

"Invertible & noninvertible symmetries  
generalized anomalies for pedestrains:  
a view from the tones"

- ➊ symmetries are important
- ➋ 't Hooft anomalies  $\Rightarrow$  IR constraints  
(QCD, Seiberg dualities, compositeness)
- ➌ 2017 - now : new 't H. anomalies  
involving generalized symm

❹ exciting from general QFT point of view

this talk : - a very old-fashioned take on  
new stuff

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↙  
M.A. talk will take a more modern  
perspective

the main points I want to make here

- (1) Some new generalized anomalies can be  
understood using canonical quantiz'n of  
YM + matter on  $T^3$  (space) w/ twisted b.c.  
 $\equiv$  't Hooft fluxes/bckgd. for 1-form symm /

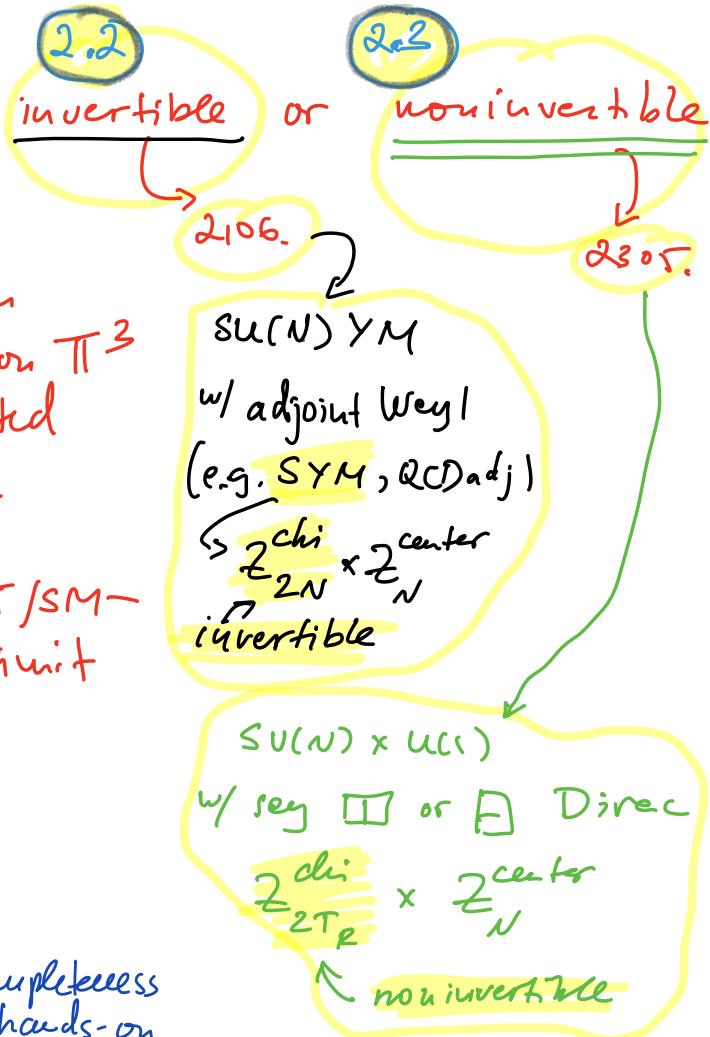
(2) anomalies between chiral symmetries  
lead to

exact degeneracies in  
the Hilbert space on  $\mathbb{T}^3$   
in appropriately twisted  
background; these hold  
at any size  $\mathbb{T}^3$

- very unusual in QFT/SM—  
& therefore in  $V \rightarrow \infty$  limit  
as well  $\Rightarrow$   
 $\Rightarrow \propto$  volume phase  
structure affected

2.1

- no pretense on generality/completeness
- focus on examples & hands-on understanding

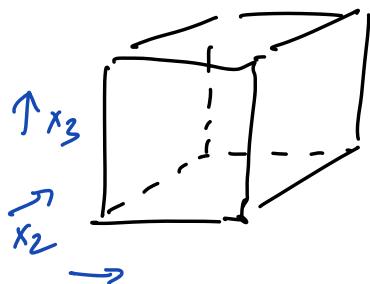


in 2.3:

mixed anomaly  
between noninvert.  
chiral & center  
 $\Rightarrow$  exact degeneracy  
(as far as I know,  
1st ex. in 4d YM)  
+ nonzero  $\langle \psi_B \psi_\eta \rangle$  !

- begin trying to be pedagogic, but necessarily skip some detail (there's too much)

- $T^3$  hamiltonian quantization



1980s: ... 't Hooft, Lüscher,  
van Baal, Witten ...

(1) center symmetry & "electric" flux sectors

(2) 't Hooft twists & "magnetic" flux sectors.

(1) center symmetry

$$\hat{T}_i, i = 1, 2, 3$$

- "gauge trfs" periodic up to center of  $SU(N)$  ( $\mathbb{Z}_N$ )

- do not act on local gauge inuts  $\text{tr}(F_{\mu\nu} F_{\alpha\beta})$ , etc...

- but:

$$\hat{W}_i = \text{tr}_{\square} P e^{\int_0^{L_i} A_i dx^i}$$

this is a  
global symm!  
(acts on  $W_i$ )

$$\hat{T}_i \hat{W}_i \hat{T}_i^{-1} = \text{tr}_{\square} P T_i(0) e^{\int_0^{L_i} A_i dx^i} T_i^{-1}(L_i) = e^{i \frac{2\pi}{N}} \hat{W}_i$$

$$\therefore \hat{T}_i \hat{w}_i \hat{T}_i^{-1} = e^{i \frac{2\pi}{N} \delta_{ij}} \hat{w}_j$$

since  $\hat{\psi}_{\text{adj.}}^{T_i} = \hat{T}_i \hat{\psi}_{\text{adj.}} T_i^{-1}$ ,  $\hat{\psi}_{\text{adj.}}^{T_i}$  &  $\hat{\psi}_{\text{adj.}}$  same b.c., so

in a theory w/ adjoints only  $[\hat{T}_i, \hat{H}] = 0$

$\therefore \hat{T}_i$  generate  $\mathbb{Z}_N^{(1)}$ : "one-form  $\mathbb{Z}_N$ " center

in  $\text{SU}(N=2k)$  w/  $\square$ , only  $\hat{T}_i^{\frac{N}{2}}$ , e.g.  $\mathbb{Z}_2^{(1)}$   
 (since  $\hat{\psi}_{\square}^{\hat{T}_i} \rightarrow \hat{T}_i \hat{\psi}_{\square} \hat{T}_i^T$ )

in  $\text{SU}(N) \times \text{U}(r)$  w/  $\square$  :  $\mathbb{Z}_N^{(1)}$   
 any  $\hat{T}_i = \hat{T}_i^{\text{SU}(N)} \otimes \hat{t}_i^{\text{U}(r)}$

$$\hat{T}_i(x + L_i) = e^{i \frac{2\pi}{N}} \hat{T}_i(x)$$

$$\hat{t}_i(x + L_i) = e^{-i \frac{2\pi}{N}} \hat{t}_i(x)$$

Whenever  $[\hat{T}_i, \hat{H}] = 0$  w/  $\hat{T}_i$  generates  $\mathbb{Z}_N^{(1)}$

label states by  $|E, e_1, e_2, e_3\rangle = |E, \vec{e}\rangle$

$$e_i \in \mathbb{Z} \text{ mod } N$$

where  $\hat{T}_i |E, \vec{e}\rangle = e^{i \frac{2\pi}{N} e_i} |E, e_i\rangle$

$|\vec{e}\rangle$  : "electric flux sectors" of  $T^3$  H.space

name 'cause

$$|e_i\rangle \rightarrow w_i |e_i\rangle$$

$$\hat{T}_i |e_i\rangle = |e_i\rangle e^{i \frac{2\pi}{N} e_i}$$

$$\begin{aligned} \hat{T}_i (w_i |e_i\rangle) &= \\ &= w_i |e_i\rangle e^{i \frac{2\pi}{N} (e_i + 1)} \end{aligned}$$

e.g. in pure YM,  $\theta \neq 0$

as  $L \rightarrow \infty$  only one sector ( $\vec{e} = 0$ ) has finite energy, others  $\sim L$  (confinement)

Done with  $\mathbb{Z}$

(1) center symmetry & "electric" flux sectors

next:

(2) 't Hooft twists & "magnetic" flux sectors.

whenever we have global  $\mathbb{Z}_N^{(1)}$  (or any  $G$ ) can turn on bckgd fields for it

$\mathbb{Z}_N^{(0)} \leftarrow 0\text{-form}$  acts on local op's  $\rightarrow$  gauge field is 1-form

$\mathbb{Z}_N^{(1)} \leftarrow 1\text{-form}$  acts on lines  $\rightarrow$  gauge field is 2-form

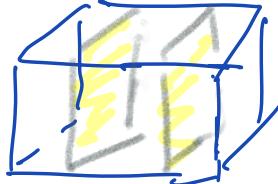
1-form field  $\rightarrow$  gauge inut. noncontractible w/ lines

2-form field  $\rightarrow$  ——— surfaces

$\mathbb{T}^3$  has 3 noncontractible 2-surfaces  
1-2, 1-3, 2-3 planes

$\oint B^{(2)}$  2-form 2<sub>n</sub> gauge field  
 12-plane =  $m_3 \in \mathbb{Z}(\text{mod } N)$   
 $\oint B^{(2)}$  =  $m_2$   
 13-plane  
 $\oint B^{(2)}$  =  $m_1$   
 23-plane  
 same for any 23 plane in  $T^3$   
 ( "topological b.c.", no 3-form flux thru any 3-volume)

these are the 3 types of topological 2-form b.c.s one can turn on



One can quantize gauge theory on  $T^3$   
 w/ any choice  $\vec{m} = (m_1, m_2, m_3)$   
 of 3 mod- $N$  integers

= "discrete magnetic fluxes"  
 or  
 "t Hooft twisted b.c."

skip detail  
use  
analogy:

if one turns on <sup>(topological)</sup> W. line background  
for 1-form gauge field gauging 0-form  
sym.

twisted b.c. (by 0-form) for  
fields charged under 0-form

analogy

turn on background  
2-form gauge field  $\oint B^{(2)}$   
is equiv. to twisted b.c. for  
gauge field in 1-2 plane  $\pi_2$ -plane

lattice  
descript.

continuum.

Summary:  
(so far)

in any gauge theory w/  
1-form  $\mathbb{Z}_N^{(1)}$  Hilbert space  
on  $T^3$ :

$|E, \vec{e}^>$

$\vec{m}$

$B^{(2)}$  topological b.c. for  $\mathbb{Z}_N^{(1)}$

twisted b.c.

eigenvalues  
of  $T_i$ ,  $\mathbb{Z}_N^{(1)}$ -generators

To study our examples, suffices to take

$$\vec{m} = (0, 0, 1) \quad , \text{ e.g. } \sum_{i=1,2,3} \vec{B}^{(2)} \neq 0 \bmod N$$

the crucial observation ('t Hooft) is  
that  $\vec{T} \cdot \vec{m} = \hat{T}_3$  is singled out.

Skip details ... when  $m_3 \neq 0$ ,

$$\hat{T}_3(\vec{x}) : \mathbb{T}^3 \rightarrow G \quad \begin{array}{l} \text{(map obeys} \\ \text{appropriate} \\ \text{gauge} \\ \text{group} \end{array} \quad \begin{array}{l} \text{to} \\ \vec{m} = (0, 0, 1) \\ \text{b.c. on } \mathbb{T}^3 \end{array}$$

has fractional winding #:

$$Q(T_3) = \int_{\mathbb{T}^3} d^3x \operatorname{tr}(T_3 d T_3^{-1})^3 \frac{1}{2\pi r^2} = \frac{m_3}{N} (\bmod \mathbb{Z})$$

explaining this necessitates an excursion into details: b.c.'s + transition functions ...

(the  $\hat{T}_1$  &  $\hat{T}_2$  do not have such property.)

Because of this,

$$\frac{1}{T_3} e^{i 2\pi \int_{\mathbb{T}^3} \text{tr}(\hat{A} d\hat{A} + \dots)} \frac{1}{T_3^{-1}}$$

is nontrivial

- later call this  $d^3x K^0$
- $\int d^3x K^0(A)$  changes by integer under large gauge trf's w/ integer winding #

because of  $Q(\frac{1}{T_3}) = \frac{m_3}{N}$ , we have

$$\frac{1}{T_3} e^{i 2\pi \int d^3x K^0(\hat{A})} \frac{1}{T_3^{-1}} = e^{i 2\pi \frac{m_3}{N}} e^{i 2\pi \int d^3x K^0(\hat{A})}$$

this operator "shifts  $\theta$  by  $2\pi$ "

$$U_1 | \psi \rangle = | \psi \rangle e^{i\theta}$$

$$U_1 (e^{i2\pi \int K^a d^3x} |\psi\rangle) =$$

$$= | \psi \rangle e^{i(\theta + 2\pi)}$$

" $\theta$ - periodicity anomaly"

Why does matter?

Ex. 1 SYM

$m_3 = 1$  from now on.

$$\partial_\mu j_f^\mu = \partial_\mu (\bar{\lambda}^a \sigma^\mu \lambda^a) = 2N \partial_\mu K_\mu$$

$$J_S^\mu = j_f^\mu - 2N K^\mu \quad \text{conserved}$$

but  $Q_S = \int d^3x (j_f^0 - 2N K^0)$  <sup>not gauge invt.</sup>

however

$$e^{i \frac{2\pi}{2N} Q_S} = e^{i \frac{2\pi}{2N} \int d^3x j_f^\circ - i 2\pi \int K^0 d^3x}$$

↓

is gauge inut. →  $\mathbb{Z}_{2N}^{(0)}$ : discrete chiral

$$\hat{X}_{2N} = e^{i \frac{2\pi}{2N} \int d^3x j_f^\circ} e^{-i 2\pi \int K^0 d^3x}$$

↗  $\hat{T}_3 \hat{X}_{2N} \hat{T}_3^{-1} = e^{-i \frac{2\pi}{N}} \hat{X}_{2N}$

(mixed 1-form center)  $\mathbb{Z}_{2N}$  chiral anomaly in  $H$ -space on  $T^3$  w/  $m_3 = 1$

- no 1-dim reps!

- recall

$$\hat{T}_3 |e_3\rangle = e^{i \frac{2\pi}{N} e_3} |e_3\rangle$$

$$\hat{T}_3 (\hat{X}_{2N} |e_3\rangle) = \underbrace{(\hat{X}_{2N} |e_3\rangle)}_{\hat{T}_3^{-1} \hat{T}_3 = \mathbb{I}} e^{i \frac{2\pi}{N} (e_3 - 1)}$$

state w/  $e'_3 = e_3 - 1$

of the same energy ( $[\hat{X}_{2N}, \hat{H}] = 0$

$$[\hat{T}_3, \hat{H}] = 0)$$

$\Rightarrow$  all states on  $T^3$

w/  $m_3=1$  are  $N$ -fold degenerate

exactly, @ any  $L^3$ !

2/06  
paper  
+ ...

• degeneracy remains as  $L \rightarrow \infty$

• if center unbroken, clustering vacua are the

$|0, e_3\rangle$  ones  $e_3 = 0, \dots, N-1$

•  $L^3 \rightarrow \infty$  shouldn't care about b.c.  $\hat{X}_{2N}$  is broken

$\Rightarrow$  all follows from anomalies.

this was my invertible ex. ( $\mathcal{Z}_{2N}^{(0)}$   
is invertible)

(similar story in YM @  $\theta = \pi$ ,  
other gauge groups w/ centers etc...).

BUT

now to the noninvertible ex:

Ex. 2 :  $SU(N) \times U(1)$  w/  $\square$  Dirac

below:  $T_R = T_A = N-2$ ;  $d_R = \frac{N(N-1)}{2}$

$$\partial_\mu j_\mu^\mu = 2T_R \partial_\mu K^\mu(A) + 2d_R \partial_\mu K^\mu(a)$$

$\uparrow$  SU(N)CS  $\uparrow$  U(1)CS

$\uparrow$  axial  $U(1)$

w/out  $U(1)$ , a  $\mathbb{Z}_{2T_R}^{(o)} \subset U(1)_A$  is

anomaly free

$X_{2T_R}$  is invert. wrt large  
transf's, like SYM

w/ the  $U(r)$ :

$$X_{2T_R}[A, a] = e^{i \frac{2\pi}{2T_R} \int d^3x j_\mu^\text{f}} e^{-i \frac{2\pi}{2T_R} \int d^3x K^0(A)} \\ \times e^{i \frac{2\pi}{2T_R} \int d^3x K^0(a) + \text{b.t.}[a]}$$

↑  
(boundary terms)

BUT: generator of  $\mathbb{Z}_{2R}^{(0)}$  not inut. under  
 $U(r)$  winding gauge transforms.

$$X_{2T_R}(A, a - d\lambda_{\vec{n}}) = X_{2T_R}(A, a) \times \\ \times e^{-i 2\pi n_z \frac{dR}{T_R} \left( m_{12} + \frac{2}{N} m_3 \right)} \\ \times \left( z \rightarrow x \rightarrow y ; \begin{matrix} 1 \mapsto 2 \mapsto 3 \\ \dots \end{matrix} \right)$$

$U(r)$  magn.  
 $d\lambda_x$

$\oint B_{12}^{(2)}$

Theory has  $\mathbb{Z}_N^{(1)}$ , w/  $U(r)$  &  $su(N)$  acting  
 in concert making  $\Psi_\Box$  invariant;  
 turned on  $\oint B_{12}^{(2)}$  &  $U(r)$  magn. flux

- to make gauge inv.,

(A.)

$\sum$  over gauge copies

or

(B.)

couple to a 3d TQFT ( both approaches  
in 2305 paper )

$$\left( " A_{T_R d_R} \left( \frac{da}{T_R} \right) " \right)$$

here: (A.)

'cause  
leads to  
anomaly !

then we define

$$\tilde{X}_{2\overline{T}_R}$$

generates

noninvertible  
 $\tilde{\sum}^{(1)}$

$$= \sum_{\vec{n} \in \mathbb{Z}} X_{2\overline{T}_R}(A, a - d\lambda_{\vec{n}})$$

$\uparrow$   
U(1) winding trf  
w/ winding  $\vec{n} \in \mathbb{Z}$

calculate ...

$$\tilde{X}_{2T_R} = X_{2T_R} \sum_{n_2 \in \mathbb{Z}} e^{-i 2\pi \frac{d_R}{T_R} n_2 \left(m_{12} + \frac{2}{N} m_3\right)}$$

Poisson resum:

$$\sum_{l_2 \in \mathbb{Z}_L} \delta \left( \frac{d_R}{T_R} \left( m_{12} + \frac{2}{N} m_3 \right) - l_2 \right)$$

notice  $\tilde{X}_{2T_R}$  vanishes unless

$$\frac{d_R}{T_R} \left( m_{12} + \frac{2}{N} m_3 \right) \in \mathbb{Z}$$

► annihilates  $U(1)$  magnetic flux  
+  $\mathbb{Z}_N^{c1}$  center flux sectors.

on the other hand, for bcds s.t.

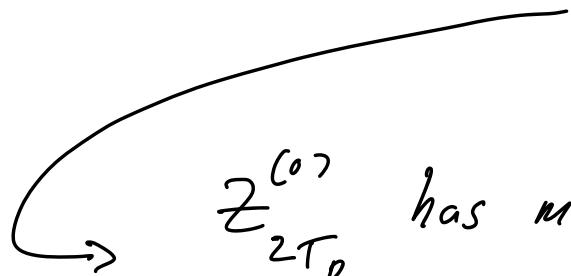
$$l_2 = \frac{d_R}{T_R} \left( u_{12} + \frac{2}{N} u_3 \right) \in \mathbb{Z}$$

operator is gauge inv. ;

hence  $X_{2T_R}$  projects  $\mathcal{H}$ -space on selected flux sectors. ; when  $l_2 \in \mathbb{Z}$ ,

op'r is gauge inv. & acts the

usual way  $\Leftrightarrow$  "noninvertible"



$Z_{2T_R}^{(0)}$  has mixed anomaly w/ U(1)

global ; after gauging

the U(1) - becomes noninvertible

[see M.A.s talk : gauging center  $m$

theories w/ mixed  $\mathbb{Z}_q^{(0)} - \mathbb{Z}_p^{(1)} \dots$   
... similar ]

Back to our gauged - U(1) example ,

what's more interesting , w/ our  
 $m_1, m_3$  U(1)-magnetic,  $\mathbb{Z}_N^{(1)}$ -center  
backgrounds , we  
can compute commutator of

$$\hat{T}_3 = \hat{T}_3 \otimes \hat{T}_3 \leftarrow \mathbb{Z}_N^{(1)} \text{ in } 3^{\text{rd}}$$

$$w/ \quad \tilde{X}_{2T_R}^{\gamma} \quad !$$

$$\tilde{J}_3 \quad \tilde{X}_{2T_K}^{\gamma} \quad T_3 = \tilde{X}_{2T_L}^{\gamma} e^{i 2\pi \left( \frac{2}{N} l_2 - \frac{m_3}{N} \right)}$$

$$w/ \quad l_2 = \frac{d_R}{T_R} \left( m_{12} + \frac{2}{N} m_3 \right)$$

(1)

mixed anomaly between  
 noninvertible  $\mathbb{Z}_{2T_R}^{\text{chiral}}$   
 &  $\mathbb{Z}_N$ -center 1-fn  
 (whenever phase nontrivial)

- in 4D YM, not aware of other exs
- straightforward to see w/  $\mathbb{T}^2$  fluxes

②

as in SYM : anomaly of  
noninvertible chiral - center  $\Rightarrow$

$\Rightarrow$  electric flux state degeneracy  
exact  $\forall L$ !

? implications for phase structure?

- on  $T^3$  turn on  $w_{12}, w_3$  fluxes  
(U(1) magnetic,  $\mathbb{Z}_n^{(1)}$   
center)
- choose them s.t. anomaly  
nontrivial, hence some  $\vec{e}$ -fluxes  
exactly degenerate,  $\forall L^3$  of  $T^3$
- this degeneracy should persist  
as  $L^3 \rightarrow \infty$ ;  $\infty - L^3$  physics

shouldn't depend on b.c., so

constrain IR physics of  $\mathbb{R}^4 \text{YM}_{+..}$

- in our ex's constraints due to noninvertible/center anomaly not as strong as in  $SU(N)$  SYM [ more akin to, e.g.  $Sp(N)$  SYM where only part of  $\infty$ -volume degeneracy revealed @  $T^3$  w/ flux]

( add some discussion, skip if )  
( no time ... )

consequences of mixed anomalies

$$\tilde{J}_3 \tilde{X}_{2T_R} \tilde{T}_3 = \tilde{X}_{T_R} e^{i2\pi \left( \frac{2}{N} l_2 - \frac{m_3}{N} \right)}$$

w/  $l_2 = \frac{d_R}{T_R} \left( m_{12} + \frac{2}{N} m_3 \right)$

- odd- $N$  phase trivial
- even- $N$   $\exists m_{12}, m_3$  s.t. phase is  $\mathbb{Z}_2$  at most

$SU(N) \times U(1)$  w/  $T_R = N-2$  (日)

when  $e^{i2\pi \left( \frac{2}{N} l_2 - \frac{m_3}{N} \right)} = -1$  (N even)

states  $e_3$  &  $e_3 + \frac{N}{2} (\text{mod } N)$  are degenerate

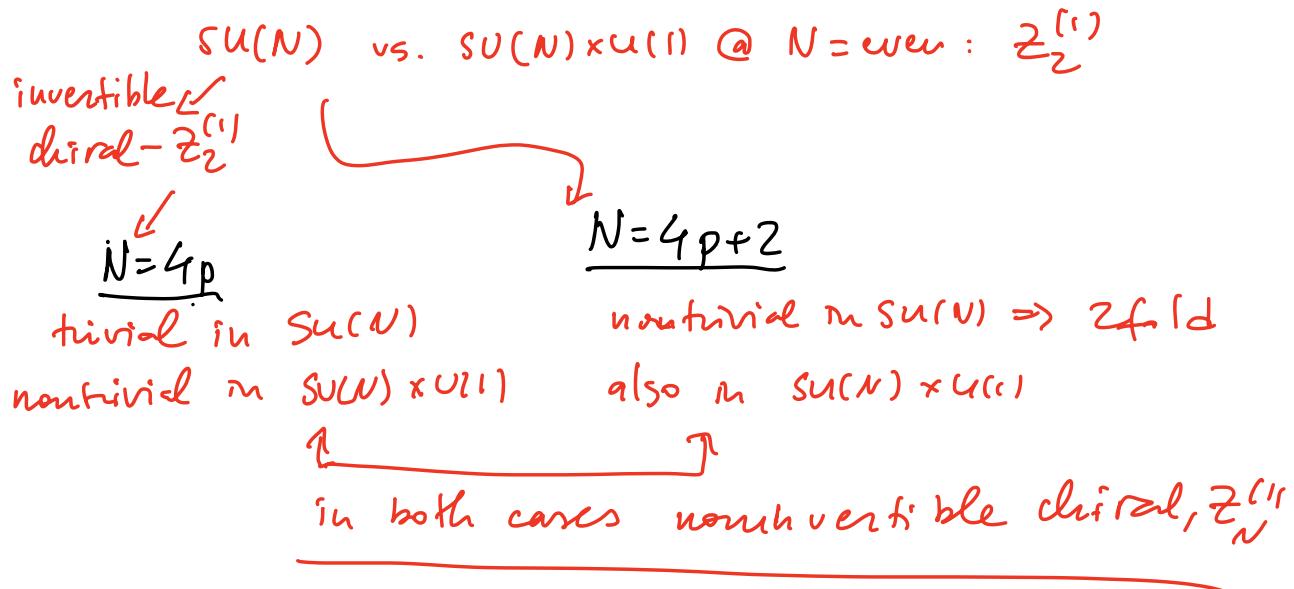
&  $L^3$  & mapped to each other by  $\tilde{X}_{2(N-2)}$ ,

they have nonzero vevs of  $(\psi_{\bar{B}} \cdot \psi_B)^{\frac{N-2}{2}}$

w/ opposite sign

$$\begin{aligned} \langle e_3 | (\psi_{\bar{B}} \cdot \psi_B)^{\frac{N-2}{2}} | e_3 \rangle &= \\ &- \langle e_3 + \frac{N}{2} | (\psi_{\bar{B}} \cdot \psi_B)^{\frac{N-2}{2}} | e_3 + \frac{N}{2} \rangle \end{aligned}$$

→ again this is an exact statement about  
the  $SU(N) \times U(1)$  theory w/  $\bar{\psi}$  Dirac,  $\not{D} L^3$



→ doesn't determine degeneracy @  $L^3 \rightarrow \infty$ ,  
in usual scenarios, there should be  
extra degeneracy emerging as  $L^3 \rightarrow \infty$

## MORAL :

- $\mathbb{T}^3$  w/ t.h. fluxes  
(backgrounds for  $\mathbb{Z}_N^{(c)}$ -center)  
is an interesting way to study  
dynamics @ finite  $L^3$
- consequences of mixed anomalies  
of discrete (parity or diiral)  
invertible & noninvertible  
symmetries &  $\mathbb{Z}_N^{(c)}$ -center  
revealed in finite  $\mathbb{T}^3$  spectrum  
⇒ exact degeneracies!- nontrivial in QFT & Stat. Mech.

① question I don't know  
the answer to: reveal  
anomaly of  $\tilde{Z}_N^{(1)} - \tilde{Z}_{2T_K}$   
noninvertible using the  
coupling to  $A_{d_R, T_R} \left(\frac{dg}{T_R}\right)$  TQFT  
... but someone better versed  
might know ...