Anomalies, tori, and new twists in the gaugino condensate

Erich Poppitz (Toronto)

with

Mohamed Anber (Durham)

2210.13568 (w/ some mention of 2307.04795 and work since)

title of 2210.13568 is **The gaugino condensate from asymmetric four-torus with twists**: sounds like a mouthful & is 70 pages long!

why now, etc. ?

$\mathcal{N}$=1 SYM: symmetries and nonrenormalization theorems

$$S_{SYM} = \frac{1}{g^2} \int_{\mathbb{T}^4} \mathrm{tr} \left[\frac{1}{2} F_{mn} F_{mn} + 2(\partial_n \bar{\lambda}_{\dot{\alpha}} + i[A_n, \bar{\lambda}_{\dot{\alpha}}]) \bar{\sigma}_n^{\dot{\alpha}\alpha} \lambda_\alpha \right]$$

G=SU(N)

chiral $U(1) : \lambda \to e^{i\alpha}\lambda$ by anomaly $\to Z_{2N}^{(0)}$

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$Z_{2N}^{(0)} \to Z_2^{(0)}$ $\quad \langle \mathrm{tr}\,\lambda^2\rangle \underset{(N=2)}{=} \pm 16\pi^2\Lambda^3$

<u>the "mother" of all exact results in SUSY, no further corrections</u>

$$\Lambda^3 = \mu^3 \frac{e^{-8\pi^2/Ng^2}}{g^2} \left(= \mu^3\, e^{-\frac{8\pi^2}{Ng_h^2(\mu)}}\right)$$ holomorphic scale

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$\mathcal{N}$=1 SYM: symmetries and nonrenormalization theorems

1983-1999: Novikov, Shifman, Vainshtein, Zakharov; Amati, Konishi, Rossi, Veneziano; Affleck, Dine, Seiberg; Cordes; Finnell, Pouliot (SQCD —> SYM on R^4); Davies, Hollowood, Khoze, Mattis;... 2014 Anber, Teeple, EP (SYM on $R^3 \times S^1$ —> SYM on R^4)

two weakly-coupled calculations of $\langle \lambda^2 \rangle$

$$Z_{2N}^{(0)} \to Z_2^{(0)} \qquad \langle \mathrm{tr}\, \lambda^2 \rangle_{(N=2)} = \pm\, 16\pi^2 \Lambda^3$$

all history…?

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chiral $U(1) : \lambda \to e^{i\alpha}\lambda$ broken by anomaly to $Z_{2N}^{(0)}$

center symmetry: $Z_N^{(1)}$, acting on Wilson loops by Z_N phase

ex. of "generalized symmetries, backgrounds, new anomalies…"

Gaiotto, Kapustin, Komargodski, Seiberg, + hundreds… 2015-

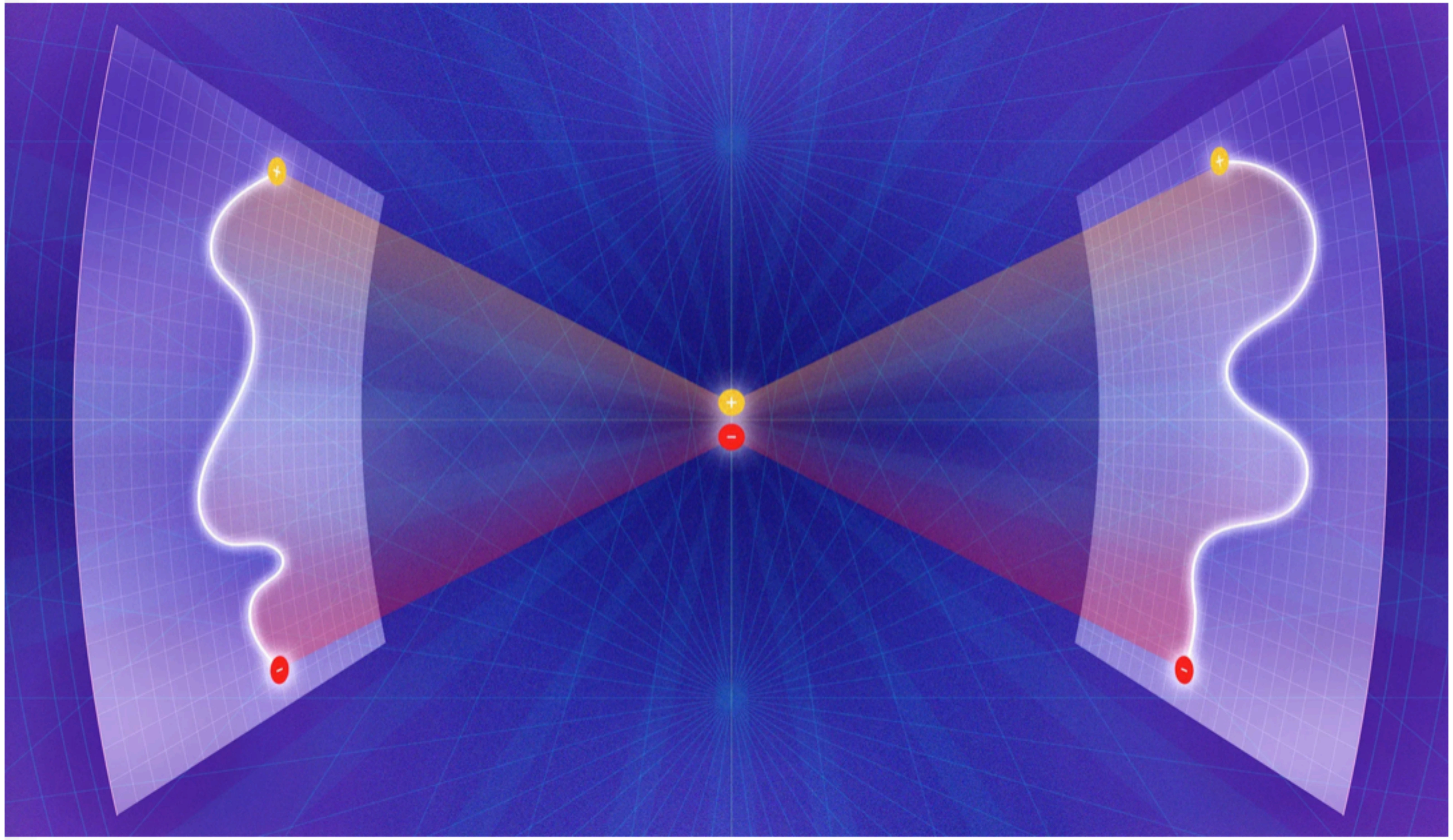
“Quanta” Spring ’23 —>

MATHEMATICAL PHYSICS

A New Kind of Symmetry Shakes Up Physics

23

So-called “higher symmetries” are illuminating everything from particle decays to the behavior of complex quantum systems.



The symmetries of 20th-century physics were built on points. Higher symmetries are based on one-dimensional lines.

Samuel Velasco/Quanta Magazine

1-form center symmetry $Z_N^{(1)}$ acts on Wilson loops

(e.g. in SYM)

$$Z_q^{(1)}$$

$$: \; W_1 \to e^{i\frac{2\pi}{q}l_1} W_1$$

$$: \; W_2 \to e^{i\frac{2\pi}{q}l_2} W_2$$

$$: \; W_3 \to e^{i\frac{2\pi}{q}l_3} W_3$$

well-known on lattice since the mid 1970’s - generalized to non-winding loops GKKS+

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center/chiral mixed anomaly!

chiral $U(1): \lambda \to e^{i\alpha}\lambda$ broken by anomaly to $Z_{2N}^{(0)}$

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ex. of "generalized symmetries, backgrounds, new anomalies…"

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new developments warrant a new look at some old studies of gaugino condensate…

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recent motivation

I. semiclassical studies of confinement…

in a controlled way, show relevance of objects of fractional $Q_{top.}$ for confinement and χSB, advocated by many (González-Arroyo,…)

semiclassical - small spaces

eg $R^2 \times T^2$ Tanizaki, Ünsal, 2022 or

Ünsal, +w/ Yaffe, w/ Shifman +…: 2007-
$R^3 \times S^1$ $SU(N) \to U(1)^{N-1}$ -… + any G SYM

2. generalized symmetries, backgrounds, new anomalies

Gaiotto, Kapustin, Komargodski, Seiberg, +… 2015-

2-form backgrounds for 1-form center also lead to fractional Q_{top}

relation of "'t Hooft twists" 't Hooft, van Baal 1980s to anomalies now clearly understood

I. semiclassical studies
of confinement…

2. generalized symmetries,
backgrounds, new anomalies

one of two weakly-coupled calculations
of $\langle\lambda^2\rangle$: continuous connection to R^4

$$\langle\lambda^2\rangle = 16\pi^2\Lambda^3$$

$\longleftrightarrow$

using this new and deeper knowledge,
revisit old (1984!) calculations of $\langle\lambda^2\rangle$ on T^4

$$\langle\lambda^2\rangle = c\ 16\pi^2\Lambda^3$$

Cohen, Gomez '84;
Shifman, Vainshtein '86

how well do we understand semiclassics in the femtouniverse?

is there continuity to infinite volume limit?

- test for condensate, in SYM, where some exact results are known

what fluctuations contribute to the gaugino condensate?

one of two weakly-coupled calculations of $\langle \lambda^2 \rangle$: continuous connection to R^4 $\longleftrightarrow$ using this new and deeper knowledge, revisit old (1984!) calculations of $\langle \lambda^2 \rangle$ on T^4

$$\langle \lambda^2 \rangle = 16\pi^2 \Lambda^3$$

$$\langle \lambda^2 \rangle = c \; 16\pi^2 \Lambda^3$$

Cohen, Gomez '84; Shifman, Vainshtein '86 in 2021, w/ Cox & Wandler studied 1-form center/0-form anomaly in YM, SYM,…, in Hamiltonian on twisted T^3 of any size. Anomaly implies exact degeneracies!

discuss on board… or… ?

we now canonically quantize of SU(N) on T^3:

't Hooft, van Baal
Lüscher, Witten,
González-Arroyo
1980's

Hilbert space with spatial 't Hooft twist $n_{12} = 1$ (e.g., suffices), $A_0 = 0$ gauge, "by the book"

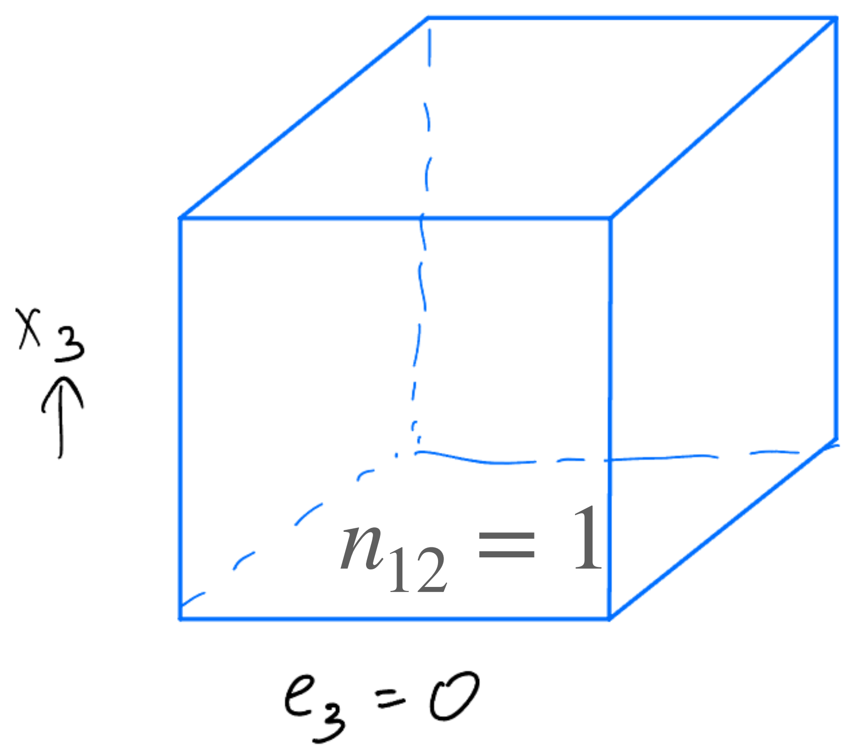
$\Psi(A)$ with $A_R = \Omega_1(A_L - id)\Omega_1^{-1}$, etc., with some chosen gauge for $\Omega_{1,2,3}$

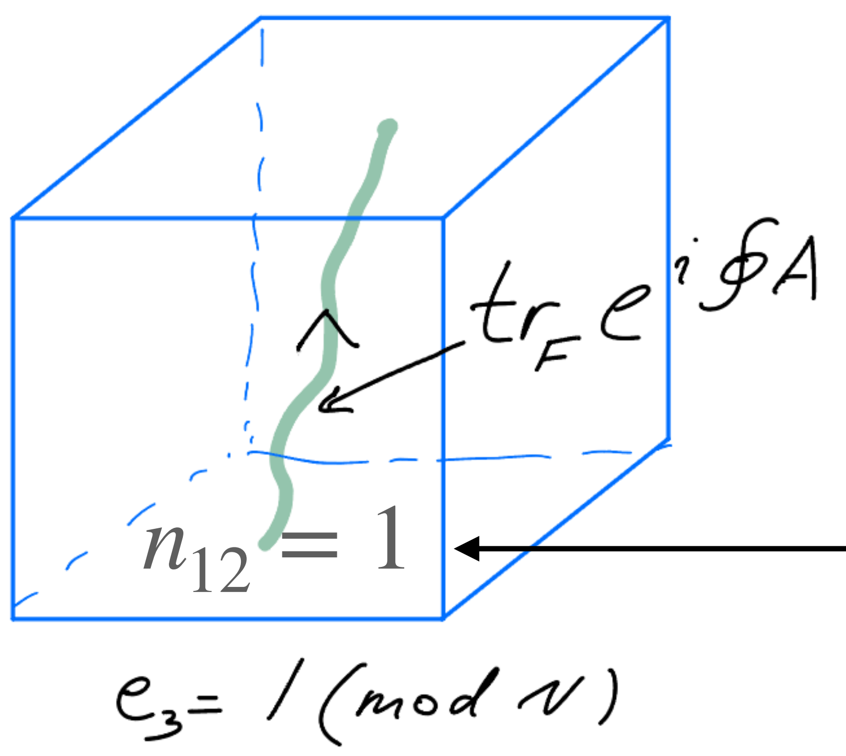
obeying $\Omega_1(L_2)\Omega_2(0) = e^{i\frac{2\pi}{N}}\ \Omega_2(L_1)\Omega_1(0)$ and no 13 and 23 twists

1-form $Z_N^{(1)}$: $\hat{T}_i, i = 1,2,3$, generated by gauge transforms (maps $T^3 \to SU(N)$)

periodic up to center element and preserving b.c. w/ Ω_i

$$[\hat{T}_i, \hat{H}] = 0 \implies |E, e_1, e_2, e_3\rangle,\ \hat{T}_i|E, e_1, e_2, e_3\rangle = |E, e_1, e_2, e_3\rangle\ e^{i\frac{2\pi}{N}e_i}$$





"electric flux sectors" (changed by winding Wilson loop)

"magnetic flux sectors" (changed by winding 't Hooft loop)

torus Hilbert space, with or without twists,
splits into N^3 electric flux sectors

$$\hat{T}_i \hat{W}_j = e^{i\frac{2\pi}{N}\delta_{ij}}\ \hat{W}_j \hat{T}_i$$

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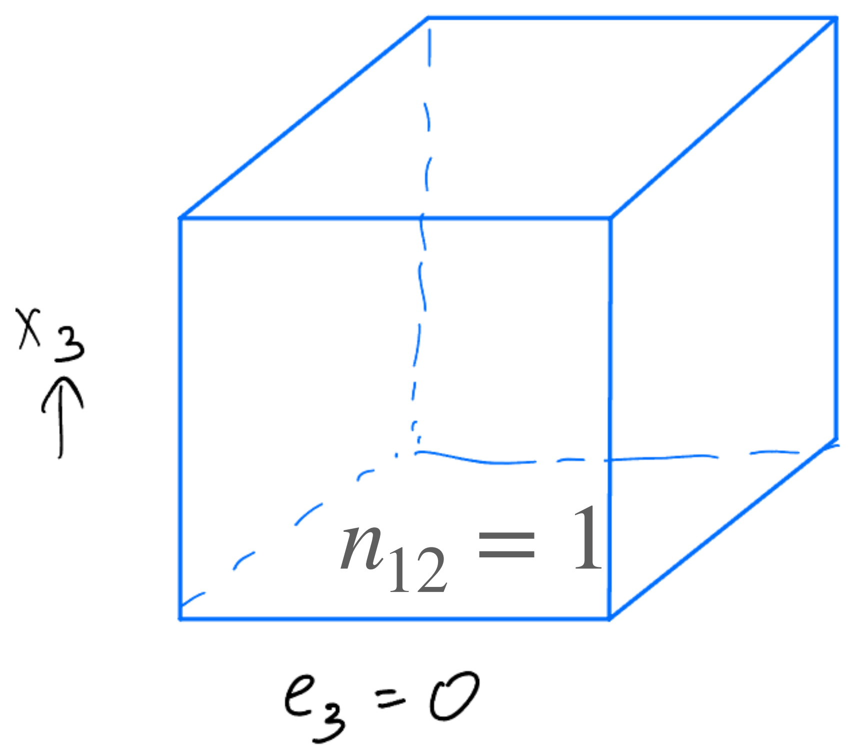

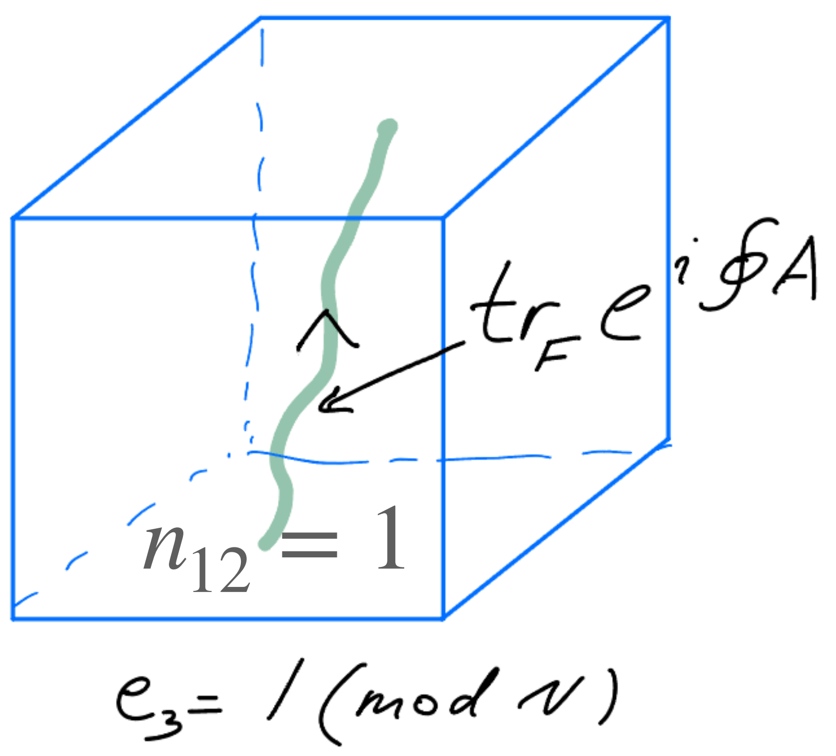

Crucial observation ('t Hooft)

$\hat{T}_3$, the $Z_N^{(1)}$ generator in the direction orthogonal to the (12) plane of the twist has winding number Q = $\dfrac{n_{12}}{N} (\mathrm{mod}\; Z)$

torus Hilbert space, with or without twists,
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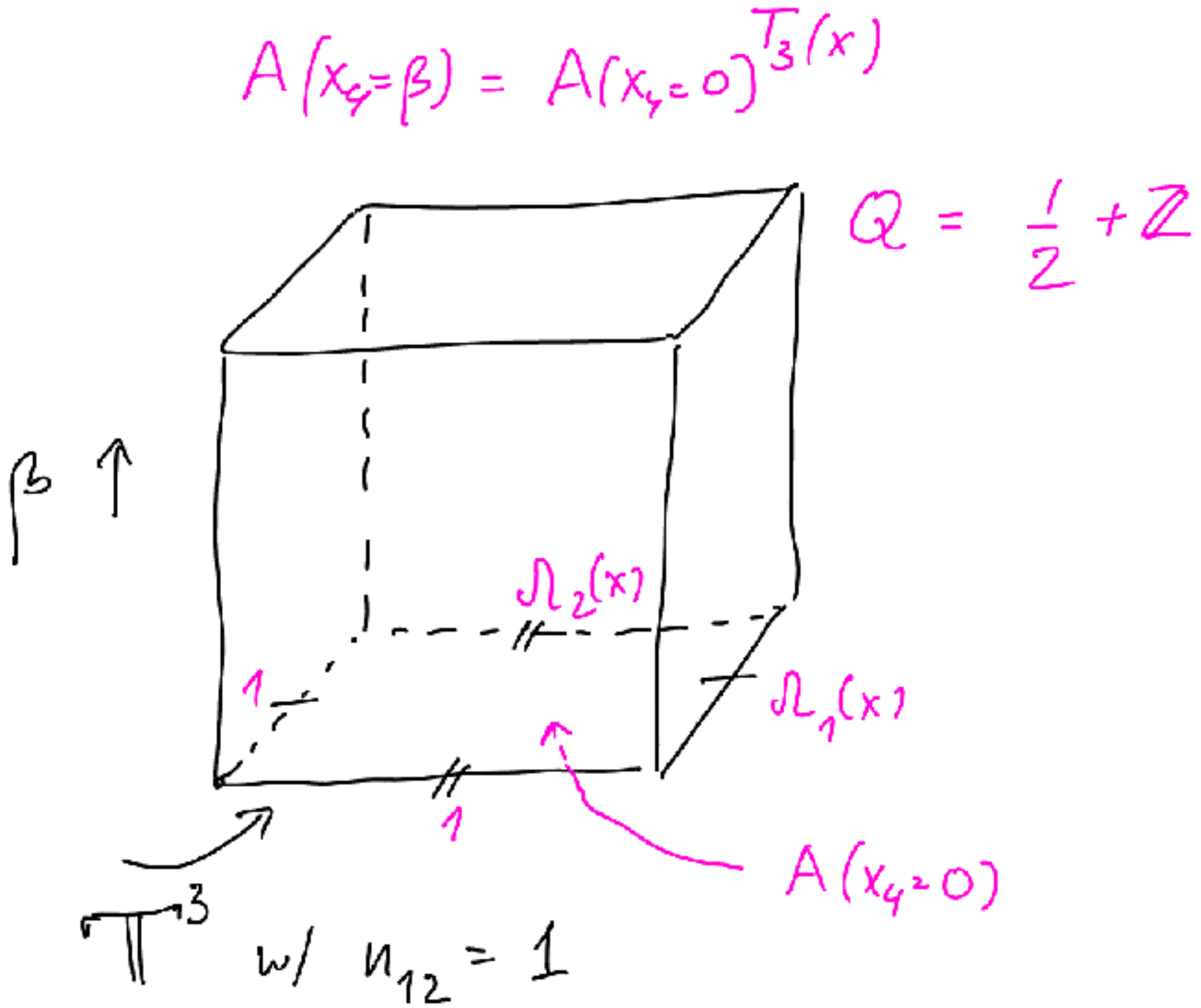


idea only (details are plentiful… see eg appx of 2106 paper w/ Cox, Wandler)

$$Q = \frac{1}{8\pi^2}\int \mathrm{tr}\, F \wedge F = \frac{1}{64\pi^2}\int d^4x F^a_{\mu\nu} F^a_{\lambda\sigma} \epsilon^{\mu\nu\lambda\sigma} = \int d^4x \partial_\mu K^\mu$$

integrand a total derivative, Q only depends on transition functions

for a 4d field configuration twisted by T_3 (denoted C) in time and n_{12} in space:

$$Q[C] = \frac{1}{24\pi^2}\int_{\mathbb{T}^3} \mathrm{tr}\,(CdC^{-1})^3$$

a direct calculation (only requires cocycle conditions, good gauge choice, not explicit form of C=T_3), then gives

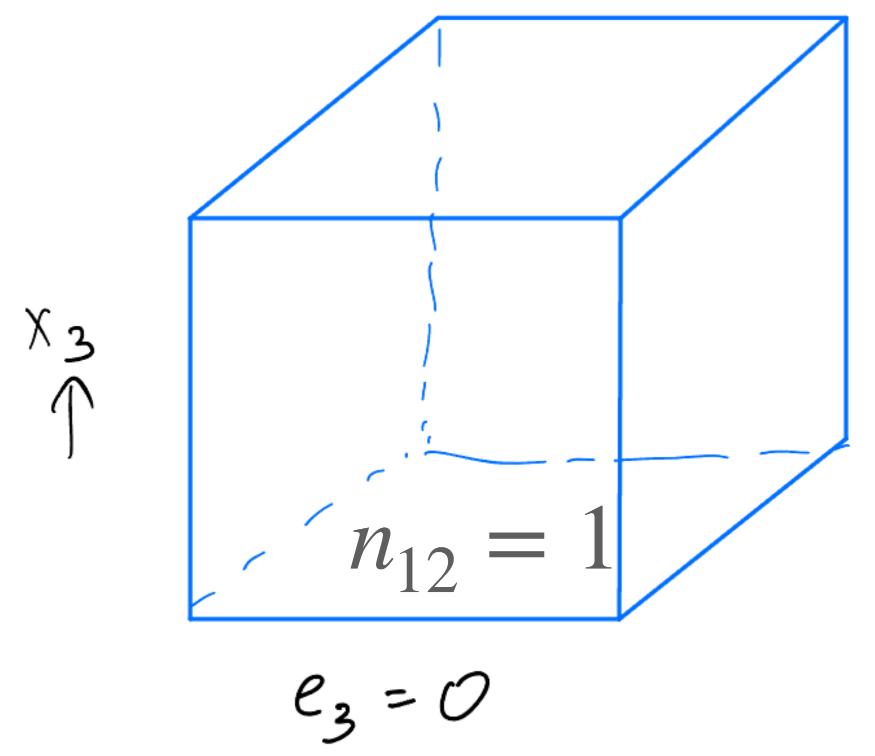
Q = $\frac{n_{12}}{N}$(mod Z) = winding of $\hat{T}_3(\vec{x})$, as map $T^3 \to SU(N)$

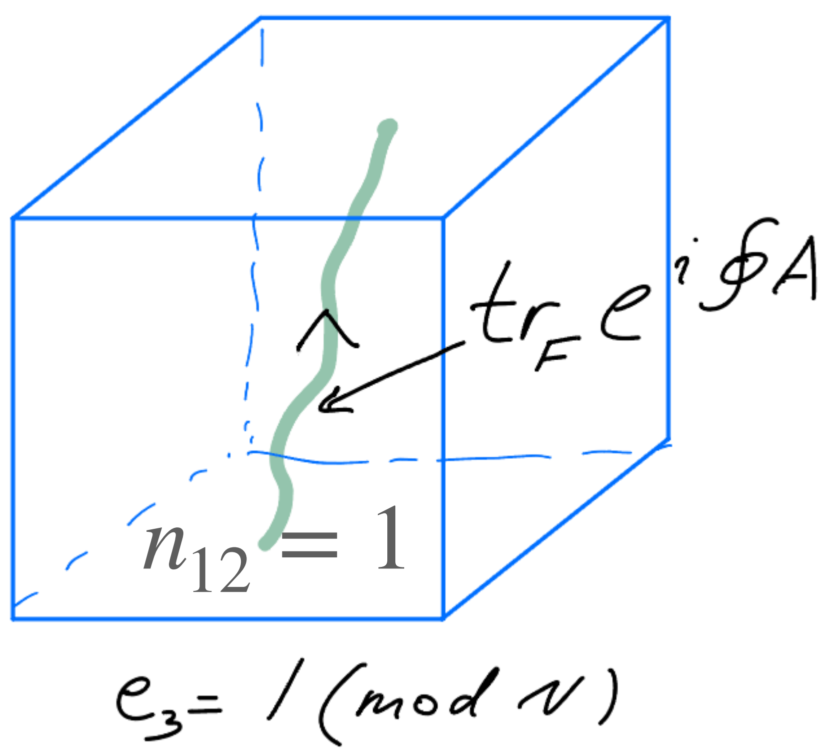
(considering 4d field configuration is a clutch ('t Hooft); equiv., can explicitly construct $\hat{T}_3(\vec{x})$ and compute winding…)

[García Pérez, González-Arroyo '92; Selivanov-Smilga '00; Wandler-EP 2211]

we now canonically quantize of SU(N) on T^3:

$$[\hat{T}_i, \hat{H}] = 0 \implies |E, e_1, e_2, e_3\rangle, \;\; \hat{T}_i |E, e_1, e_2, e_3\rangle = |E, e_1, e_2, e_3\rangle \, e^{i\frac{2\pi}{N} e_i}$$





torus Hilbert space, with or without twists, splits into N^3 electric flux sectors

<u>Crucial observation ('t Hooft)</u>

$\hat{T}_3$, the $Z_N^{(1)}$ generator in the direction orthogonal to the (12) plane of the twist has winding number Q = $\dfrac{n_{12}}{N} (\mathrm{mod}\ Z)$

but then, since the change of CS functional is the winding number

$$\implies \hat{T}_3 \, e^{i2\pi \int_{T^3} \mathrm{tr}(\hat{A} d\hat{A} + \ldots)} \, \hat{T}_3^{-1} = e^{i\frac{2\pi}{N}} \, e^{i2\pi \int_{T^3} \mathrm{tr}(\hat{A} d\hat{A} + \ldots)}$$

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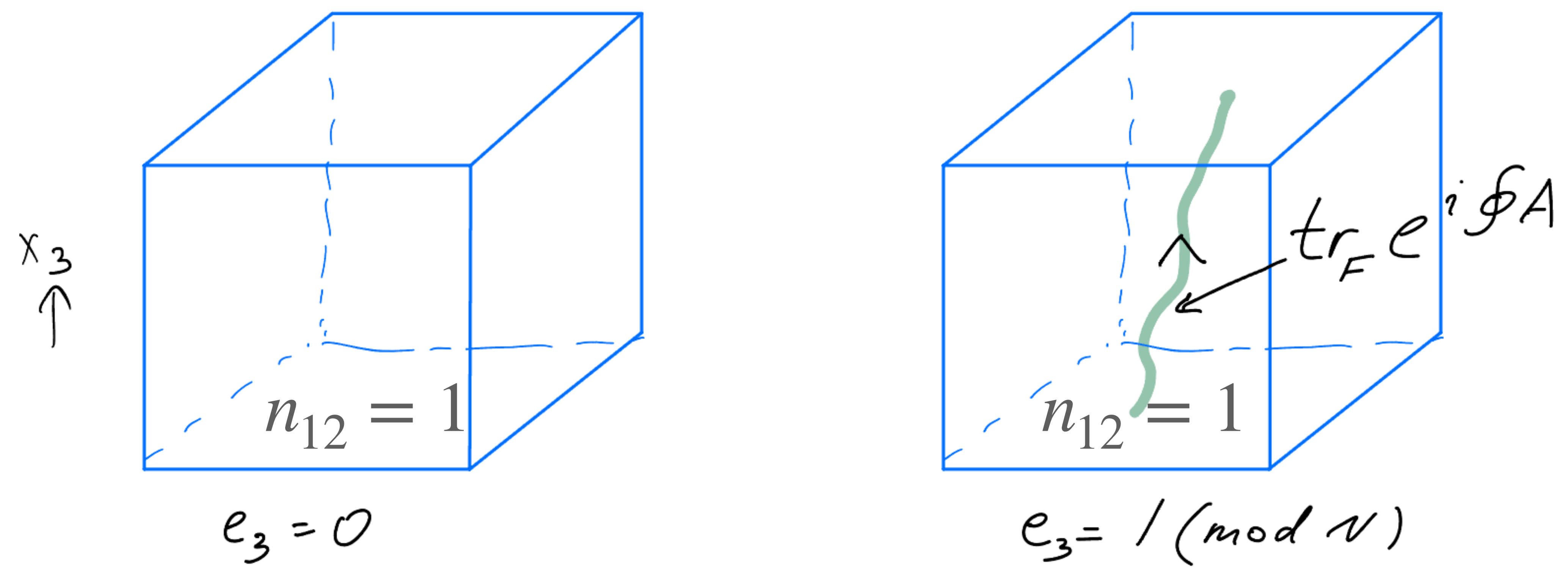


i.e., operator shifting θ angle by 2π does not commute with 1-form center symmetry in the direction orthogonal to the twist

- Pierre van Baal PhD thesis, 1984, Ch 3, unpublished! (**)

- "theta-periodicity anomaly"… [GKKS+] ~ 2010's (in Euclidean)

(**) admittedly, while commutation relation appears there, its significance as an anomaly and implications for large volume theory was not appreciated back then… (why?)

we now canonically quantize of SU(N) on T^3:

consider SU(N) with adjoints, for definiteness take SYM, $\underline{n_f = 1}$ below:

$$\partial_\mu \hat{j}^\mu_f = \partial_\mu(\hat{\lambda}^{a\,\dagger}\bar{\sigma}^\mu \hat{\lambda}^a) = 2n_f N \partial_\mu \hat{K}^\mu \longrightarrow$$ R-current not conserved

$$\hat{J}^\mu_5 = \hat{j}^\mu_f - 2n_f N \hat{K}^\mu$$

$$\hat{Q}_5 = \int d^3x \hat{J}^0_5 = \int d^3x \hat{j}^0_f - 2n_f N \int d^3x \hat{K}^0$$

$\longrightarrow$ conserved but not gauge invariant

$$\hat{X}_{2N} = e^{i\frac{2\pi}{2N}\hat{Q}_5} = e^{i\frac{2\pi}{2N}\int d^3x \hat{j}^0_f}\, e^{-i2\pi\int d^3x \hat{K}_0} \longrightarrow$$ gauge invariant operator of $Z^{(0)}_{2N}$ discrete R-symmetry

$$\hat{T}_3\, \underset{\uparrow \int d^3x \hat{K}_0}{e^{i2\pi\int_{T^3}\mathrm{tr}(\hat{A}d\hat{A}+\ldots)}}\, \hat{T}_3^{-1} = e^{i\frac{2\pi}{N}}\, e^{i2\pi\int_{T^3}\mathrm{tr}(\hat{A}d\hat{A}+\ldots)} \Longrightarrow \hat{T}_3\, \hat{X}_{2N}\, \hat{T}_3^{-1} = e^{-i\frac{2\pi}{N}}\, \hat{X}_{2N}$$

<u>mixed 0-form/1-form anomaly</u>

Ex 1.: *SYM on twisted T^3 - invertible chiral/center anomaly*

Hilbert space with spatial ‘t Hooft twist $n_{12} = 1$ (e.g., suffices); SYM has two global symmetries, $\hat{T}_3$ and $\hat{X}_{2N}$, 1-form and 0-form, invertible (=normal unitary operators on Hilbert space) commute with Hamiltonian, but not with each other:

$$\hat{T}_3\,\hat{X}_{2N}\,\hat{T}_3^{-1} = e^{-i\frac{2\pi}{N}}\,\hat{X}_{2N} \longrightarrow \hat{X}_{2N}\,|\,E, e_3\rangle = |\,E, e_3 - 1\rangle$$

action of chiral symmetry changes flux of state (the one in 3rd direction, for 12 twist)

all energy levels on the twisted T^3 are N-fold degenerate, exact degeneracy, for any volume, provided $n_{12} = 1$!

[Cox, Wandler, EP 2106]

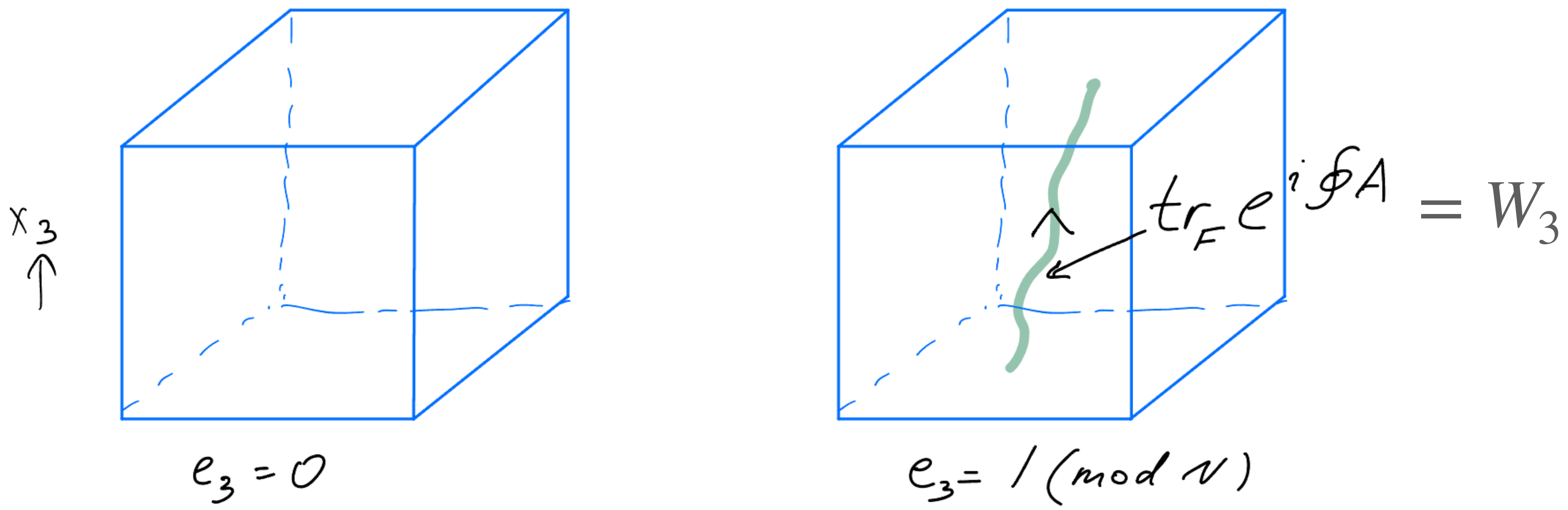
as volume goes to infinity, if theory confines (center unbroken), clustering ground states are the lowest energy degenerate flux states, related by broken discrete chiral symmetry - here, a consequence of the mixed anomaly!

does not require SUSY, similar degeneracies in non-SUSY QCD(adj)

exact degeneracies less severe if gauge group has smaller center… SP, Spin, E6, E7

[Cox, Wandler, EP 2106]

in 2021, w/ Cox & Wandler studied 1-form center/0-form anomaly in YM, SYM,…, in Hamiltonian on twisted T^3 of any size. Anomaly implies exact degeneracies!



$SU(2)$ **SYM, for simplicity**

$e_3 \in \{0,1\}$ $\hat{T}_3$**-eigenvalue:** $\hat{T}_3 | e_3 \rangle = e^{i\pi e_3} | e_3 \rangle$

$\hat{T}_3$ **- generator of Z_2 center symmetry along** $(0,0,1)$

$$\hat{T}_3\, \hat{W}_3 = -\, \hat{W}_3\, \hat{T}_3$$

$\hat{X}$ **- generator of Z_4 chiral symmetry**

Hilbert space with spatial ‘t Hooft twist $\underline{n_{12} = 1}$: $| \ldots \rangle_{(n_{12})}$

$| E, e_3 = 1 \rangle_{(n_{12})}$ **degenerate w/** $| E, e_3 = 0 \rangle_{(n_{12})}$ **for all E, any size T^3** **interchanged by chiral symmetry**

---

anomaly: $\hat{T}_3\, \hat{X} = (-)^{n_{12}}\, \hat{X}\, \hat{T}_3$

$$[\hat{X}, \hat{H}] = [\hat{T}_3, \hat{H}] = 0 \implies \hat{X} | E, e_3 = 0 \rangle_{(n_{12})} \sim | E, e_3 = 1 \rangle_{(n_{12})}$$

(phase depends on whether B or F)

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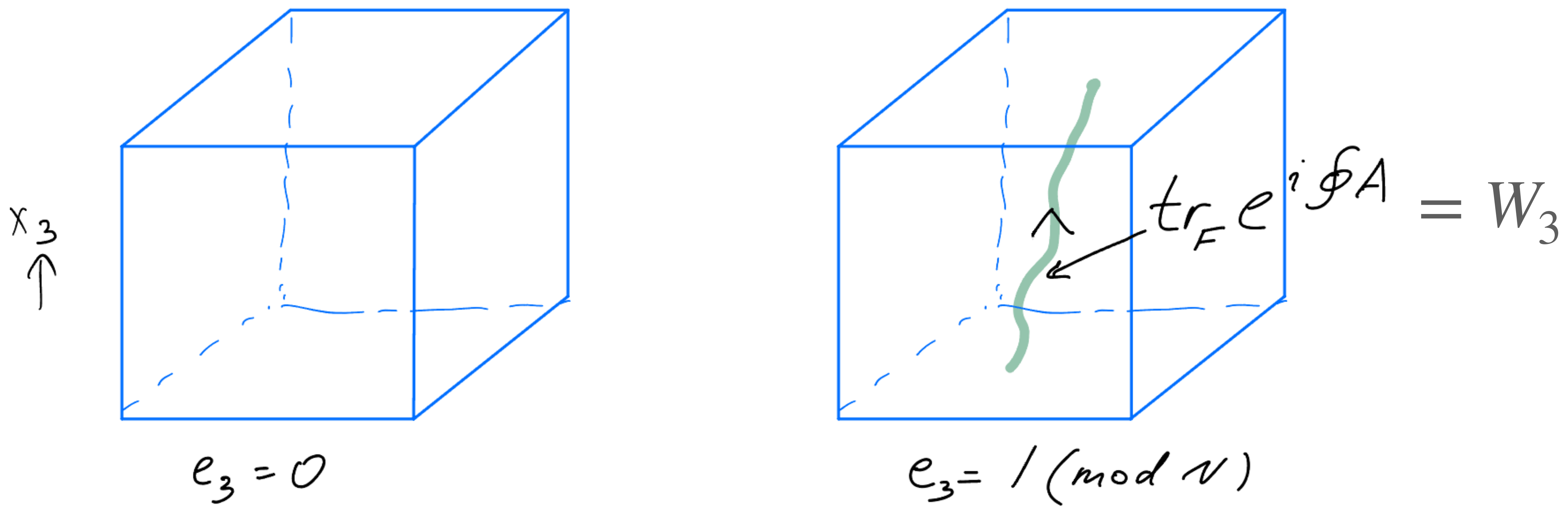
remarks on infinite vs. finite volume in ’t Hooft flux $n_{12} = 1$ background



Assuming confinement (unbroken center) -> broken chiral

$$|E=0, e_3 = 1\rangle_{(n_{12})}$$
$$|E=0, e_3 = 0\rangle_{(n_{12})}$$

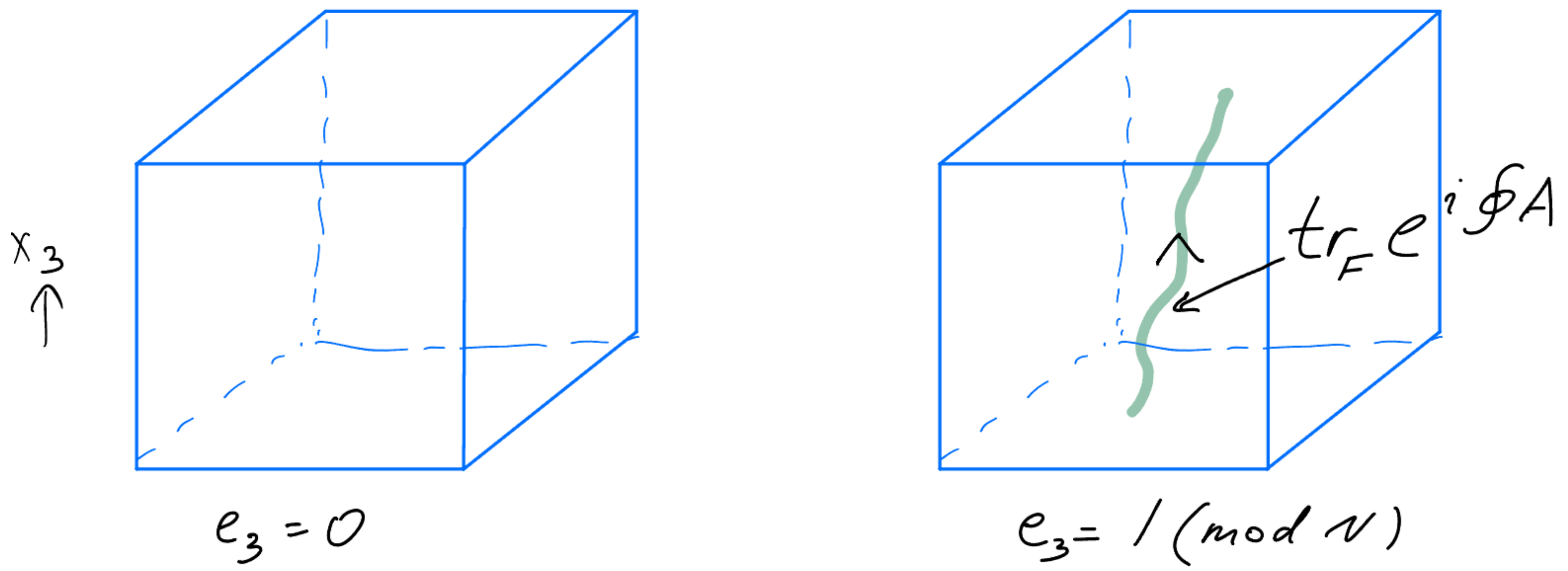
two clustering vacua in infinite volume limit

twisted b.c. should be irrelevant in gapped theory in ∞ volume

lattice pure-YM, $\theta = 0$: string tensions, glueballs
agree $V \gg V_0$ twist vs no twist

[*Teper, Stephenson ’89,’91*]

remarks on infinite vs. finite volume in ’t Hooft flux $n_{12} = 1$ background



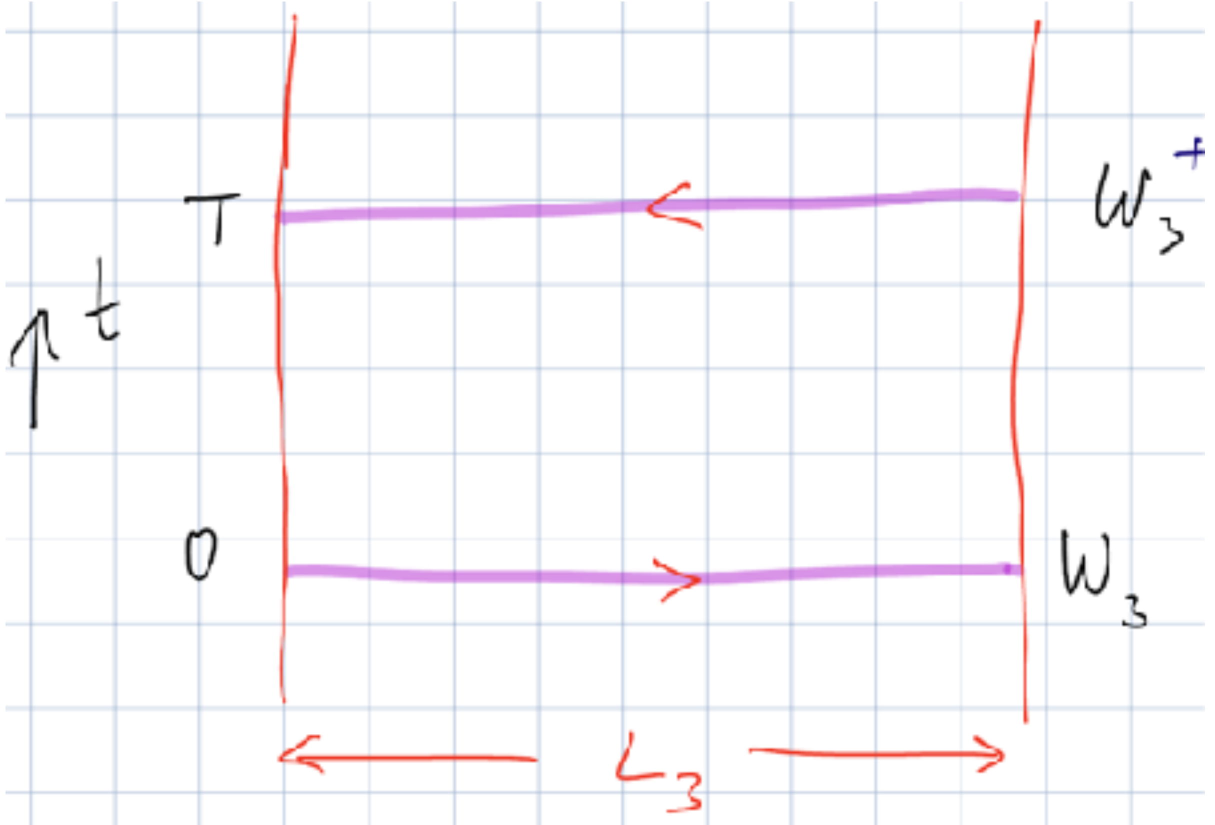
$= W_3$

Assuming confinement (unbroken center) -> broken chiral

$$|E=0, e_3=1\rangle_{(n_{12})}$$
$$|E=0, e_3=0\rangle_{(n_{12})}$$

two clustering vacua in infinite volume limit

$${}_{(n_{12})}\langle 0, e_3 \,|\, W_3^\dagger(\vec{x}_{12}, T)\; W_3(\vec{x}_{12}, 0) \,|\, 0, e_3\rangle_{(n_{12})}\Big|_{T\to\infty} = \text{exact} = |{}_{(n_{12})}\langle 0, e_3 \,|\, W_3(\vec{x}_{12}, 0) \,|\, 0, e_3 + 1\rangle_{(n_{12})}|^2$$



$\neq 0$ for $L_{1,2,3} < \infty$ **(“perimeter,” “broken” T_3)**

for $L_{1,2} \to \infty$ m-x element expected to $\to 0$
by clustering $(W_3(\vec{x}_{12}, 0)$ local, at $L_3 < \infty)$ **(area law, unbroken T_3)**

armed with this, consider condensate, $\lambda^2 \equiv \mathrm{tr}\lambda^2$:

T_3 eigenvalue

inserts n_{34}=1 twist

$$\langle \lambda^2 \rangle_{n_{12},n_{34}} = \mathrm{Tr}_{\mathscr{H}_{n_{12}}} \, e^{-\beta H}(-1)^F \, \hat{T}_3 \, \lambda^2 = \sum_{E;e_3=0,1} (-)^F e^{-\beta E} \, \underline{(-1)^{e_3} \, \langle E, e_3 | \lambda^2 | E, e_3 \rangle_{(n_{12})}}$$

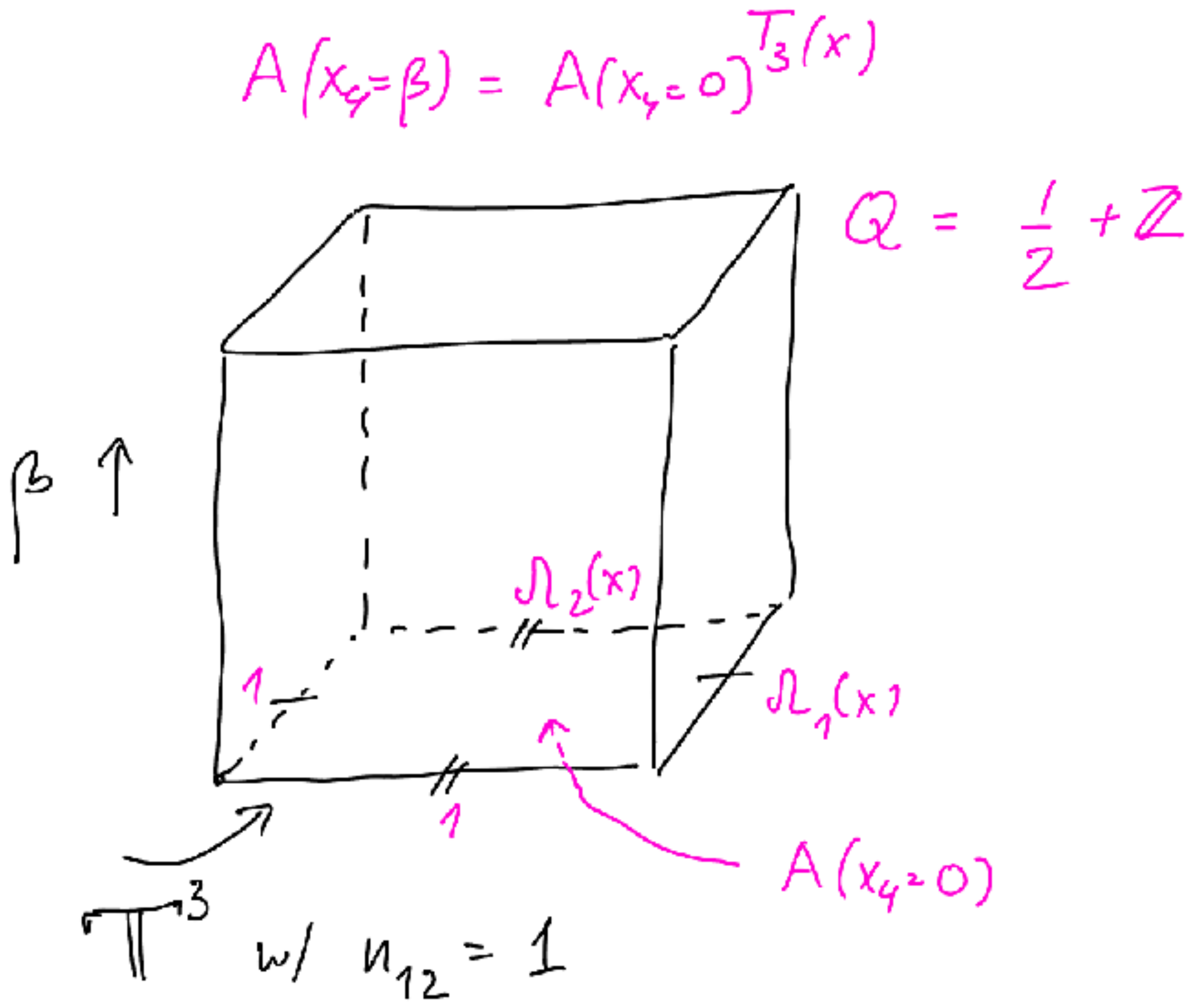
hence this product is same for e_3=0,1

$\hat{X}|E,0\rangle_{(n_{12})} \sim |E,1\rangle_{(n_{12})}$ and $\hat{X}\,\lambda^2\,\hat{X}^\dagger = -\lambda^2$ imply that λ^2 has opposite signs in degenerate flux states

$$\langle \lambda^2 \rangle_{n_{12},n_{34}} = 2 \sum_E (-)^F e^{-\beta E} \, \langle E,0 | \lambda^2 | E,0 \rangle_{(n_{12})}$$

normalize by path integral without λ^2 and $\hat{T}_3$ (i.e. no n_{34} twist, only n_{12}), i.e. Witten index

$$\langle 1 \rangle_{n_{12},0} = \mathrm{Tr}_{\mathscr{H}_{n_{12}}} \, e^{-\beta H}(-1)^F = \sum_{E;e_3=0,1} (-)^F e^{-\beta E} \, \langle E, e_3 | E, e_3 \rangle_{(n_{12})} = 2$$



$Q \in Z + 1/2$

$$\frac{\mathrm{Tr}_{\mathscr{H}_{n_{12}}} e^{-\beta H}(-1)^F \hat{T}_3\, \lambda^2}{\mathrm{Tr}_{\mathscr{H}_{n_{12}}} e^{-\beta H}(-1)^F} = \frac{\int_{n_{12}=1, n_{34}=1} \mathscr{D}A\mathscr{D}\lambda\, e^{-S}\, \lambda^2}{\int_{n_{12}=1, n_{34}=0} \mathscr{D}A\mathscr{D}\lambda\, e^{-S}}$$

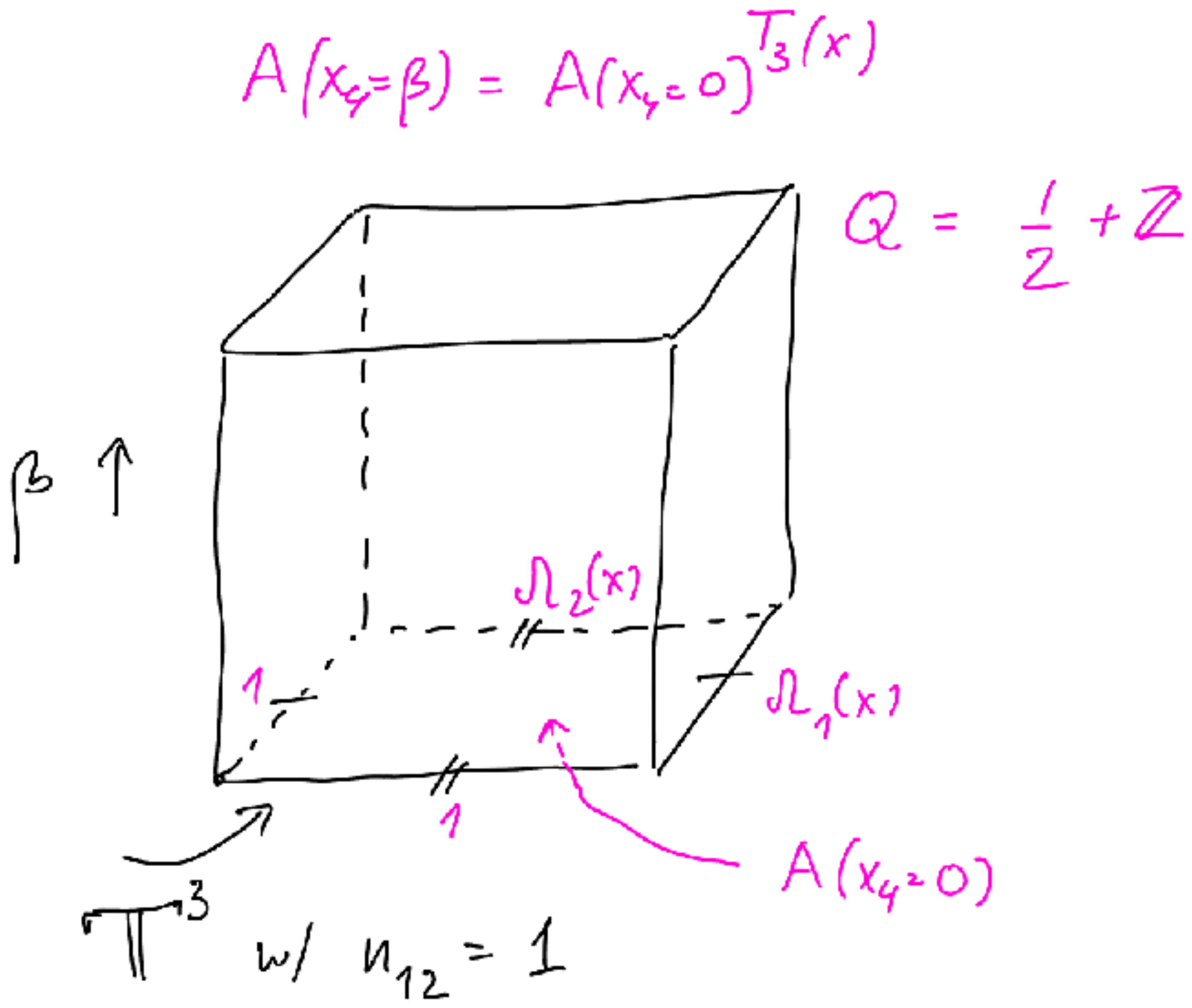
$Q \in Z$

$$\equiv \langle \lambda^2 \rangle = \sum_E (-)^F e^{-\beta E} \langle E, 0 | \lambda^2 | E, 0 \rangle_{(n_{12})}$$

semiclassical expansion expected to hold at small T^4 ("femtouniverse")

$Q = \frac{1}{2}$, the leading contribution to numerator, will have two undotted λ zero modes

we shall discuss this calculation… but first the big picture



$Q \in Z + 1/2$

$$\frac{\mathrm{Tr}_{\mathscr{H}_{n_{12}}}\, e^{-\beta H}(-1)^F \hat{T}_3\, \lambda^2}{\mathrm{Tr}_{\mathscr{H}_{n_{12}}}\, e^{-\beta H}(-1)^F} = \frac{\int_{n_{12}=1,n_{34}=1} \mathscr{D}A\mathscr{D}\lambda\, e^{-S}\, \lambda^2}{\int_{n_{12}=1,n_{34}=0} \mathscr{D}A\mathscr{D}\lambda\, e^{-S}}$$

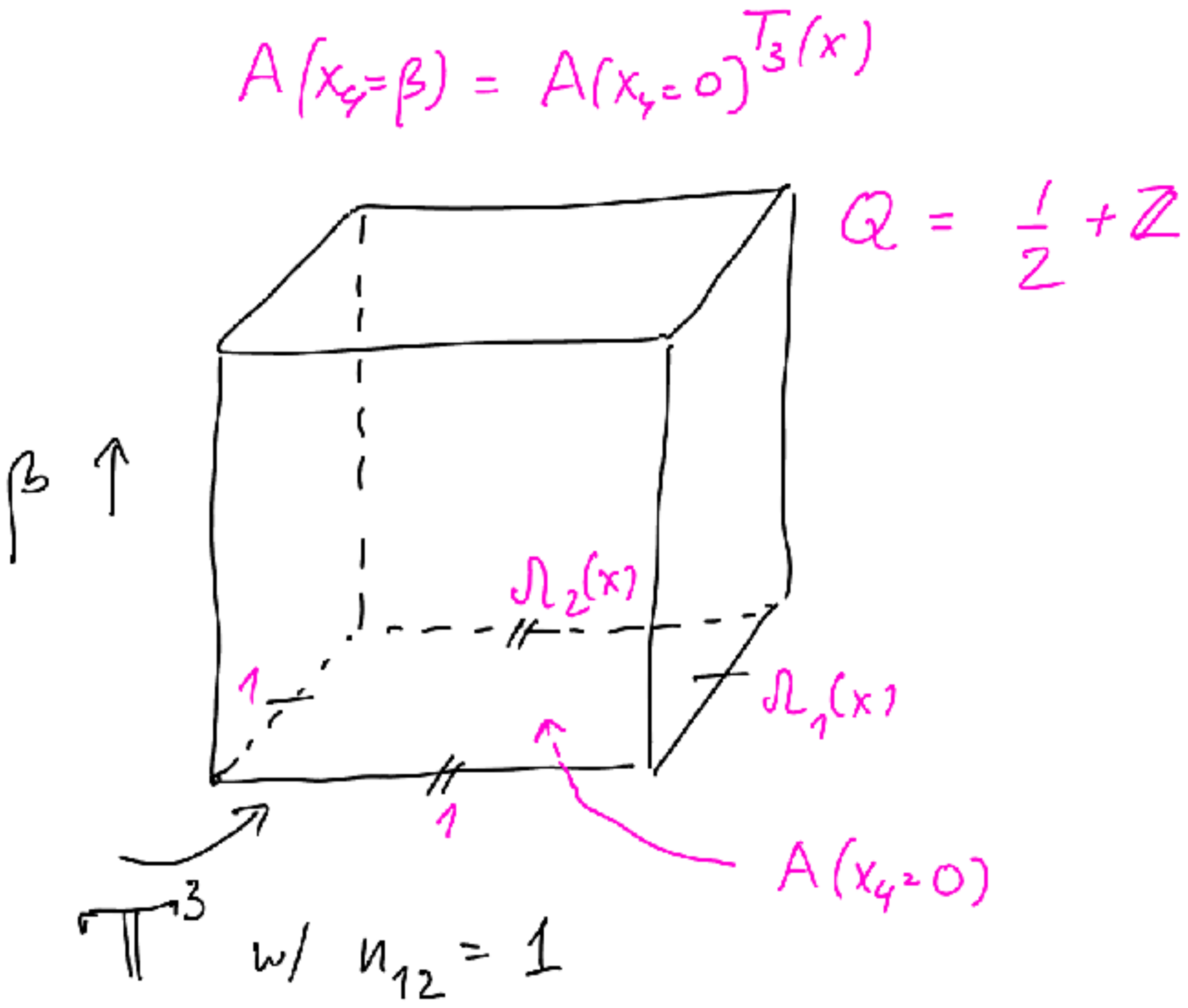
$Q \in Z$

$$\equiv \langle \lambda^2 \rangle = \sum_E (-)^F e^{-\beta E} \langle E,0 | \lambda^2 | E,0 \rangle_{(n_{12})}$$

take β infinite: only E=0

take $L_{1,2,3}$ infinite:
R^4 gaugino condensate in one of the vacua

$Q \in Z + 1/2$



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take β infinite: only E=0

take $L_{1,2,3}$ infinite:
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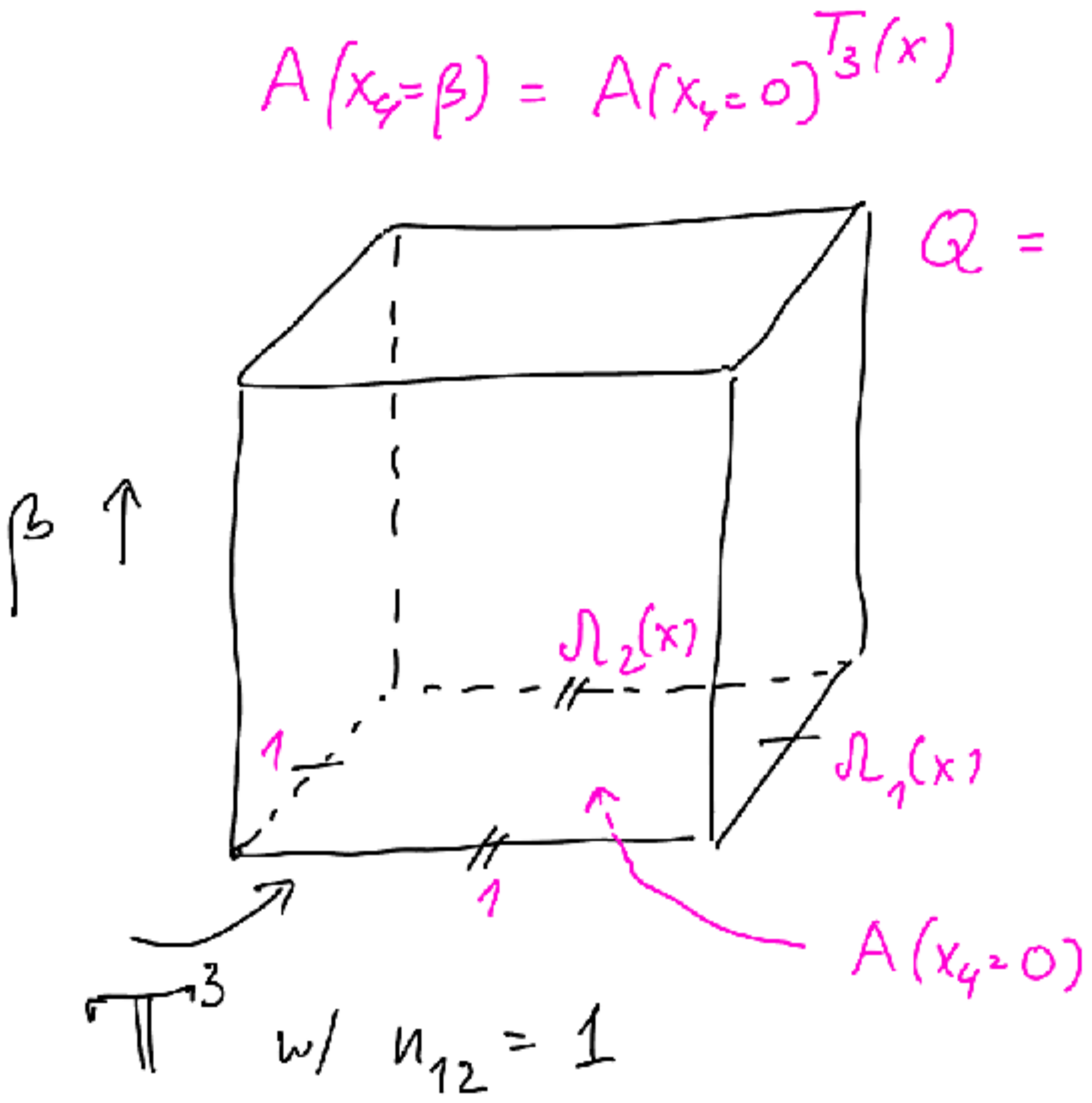
semiclassical calculation in femtouniverse limit
- made assumptions, stated later!

+ argue that result is L_μ, g_{YM}-independent



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$Q = \dfrac{1}{2}$, the leading semiclassical contribution to numerator, w/ two undotted λ zero modes.

what are these instantons?

't Hooft, 1981, $Q = \dfrac{1}{2}$ constant flux background

BPS if symmetric T^4: $L_1 L_2 = L_3 L_4$

$$\bar{A}_n(x,z) = \bar{A}^3_n(x,z)\frac{\tau^3}{2} : \quad \bar{A}^3_1 = \frac{2\pi x_2}{L_1 L_2} + \frac{z_1}{L_1}, \quad \bar{A}^3_2 = \frac{z_2}{L_2}, \quad \bar{A}^3_3 = \frac{2\pi x_4}{L_3 L_4} + \frac{z_3}{L_3}, \quad \bar{A}^3_4 = \frac{z_4}{L_4}.$$

moduli

$$F^{(0)}_{mn} = \frac{\tau^3}{2}\begin{pmatrix} 0 & -\frac{2\pi}{L_1 L_2} & 0 & 0 \\ \frac{2\pi}{L_1 L_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2\pi}{L_3 L_4} \\ 0 & 0 & \frac{2\pi}{L_3 L_4} & 0 \end{pmatrix}$$

Commun. Math. Phys. 81, 267–275 (1981)

Some Twisted Self-Dual Solutions for the Yang-Mills Equations on a Hypertorus*

such an action. All our solutions will be represented in a suitably chosen gauge that makes them look essentially translationally invariant and Abelian. However, considering the difficulty we had in finding them it looked worth-while to publish the result.

… SU(N) generalizations

't Hooft, 1981, $Q = \dfrac{1}{2}$ constant flux background

BPS if symmetric T^4: $L_1L_2 = L_3L_4$

BPS - minimum action for given Q
- preserves 1/2 SUSY

(SYM: B/F det's of nonzero modes cancel, up to power of PV regulator mass)

attempting symmetric T^4 … all looks bad!

- find 4 λ and 2 $\bar{\lambda}$ zero modes

(explicit, 2210.13568)

- these source gauge field EOM… lifted? how?

(we don't know!)

- $L_1L_2 = L_3L_4$ does not allow taking some interesting limits, e.g., $R^2 \times T^2_{n_{12}}$ *Tanizaki Ünsal 2022* 't Hooft, 1981, $Q = \frac{1}{2}$ constant flux background

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Tanizaki Ünsal 2022

Cohen, Gomez 1984 gave an expression using this solution ("toron") ***unaware (?) of subtleties mentioned, or of coefficient.***

In any case, since Hilbert space at finite $T^3_{n_{12}}$ was not understood at the time, interpretation would have been difficult.

’t Hooft, 1981, $Q=\frac{1}{2}$ constant flux background

BPS if symmetric T^4: $L_1L_2 = L_3L_4$

attempting symmetric T^4 … all looks bad! $\longrightarrow$

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Tanizaki Ünsal 2022

González-Arroyo, Pérez, Pena 2000

deform the symmetric T^4, impose BPS:

- only 2 λ zero modes

- no source term in YM field EOM

- $L_1L_2 \neq L_3L_4$, so can take limits

Sounds fantastic!?

There is "bad news," too: deformed-T^4 analytic BPS solution is only known to leading order in

$$\Delta = \frac{L_3 L_4 - L_1 L_2}{\sqrt{V}}$$

for SU(2), there is numerical evidence for uniqueness and convergence upon comparing to "exact" (=numerical) solution for $\Delta \leq 0.08\ldots$ so, for now, we stick with SU(2)

Remark:

If there were general statements known about the moduli space of $Q = \frac{r}{N}$ instantons on T^4, one could do certain calculations in SYM only using this knowledge (not explicit form of solutions) as integrals for some correlators reduce to those over bosonic and fermionic moduli.

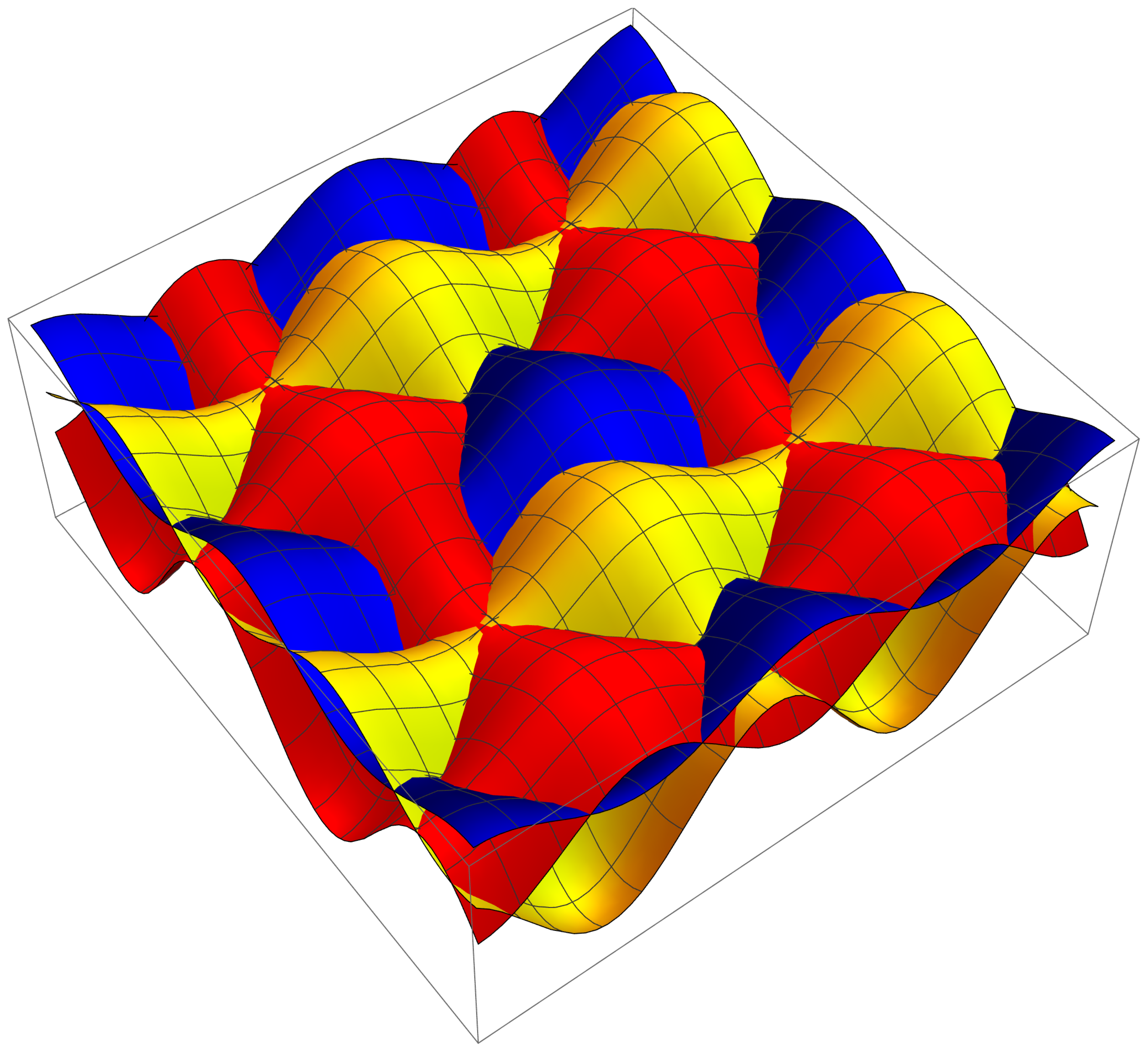
Alas…not known!

hence, we proceed by "trial and error" (consistency)

(as I'll discuss, our results may be taken to suggest that it is here where we likely need help!)

As an aside

at order Δ^1, gauge invariant densities (constant at Δ^0) acquire x-dependence



this is Q=3/N, in SU(N>3), 12 moduli are positions of 3 lumps
(yellow, red, blue; 2-torus shown doubled in size)

see Anber, EP 2307.04975

$$\Delta = \frac{L_3 L_4 - L_1 L_2}{\sqrt{V}}$$

Anber, EP 2210.13568:

deforming the symmetric T^4, we find

- only 2 λ (no $\bar{\lambda}$) zero modes
- four translational moduli z_n

explicit expressions to O(Δ)

SUSY

all orders
Δ-independence
of action

- *measure Δ-independent to all orders*
- *condensate Δ-independent to all orders*

argument assumes
convergence (+ uses SUSY)

$$\int(\text{moduli})\ \mathrm{tr}\,\lambda\lambda\Big|_{\substack{Q=\frac{1}{2}\\ \text{instanton}}} \sim \frac{1}{4g^2}\int d^4x\ \mathrm{tr}\,F^2\Big|_{\substack{Q=\frac{1}{2}\\ \text{instanton}}} = \frac{1}{2}\,S_{\substack{BPST\\ \text{inst.}}}$$

$$\left(=\frac{4\pi^2}{g^2}\right)$$

pure YM, Hamiltonian argument:

$$\langle W_1 \rangle_{n_{12},n_{34}} = \mathrm{Tr}_{\mathcal{H}_{n_{12}}} e^{-\beta H_\theta} \hat{T}_3 W_1 = 0, \text{ as } \langle E, \vec{e} \,|\, W_1 \,|\, E, \vec{e} \rangle = 0$$

Most importantly: range of moduli?

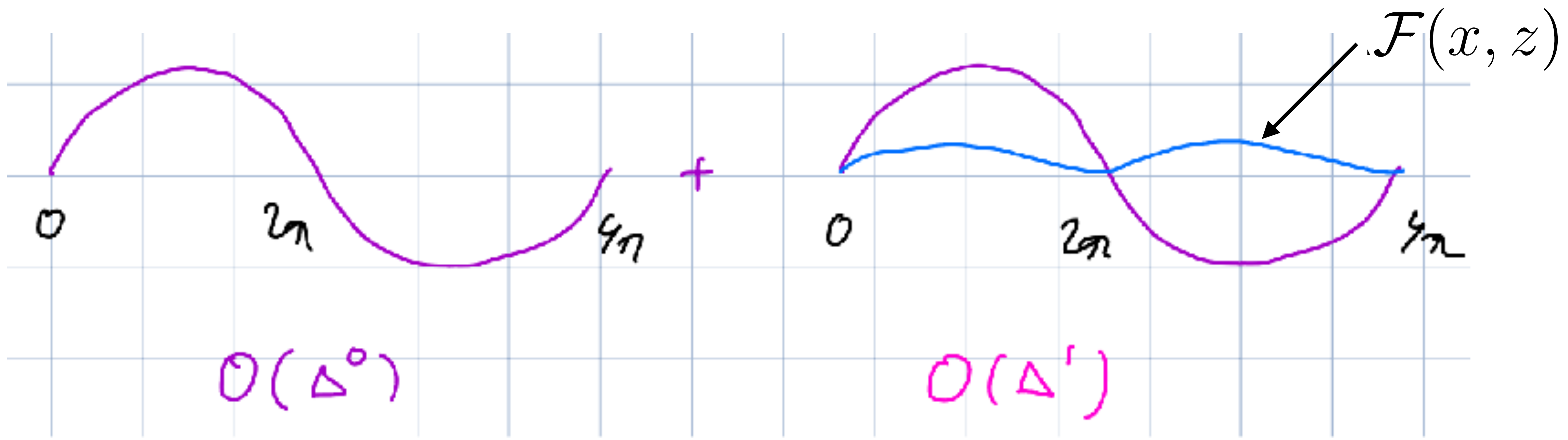
- to find range of z_n moduli, require $\langle W_\mu \rangle = 0$ in pure-YM theory in femtouniverse with twists (use uniqueness):

$$e^{-\frac{4\pi^2}{g^2} - i\frac{\theta}{2}} \frac{V}{g^4} \int_M \prod_{k=1}^{4} dz_k \, W(x, z, C_{n_1,n_2,n_3,n_4}) + \text{h.c.} = 0 \; (\forall x, \theta) \quad \text{iff } z_k \in (0, 4\pi)$$

winding loop in Q=1/2
self-dual background

$$\begin{aligned} &W(x, C_{n_1,n_2,n_3,n_4}) \\ &= 2\cos\left[\frac{1}{2}\left(n_1(z_1 + \frac{2\pi x_2}{L_2}) + n_2(z_2 - \frac{2\pi x_1}{L_1}) + n_3(z_3 + \frac{2\pi x_4}{L_4}) + n_4(z_4 - \frac{2\pi x_3}{L_2})\right)\right] . \\ &\times [1 + \Delta \mathcal{F}(x, z)] \; . \end{aligned} \qquad (5.5)$$

f-n of $z_1 + \dfrac{2\pi x_2}{L_2}$, etc., 2π periodic



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winding loop in Q=1/2
self-dual background

- range of moduli found by demanding vanishing of Wilson loop vevs in pure-YM, is equivalent to that found by demanding that there exist gauge invariants, evaluated in solution background, differentiate between all points $(0,4\pi)$ - i.e., we are not integrating over gauge equivalent values of moduli

Remark: Range of z_n moduli $(0,4\pi)$ means that instanton wraps twice around each direction of torus.
Local gauge invariants identify $z \sim z + 2\pi$, but ones dressed by Wilson loops see difference.

Recall what we compute (factor of 2 from Witten index already divided out)

$$\langle \lambda^2 \rangle = \sum_E (-)^F e^{-L_4 E} \langle E,0 | \lambda^2 | E,0 \rangle \Big|_{n_{12}=1,\ V_3=L_1L_2L_3,\ \frac{L_3L_4 - L_1L_2}{\sqrt{L_1L_2L_3L_4}} \ll 1,\ L_i\Lambda \ll 1}$$

all qualifications stated!

Collecting everything, we find

$$\langle \lambda^2 \rangle = 32\pi^2\Lambda^3 = 2 \times \underline{\underline{16\pi^2\Lambda^3}}$$

two times the $R^4, R^3 \times S^1$ result of weak-coupling calculations, all use same def. of scale $\Lambda^3 = \frac{M_{PV}^3}{g^2} e^{-\frac{4\pi^2}{g^2}}$

Recall what we compute (factor of 2 from Witten index already divided out)

$$\langle \lambda^2 \rangle = \sum_E (-)^F e^{-L_4 E} \langle E,0 | \lambda^2 | E,0 \rangle \Big|_{n_{12}=1,\ V_3 = L_1 L_2 L_3,\ \frac{L_3 L_4 - L_1 L_2}{\sqrt{L_1 L_2 L_3 L_4}} \ll 1,\ L_i \Lambda \ll 1}$$

to get to R^4, say, take $L_4 \to \infty$, obtaining

$$\langle \lambda^2 \rangle = (-)^F \langle 0,0 | \lambda^2 | 0,0 \rangle_{n_{12}=1,\ V_3 = L_1 L_2 L_3}$$

then, take $V_3 \to \infty$

there's a discrepancy only if "nothing happens" while these limits are taken

Collecting everything, we find

$$\langle \lambda^2 \rangle = 32\pi^2 \Lambda^3 = 2 \times \underset{R^4,\, R^3 \times S^1}{\underline{\underline{16\pi^2 \Lambda^3}}}$$

argue that result is L_μ, g_{YM}-independent?

Recall what we compute (factor of 2 from Witten index already divided out)

$$\langle \lambda^2 \rangle = \sum_E (-)^F e^{-L_4 E} \, \langle E,0 | \lambda^2 | E,0 \rangle \Big|_{n_{12}=1,\ V_3=L_1L_2L_3,\ \frac{L_3L_4 - L_1L_2}{\sqrt{L_1L_2L_3L_4}} \ll 1,\ L_i\Lambda \ll 1}$$

Collecting everything, we find

$$\langle \lambda^2 \rangle = 32\pi^2\Lambda^3 = 2 \times \underbrace{\underline{\underline{16\pi^2\Lambda^3}}}_{R^4,\, R^3 \times S^1}$$

<u>Holomorphy on T^4?</u>

$$\Lambda^* \frac{d}{d\Lambda^*} \langle \lambda^2 \rangle \sim \langle \lambda^2 F^* \rangle \sim \langle \lambda^2 \bar{Q}_{\dot\alpha} \bar\psi^{\dot\alpha} + \lambda^2 \bar\psi^{\dot\alpha} \bar{Q}_{\dot\alpha} \rangle \sim \langle \bar{Q}_{\dot\alpha} \lambda^2 \bar\psi^{\dot\alpha} + \lambda^2 \bar\psi^{\dot\alpha} \bar{Q}_{\dot\alpha} \rangle \underset{\text{on } R^4}{=} 0$$

T^3: for each E, e_3, $\sum_{\text{over states w/ given } E,\, e_3} (-)^F \langle E | X_{\dot 2} \bar{Q}_{\dot 1} + \bar{Q}_{\dot 1} X_{\dot 2} | E \rangle = 0$, states $\in$ reps. of $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = \delta_{\alpha\dot\beta} E$

—> holomoprphy on T^4 as well, $\langle \lambda^2 \rangle = c\Lambda^3$, holomorphy -> no $L|\Lambda|$-dependence

holomorphy argument appears known/ obvious to S.&V., the authors of 1986 "Solution of anomaly puzzle…"

Holomorphy wrt Λ leaves open dependence on dim-less ratios, like Δ, but seen not to occur…

thus, we seem to have a problem…

- we made an algebraic mistake (all factors spelled out in glory detail in paper)
- there is a loophole in L_i-independence argument?
- misidentified moduli space? (missed some global identification? **need rationale?**)
- other backgrounds contribute?

- to boot, using one (no numeric study of uniqueness here!) of 't Hooft SU(N) solutions (+ Δ…) we find

$$\langle \lambda^2 \rangle = N \times 16\pi^2 \Lambda^3$$

N times the $R^4, R^3 \times S^1$ weak coupling instanton result,
in the usual normalization (N-fold degeneracy divided out, as in SU(2))

SUMMARY:

one of two weakly-coupled calculations of $\langle \lambda^2 \rangle$: continuous connection to R^4 $\longleftrightarrow$ using this new and deeper knowledge, revisit old (1984!) calculations of $\langle \lambda^2 \rangle$ on T^4

$\langle \lambda^2 \rangle_{R^4}$ $\quad$ $\langle \lambda^2 \rangle_{T^4}$

$$\underline{\underline{\langle \lambda^2 \rangle_{T^4} = 2 \times \langle \lambda^2 \rangle_{R^4} \text{ for SU(2)}}}$$

why?

important for pushing & checking 'adiabatic continuity' program qualitatively

FUTURE:

wish for better understanding of fractional charge instantons, semiclassics, and their role in gauge dynamics (for which some evidence has accumulated)

input from math-phys/string?

(as in Dp-4 inside Dp <-> ADHM…; fractionalization of BPST on Coulomb branch)