

SU(N) connection $A = A_\mu dx^\mu, A_\mu(x) = \sum_{a=1}^{N^2-1} A_\mu^a(x) T^a$

field strength $F = dA + i A \wedge A$

self dual $F = * F$

transition functions

$$A(x + \hat{e}_\nu L_\nu) = \Omega_\mu(x)(A(x) - id)\Omega_\mu^{-1}(x)$$

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SU(N) connection

$$A = A_\mu dx^\mu, A_\mu(x) = \sum_{a=1}^{N^2-1} A_\mu^a(x) T^a$$

base

field strength

$$F = dA + i A \wedge A$$

“SU(N) principal bundle”

self dual

$$F = * F$$

transition functions

$$A(x + \hat{e}_\nu L_\nu) = \Omega_\mu(x) (A(x) - id) \Omega_\mu^{-1}(x)$$

a quick review of poor man's twisted bundle - $SU(N)/\mathbb{Z}_N$ bundle

The diagram shows a square on a torus \mathbb{T}^2 with axes x_1 and x_2 . The square's vertices are labeled 0 (bottom-left), B (bottom-right), L_1 (top-right), and L_2 (top-left). The left edge is labeled L , the right edge is labeled R , and the top edge is labeled T . A vertical arrow labeled $\Omega_2(x_1)$ points down from L_2 to the top edge. A horizontal arrow labeled $\Omega_1(x_2)$ points left from the right edge to L_1 . The equations $A_T = \Omega_2(A_B - id)\Omega_2^{-1}$ and $A_R = \Omega_1(A_L - id)\Omega_1^{-1}$ are associated with the top and right edges respectively. A coordinate system \mathbb{T}^2 is shown at the bottom left.

$$A_T = \Omega_2(A_B - id)\Omega_2^{-1} \implies \Omega_1(L_2)\Omega_2(0) = e^{i\frac{2\pi}{N}n_{12}} \Omega_2(L_1)\Omega_1(0)$$

$$A_R = \Omega_1(A_L - id)\Omega_1^{-1}$$

$$\Omega_\mu(L_\nu)\Omega_\nu(0) = e^{i\frac{2\pi}{N}n_{\mu\nu}} \Omega_\nu(L_\mu)\Omega_\mu(0)$$

$$Q = \frac{1}{8\pi^2} \int_{\mathbb{T}^4} \text{tr } F \wedge F = -\frac{\text{Pf}(n)}{N} \pmod{1}$$

't Hooft '81; van Baal '82

$$A(x + \hat{e}_\nu L_\nu) = \Omega_\mu(x)(A(x) - id)\Omega_\mu^{-1}(x)$$

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$$\downarrow F = *F$$

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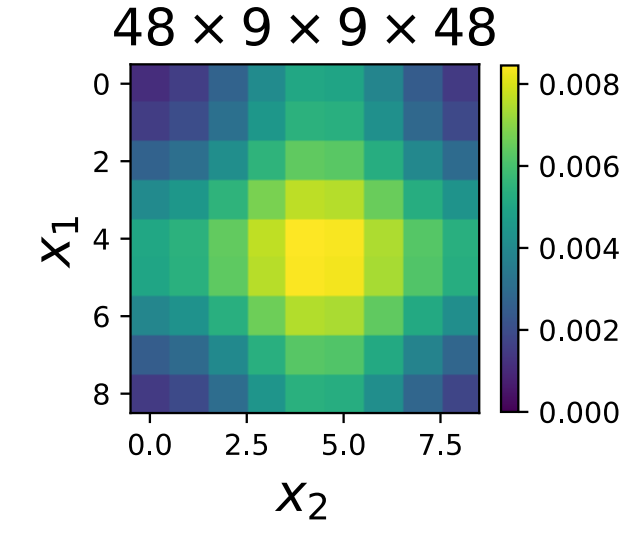
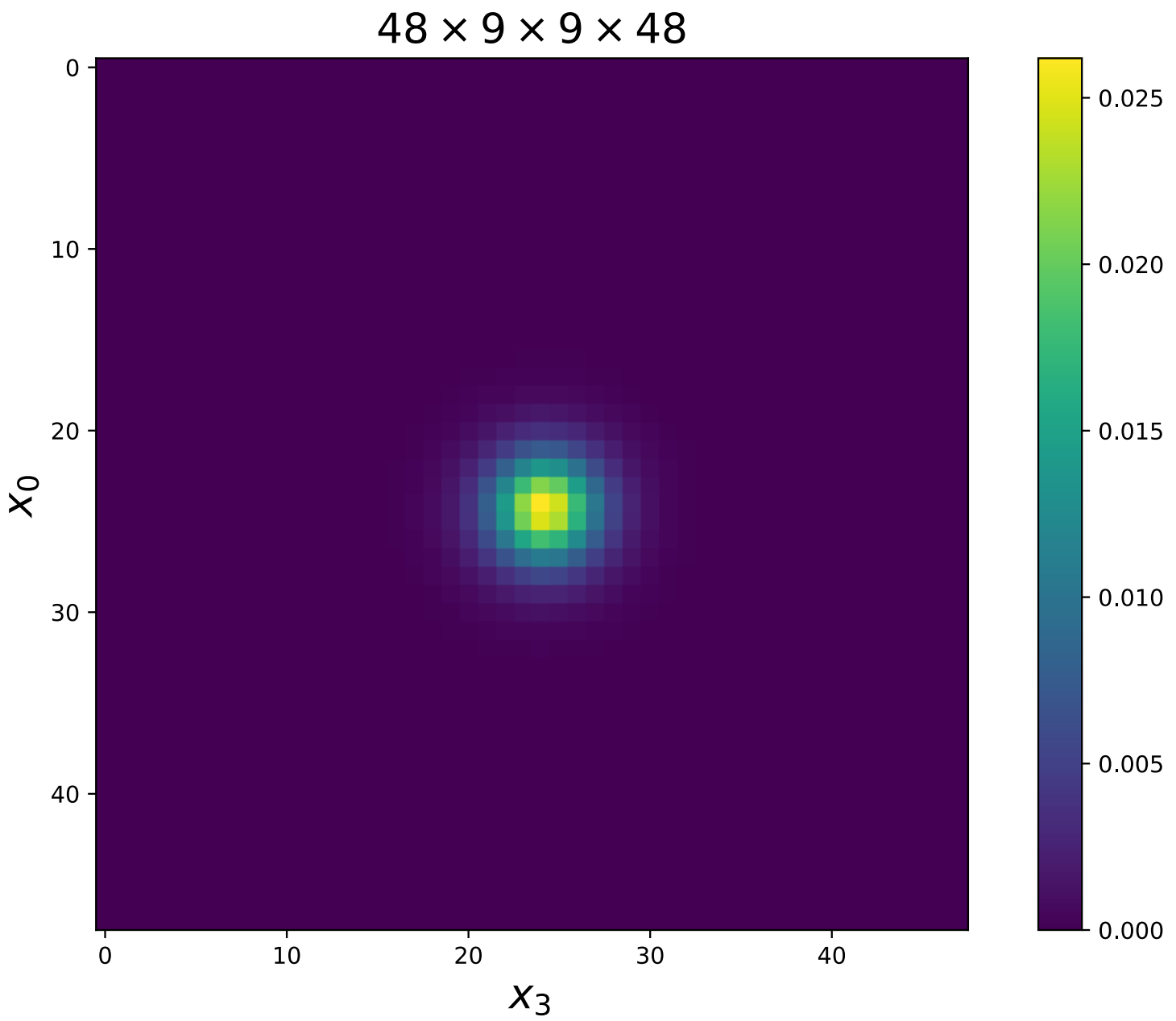
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action: $\frac{8\pi^2}{g^2 N}$

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size: $L_{\mathbb{T}_{small}^2}$



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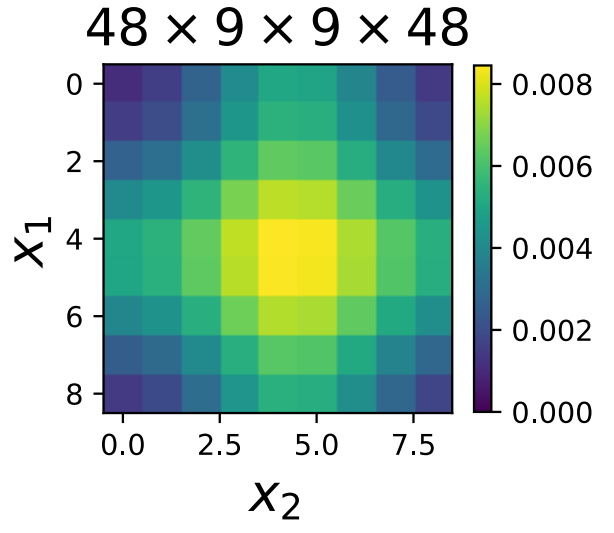
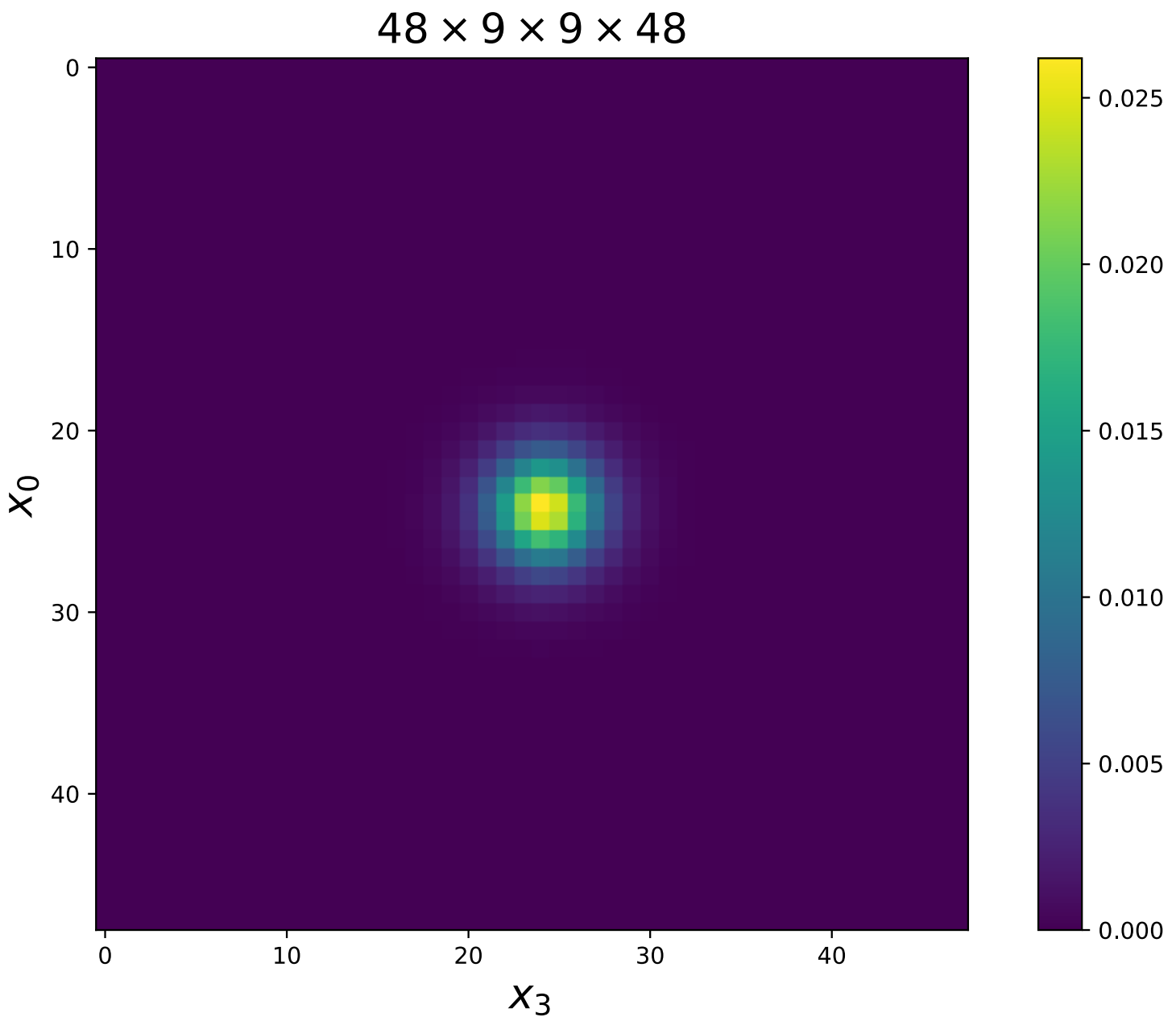
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SU(N) solutions we study depend on $k, r \in \mathbb{N}$; where $N = k + \ell, \ell \in \mathbb{N}$

$$n_{12} = -r, \quad n_{34} = 1 \quad Q_{SU(N)} = \frac{r}{N}, \quad r \in \mathbb{N}, \quad r \text{ is arbitrary, } Q \text{ can be also an integer, including 1}$$

$$F_{12} = \begin{pmatrix} -\frac{2\pi\ell r}{NkL_1L_2}I_k & 0 \\ 0 & \frac{2\pi r}{NL_1L_2}I_\ell \end{pmatrix}, \quad F_{34} = \begin{pmatrix} -\frac{2\pi}{NL_3L_4}I_k & 0 \\ 0 & \frac{2\pi k}{N\ell L_3L_4}I_\ell \end{pmatrix}$$

deceptively simple!

field strength abelian
transition functions NOT

$$F_{12} = F_{34} \implies \frac{L_1L_2}{L_3L_4} = \frac{\ell r}{k}$$

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$F_{12} = F_{34} \implies \frac{L_1L_2}{L_3L_4} = \frac{\ell r}{k}$ explicitly, $\forall \mu :$

$$\phi = \text{diag}(\underbrace{a^1, a^2, \dots, a^k}_k, \underbrace{a, a, \dots, a}_\ell)$$

where $a^c = a^{[c-r]_k}$, $a^1 + a^2 + \dots + a^k + \ell a = 0$
 $[c-r]_k \equiv c-r \pmod k, [k]_k \equiv k$

moduli?

- constant connections must commute with Ω_μ
- $\exists 4\text{gcd}(k, r)$ such moduli

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“missing moduli”:

unless $\gcd(k, r) = r$, not enough to satisfy index theorem

- $\exists 4\gcd(k, r)$ such moduli

A. Schwarz; E. Weinberg; C. Taubes '77-'82 $\dim(\text{moduli}) = 4NQ = 4r$

recap, so far:

SU(N) solutions we study depend on $k, r \in \mathbb{N}$; where $N = k + \ell, \ell \in \mathbb{N}$

$$Q_{SU(N)} = \frac{r}{N}, \quad r \in \mathbb{N}, \quad F_{12} = F_{34} \quad (F = * F) \quad \frac{L_1 L_2}{L_3 L_4} = \frac{\ell r}{k}$$

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next, what we understand:

1. $\gcd(k, r) = r$ understand dimensionality and global structure for $r=k$
(only possible if $r < N$, so all $Q \geq 1$ not understood!)

2. $\gcd(k, r) \neq r$ from field theory: *Anber, Cox, EP, 2504*

understand dimensionality  from string theory D-branes: *EP, 2604*... much easier!

global structure unknown!

- integral over moduli space crucial for physics applications

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A. Schwarz; E. Weinberg; C. Taubes '77-'82 $\dim(\text{moduli}) = 4NQ = 4r$

Atiyah, Hitchin, Drinfeld, Manin, '79

described moduli space for $Q \in \mathbb{Z}$ on \mathbb{R}^4

next, what we understand:

ADHM

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1. $\gcd(k, r) = r$

Anber, EP 2307

only possible if $r < N$, so all $Q \geq 1$ not understood!

claim 1: moduli space is that of flat connections only

$$\phi = \text{diag}(\underbrace{a^1, a^2, \dots, a^k}_k, \underbrace{a, a, \dots, a}_\ell), \quad \forall \mu = 1, 2, 3, 4$$

claim 2: globally, for $r = k$, $\prod_{\mu=1}^4 \frac{S^1_\mu \times \Gamma_r^{SU(r)}}{\mathbb{Z}_r} / S_r$

all quotients freely acting

using this, calculated ‘gaugino condensate’ in SYM on \mathbb{T}^4 , agrees with \mathbb{R}^4 calculation...

2. gcd(k, r) ≠ r

understand dimensionality

from field theory: [Anber, Cox, EP, 2504](#)

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$$\bar{A} \rightarrow \bar{A} + a, \quad *D* \wedge a = 0$$

known solution

perturbation

gauge condition

eqn. for self-dual manifold

$$F(\bar{A} + a) = *F(\bar{A} + a)$$

quadratic in a , study by expanding in nonlinearity,

~ [A. Schwarz; E. Weinberg; C. Taubes '77-'82](#)

but with a 'twist'... (later!)

result:

2.1. for $\text{gcd}(k, r) = r$, show that only $a = \text{const}$.

previously obtained that, globally, $\prod_{\mu=1}^4 \frac{S^1_{\mu} \times \Gamma_r^{SU(r)}}{\mathbb{Z}_r} / S_r$ + some physics application

result:

2.1. for $\gcd(k, r) = r$, show that only $a = \text{const}$.

2.2. $\gcd(k, r) \neq r$, **show to linear order in a** , that space of solutions includes the $4 \gcd(k, r)$ constant holonomies + $2r$ complex $q_i^a, \tilde{q}_a^i, a = 1, \dots, r/\gcd(k, r), i = 1, \dots, \gcd(k, r)$

$$\sum_{a=1}^{\frac{r}{g}} \tilde{q}_a^i q_i^a = 0, \quad i = 1, \dots, g \quad g \text{ complex constraints} \quad g \equiv \gcd(k, r)$$

$$\sum_{a=1}^{\frac{r}{g}} |\tilde{q}_a^i|^2 - |q_i^a|^2 = 0 \quad g \text{ real constraints} \quad +g \text{ rephrasing}$$

$\dim(\text{moduli}) = 4g + 4r - 2g - g - g = 4r$, as per index theorem ($= 4NQ, Q=r/N$)

the good news: satisfying, also did numerical check for $Q=2/3, r=2, k=1, \text{SU}(3)$ ($\gcd(k,r)=1$)

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The 'twist'...

... four technical slides follow

The 'twist'...

$$\bar{A} \rightarrow \bar{A} + a, \quad *D(\bar{A}) \wedge *a = 0$$

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maps to:

$\bar{D}\lambda = \epsilon\lambda^2$, \bar{D} - Dirac operator in background \bar{A} , λ is a two-spinor, function of a , $\epsilon = 1$ formal expansion

solve in formal power series:

$$\lambda = \lambda_1 + \epsilon\lambda_2 + \epsilon^2\lambda_3 + \dots + \epsilon^{n-1}\lambda_n + \dots$$

$$\bar{D}\lambda = \epsilon\lambda^2 \rightarrow \bar{D}\lambda_1 = 0 \quad \text{solve: } \lambda_1 = \sum_{i=1}^{2r} c_i \phi_i(x)$$

D is the adjoint of \bar{D}

OK if $\bar{D}D$ invertible!

(regular implicit function ∞ dim)

$$\bar{D}\lambda_2 = \lambda_1^2 \rightarrow \lambda_2 = D \frac{1}{\bar{D}D} \lambda_1^2$$

proceeds to all orders to find λ_n
as for **BSPT** on \mathbb{R}^4

$$\bar{D}\lambda_n = f(\lambda_1, \dots, \lambda_{n-1}), \quad n > 2.$$

But in our \mathbb{T}^4 background $\bar{D}D$ not invertible!

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D also has $4g$ zero modes

$$\bar{D}\lambda_2 = \lambda_1^2$$

impose orthogonality of r.h.s. to D zero modes: \rightarrow the linearized moduli space eqns
 $4g$ constraints on the $4r$ real components of c_i

$$\sum_{a=1}^{\frac{r}{g}} \tilde{q}_a^i q_i^a = 0, \quad i = 1, \dots, g$$

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the linearized moduli space eqns

the good news:

*Anber, Cox, EP, 2504
 EP, 2604*

QFT = branes!
 + one numerical check

$$\sum_{a=1}^g \tilde{q}_a^i q_i^a = 0, \quad i = 1, \dots, g$$

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$$\bar{D}\lambda_2 = \lambda_1^2$$

$$\lambda_2 = D \frac{1}{\bar{D}D} \Big|_{\perp} \lambda_1^2$$

so can invert on space \perp zero modes..

BUT:

$$\bar{D}\lambda_n = f(\lambda_1, \dots, \lambda_{n-1}), \quad n > 2.$$

higher orders... new constraints? global structure?

except $\gcd(k,r)=r$

END: The 'twist'...

result:

2.1. for $\gcd(k, r) = r$, show that only $a = \text{const}$.

globally, $\prod_{\mu=1}^4 \frac{S^1_{\mu} \times \Gamma_r^{SU(r)}}{\mathbb{Z}_r} / S_r$ + some physics application

2.2. $\gcd(k, r) \neq r$, show to linear order in a , that space of solutions includes the $4 \gcd(k, r)$ constant holonomies + $2r$ complex $q_i^a, \tilde{q}_a^i, a = 1, \dots, r/\gcd(k, r), i = 1, \dots, \gcd(k, r)$

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the bad news: perturbation calculation no info on **global structure**

- important for physics applications

- physicists commonly take infinite volume limit $\mathbb{T}^4 \rightarrow \mathbb{R}^4$

solutions with Q -integer should approach ADHM...how? when?

moduli space of $F = *F, Q = r/N, N = k + \ell; r, k \in \mathbb{N}$ on \mathbb{T}^4 :

2.1. for $\gcd(k, r) = r$, show that only $a = \text{const}$. globally, $\prod_{\mu=1}^4 \frac{S^1_{\mu} \times \Gamma_r^{SU(r)}}{\mathbb{Z}_r} / S_r$ + some physics application

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- important for physics applications

- physicists commonly take infinite volume limit $\mathbb{T}^4 \rightarrow \mathbb{R}^4$

solutions with Q -integer should approach ADHM...how? when?

**summary,
outlook
math problem?**