Anomalies, tori, and new twists in the gaugino condensate

Erich Poppitz

based on works with

Andrew Cox, F. David Wandler (Toronto)

2106.11442

and Mohamed Anber (Durham)

2210.13568 (w/ some mention of 2307.04795 and work since)

the big picture:

problem of determining the IR phases of gauge theories is complex

matching of anomalies (’t Hooft) constrains any IR fantasy one might have

old story, eg massless QCD: pions!; preons; Seiberg dualities…

the new stuff:

there are new ’t Hooft anomalies,
thus new constraints on IR behavior,
that were missed in the 1980s,
involving higher form symmetries

Gaiotto, Kapustin, Komargodski,Seiberg: 2014-… [GKKS+]

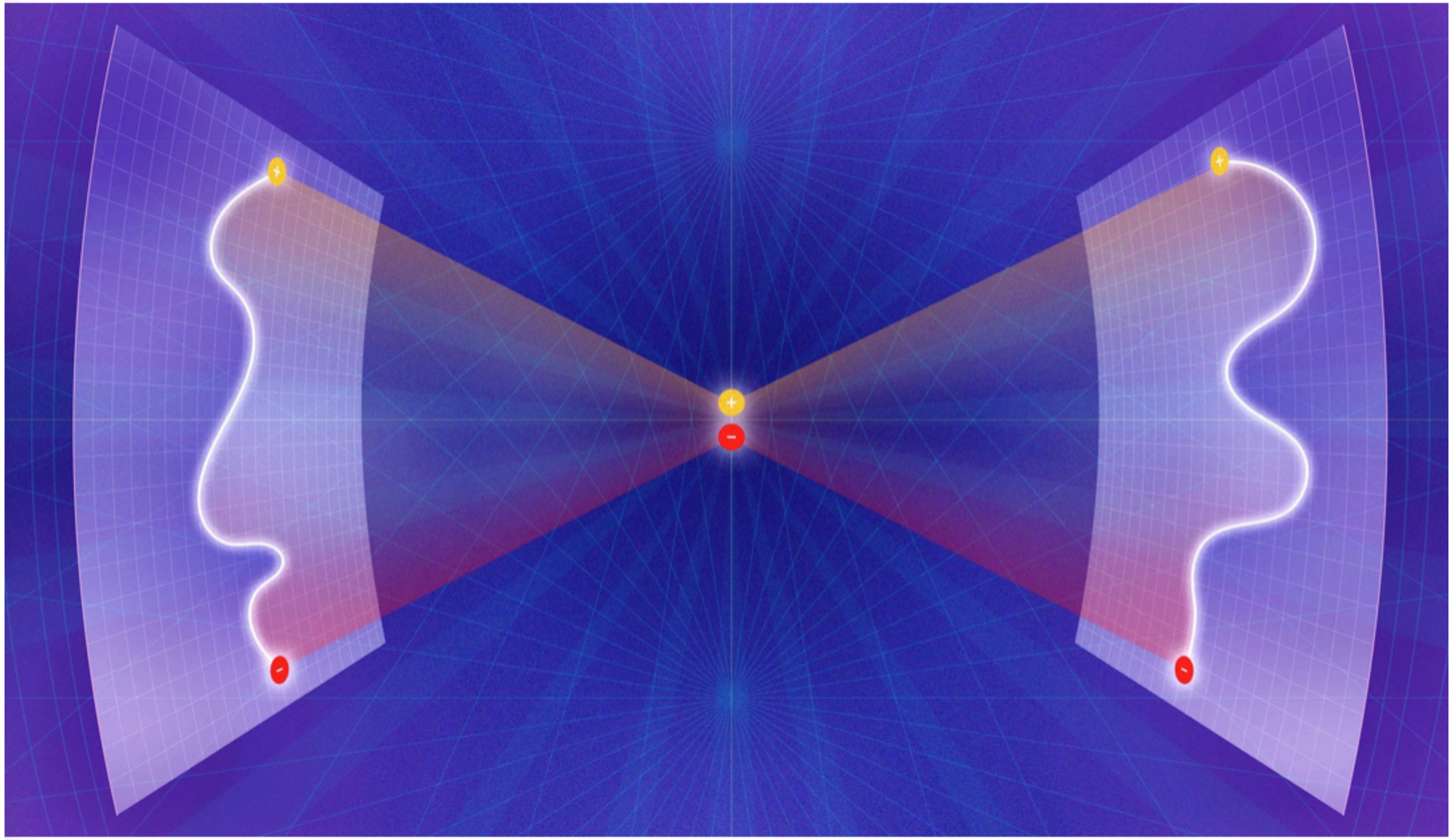
“Quanta”
Spring ’23 —>

MATHEMATICAL PHYSICS

A New Kind of Symmetry Shakes Up Physics

23

So-called “higher symmetries” are illuminating everything from particle decays to the behavior of complex quantum systems.



The symmetries of 20th-century physics were built on points. Higher symmetries are based on one-dimensional lines.

Samuel Velasco/Quanta Magazine

any hype aside, this is exciting from a general QFT point of view

as it gives a new nonperturbative tool to study gauge theories

this talk

1. HOW MIXED ANOMALIES BETWEEN CHIRAL (invertible or not) AND CENTER SYMMETRY ("1-form") ARISE IN HILBERT SPACE OF GAUGE THEORY ON TORUS & WHAT THEY IMPLY

2. APPLICATION TO SYM: SEMICLASSICS ON $\mathbb{T}^4$ AND THE GAUGINO CONDENSATE *vs.* SEMICLASSICS ON $\mathbb{R}^4, \mathbb{R}^3 \times \mathbb{S}^1$

remarks

1. HOW MIXED ANOMALIES BETWEEN CHIRAL (invertible ~~or not)~~ AND CENTER SYMMETRY (“1-form”) ARISE IN HILBERT SPACE OF GAUGE THEORY ON TORUS & WHAT THEY IMPLY

- no time for noninvertible anomaly (Anber, EP 2305.14425)

- will largely use language established by 1980

  will review w/out details, as not used by many *…intimately familiar to many at IFT!*

  ↑

  1st part of talk!

remarks

2. APPLICATION TO SYM: SEMICLASSICS ON $\mathbb{T}^4$ AND THE GAUGINO CONDENSATE *vs.* SEMICLASSICS ON $\mathbb{R}^4, \mathbb{R}^3 \times \mathbb{S}^1$

- motivated by recent work on semiclassical confinement and continuity to $\mathbb{R}^4$: $\mathbb{R}^3 \times \mathbb{S}^1$ (monopole-instantons, Ünsal et al, 2007+) or $\mathbb{R}^2 \times \mathbb{T}^2$ (center vortices, Tanizaki-Ünsal, 2022+)

  authors argue "adiabatic continuity" to $\mathbb{R}^4$: test in SYM !

- work at IFT on confinement and fractional instantons: García Pérez, González-Arroyo 1990's+

- anomaly and Hilbert space allow to revisit old $\mathbb{T}^4$ calculation of gaugino condensate (Cohen, Gómez, 1984; Shifman, Vainshtein 1986) and improve/confront with other existing calculations

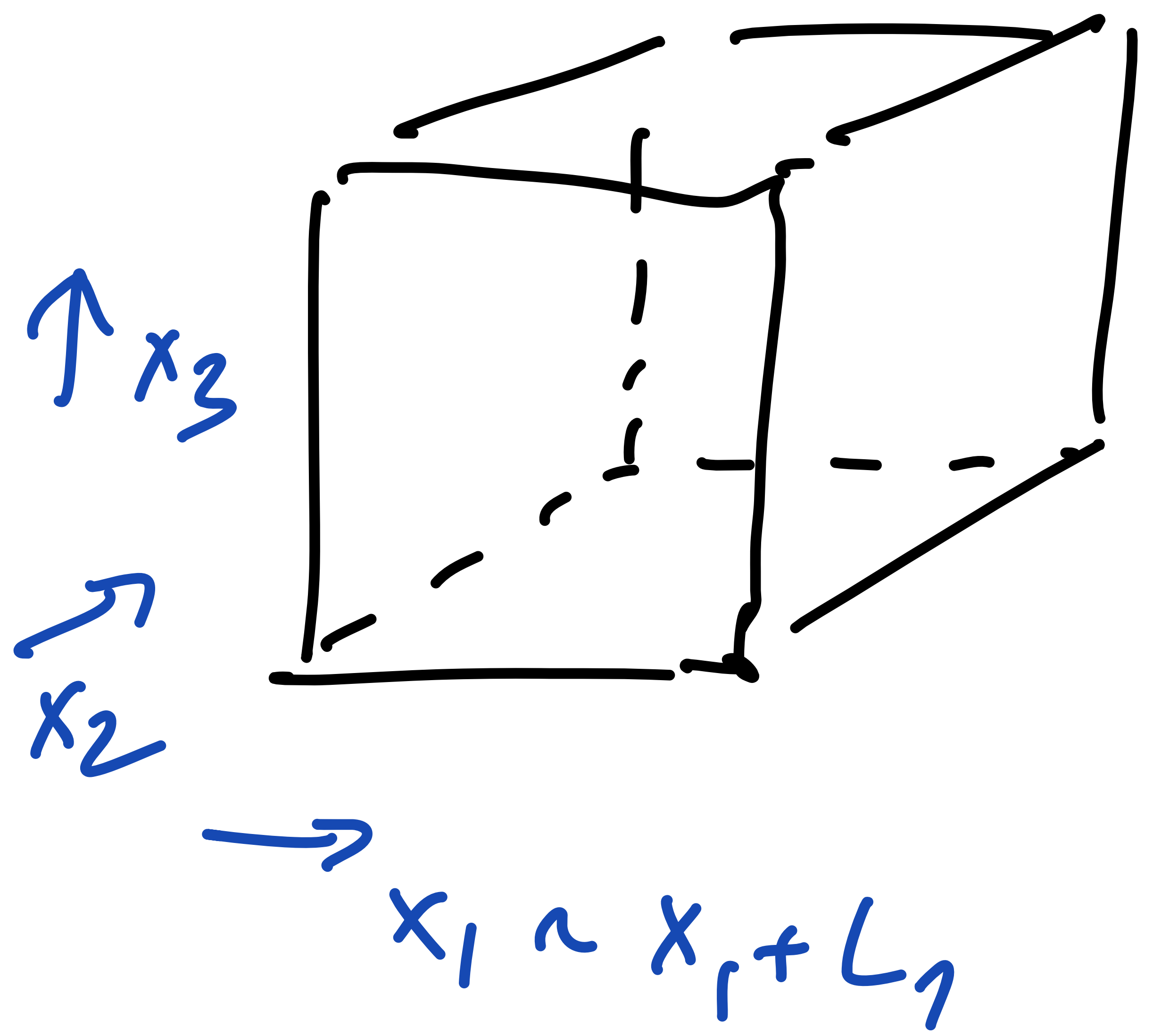
—> some puzzles remain!

2nd part of talk

1. reminder of old-fashioned language (~1980)

use Hamiltonian quantization on $\mathbb{T}^3$:

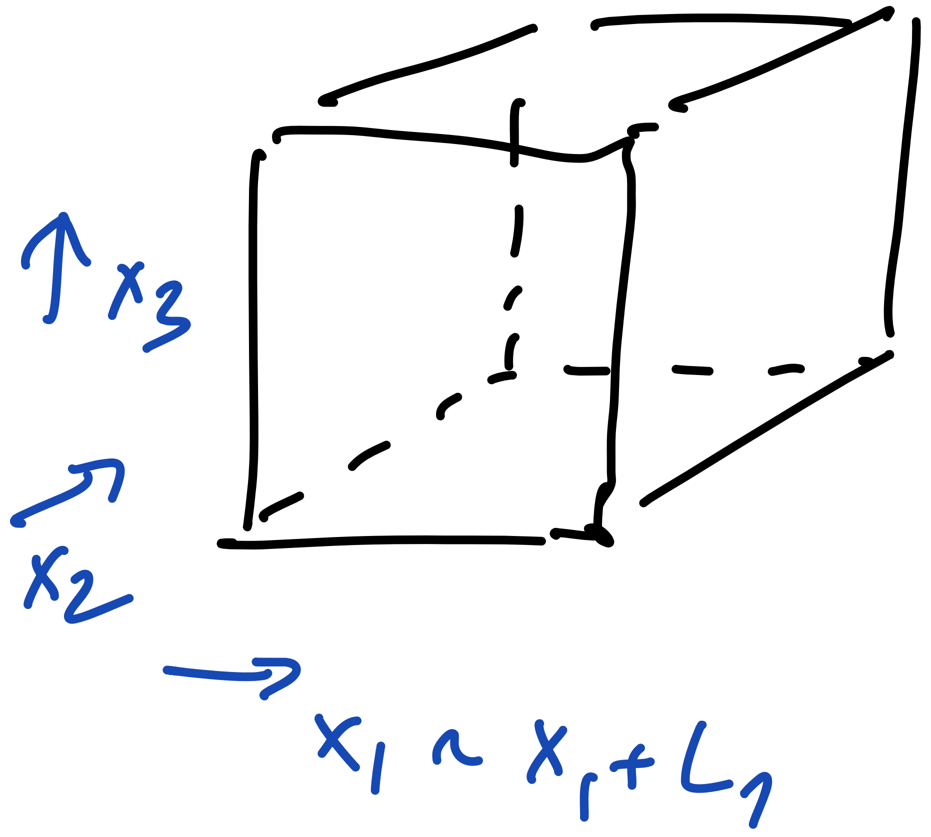
$A_0 = 0$ gauge, states $\Psi[A]$ invariant under time-independent gauge transforms (Gauss' law)



1. reminder of old-fashioned language (~1980)

1.1 center symmetry: $\hat{T}_i, i = 1{,}2{,}3$: "gauge" transforms periodic in x_i up to a center element

$$\hat{T}_i(\vec{x} + \vec{e}_j L_j) = \hat{T}_i(\vec{x})\, e^{i\frac{2\pi}{N}\delta_{ij}}$$



only acts on winding Wilson loops in fundamental

$$\hat{W}_i = \mathrm{tr}_F \mathscr{P}\, e^{i\int_0^{L_i} \hat{A}_i dx^i} \longrightarrow \hat{T}_i \hat{W}_j \hat{T}_i^{-1} = e^{i\frac{2\pi}{N}\delta_{ij}}\, \hat{W}_j$$

- time-direction version familiar from deconfinement transition in pure YM

- modern language: $\mathbb{Z}_N^{(1)}$ 1-form symmetry, only acts on line operators, not on local gauge invariants like $\mathrm{tr}\, F_{\mu\nu} F_{\lambda\sigma} \ldots$

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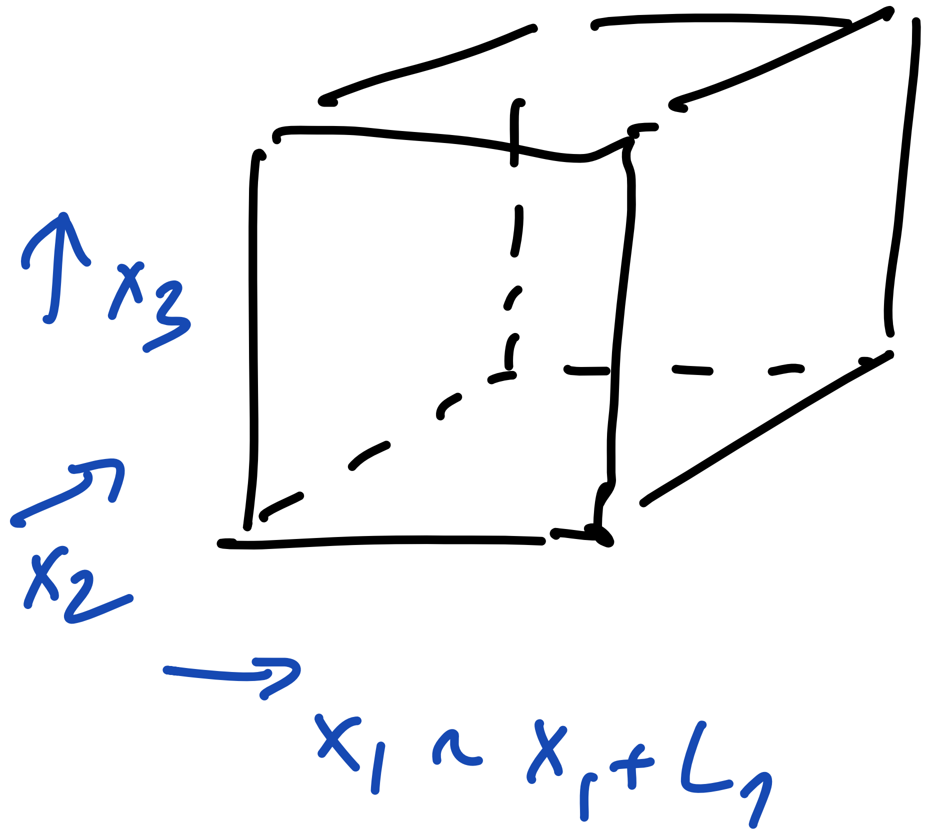

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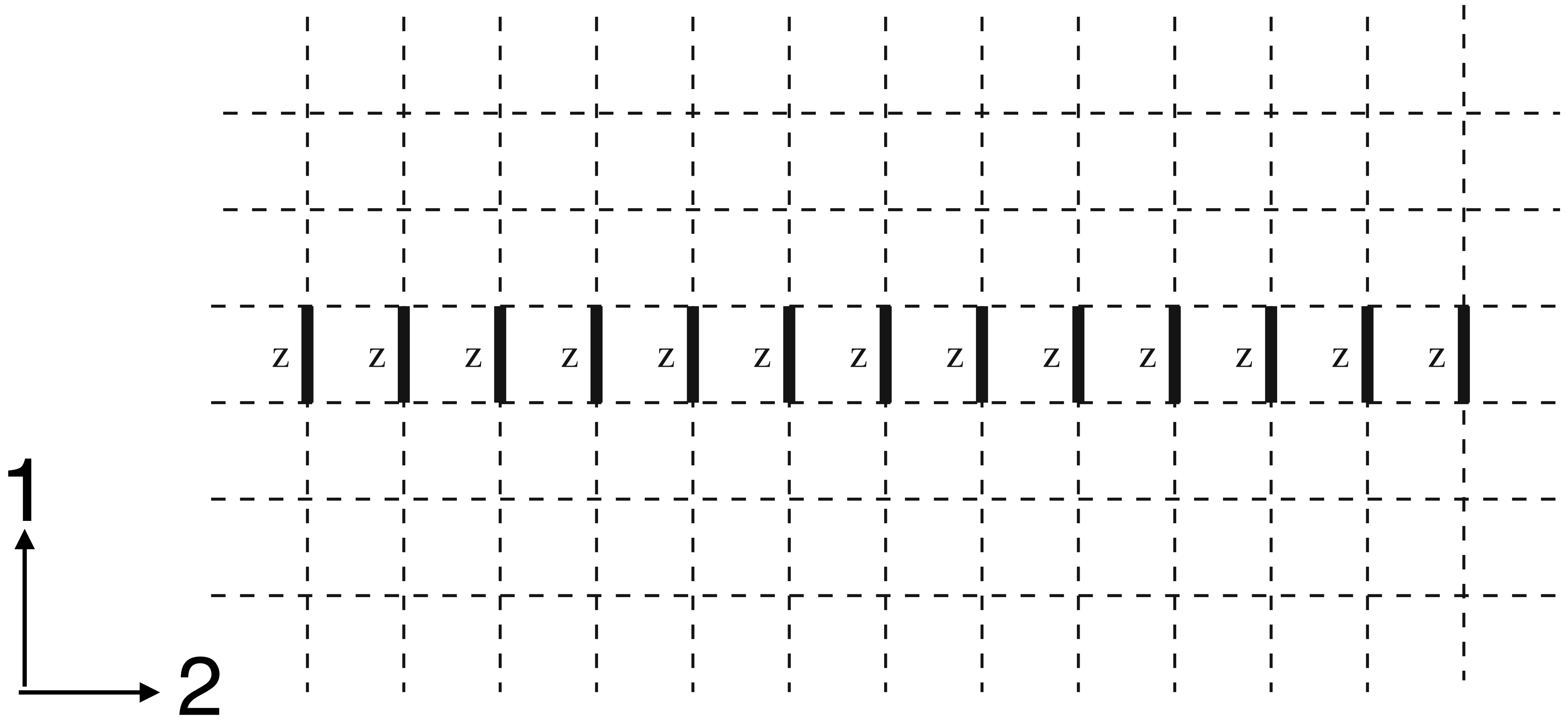
on lattice, $\hat{T}_1$ multiplies by $z = e^{i\frac{2\pi}{N}}$ shown link fields in direction 1 (for all x_3, x_4)

- all nonwinding closed loops invariant
- winding loops transform by z



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if the SU(N) theory has adjoint fields only, $\mathbb{Z}_N^{(1)}$ remains a symmetry, since

$\hat{\Psi}_{adj} \rightarrow \hat{T}_i \hat{\Psi}_{adj} \hat{T}_i^{-1}$ so transformed field has same b.c. ($\hat{T}$ and $\hat{T}^{-1}$ phases cancel)

if matter representation has nontrivial N-ality (transforms under center), the story changes (will not need for this talk)

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in each of these cases, the appropriate $\hat{T}_i$ obey

$[\hat{T}_i, \hat{H}] = 0$ so we can label states in $\mathbb{T}^3$ Hilbert space

by "electric flux" quantum numbers $|E, e_1, e_2, e_3\rangle = |E, \vec{e}\rangle$

$\hat{T}_i |E, \vec{e}\rangle = |E, \vec{e}\rangle\, e^{i\frac{2\pi}{N}e_i}$, three (mod N) integers

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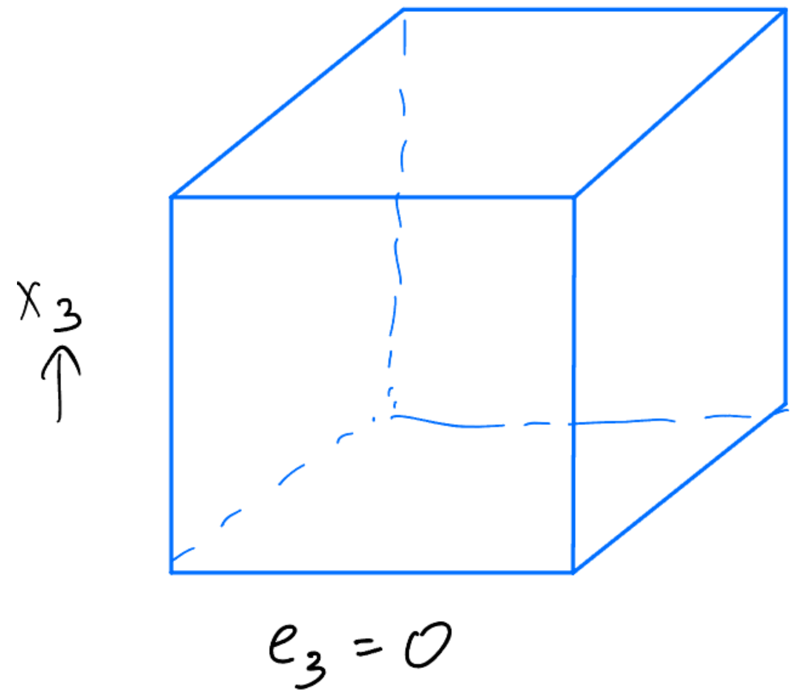
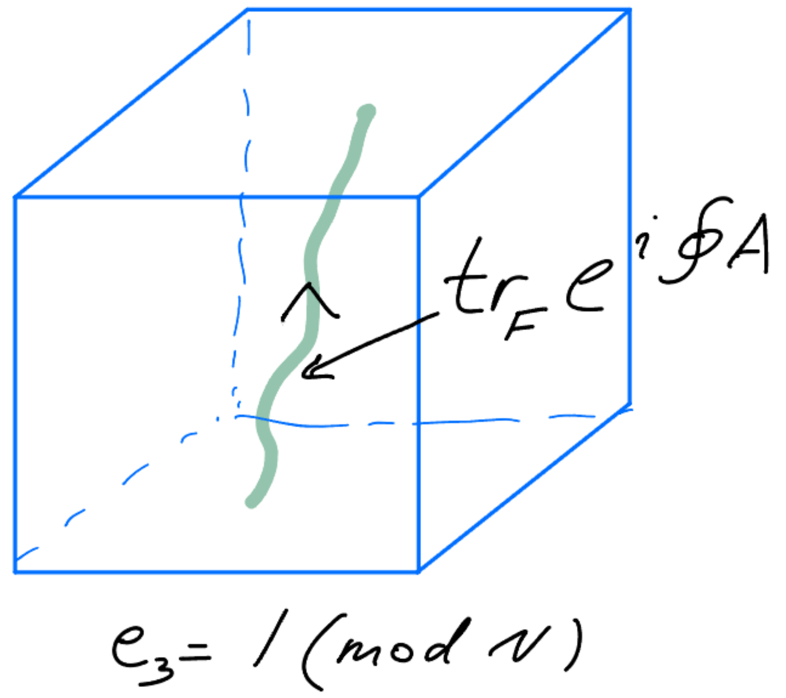
$$[\hat{T}_i, \hat{H}] = 0 \qquad\qquad |E, e_1, e_2, e_3\rangle = |E, \vec{e}\rangle$$

1.2 electric flux sectors in Hilbert space on $\mathbb{T}^3$

value of e_i is changed by one unit by acting with $\hat{W}_i$ on state:

$$\hat{T}_i\,(\hat{W}_i|\vec{e}\rangle) = (\hat{W}_i|\vec{e}_i\rangle)\, e^{i\frac{2\pi}{N}(e_i+1)}$$





in pure YM, at $\theta \neq \pi$, as $L \to \infty$, only one electric flux sector ($\vec{e} = 0$) has finite energy, while all others have energy $\sim L$ with coefficient given by the k-string tension; studied much on and off the lattice:

't Hooft '80, Lüscher '82, van Baal, Witten, González-Arroyo…

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1.2 electric flux sectors in Hilbert space on $\mathbb{T}^3$

value of e_i is changed by one unit by acting with $\hat{W}_i$ on state:

"whenever you have global symmetry, it pays to introduce a background gauge field for it"

(Seiberg)

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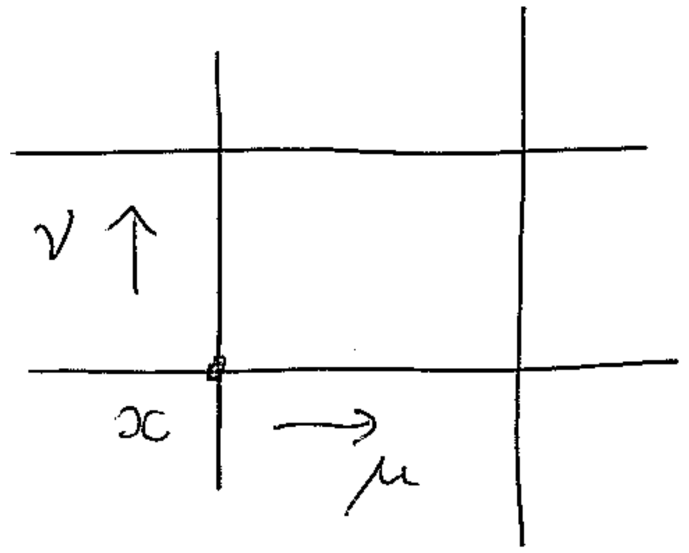
value of e_i is changed by one unit by acting with $\hat{W}_i$ on state:

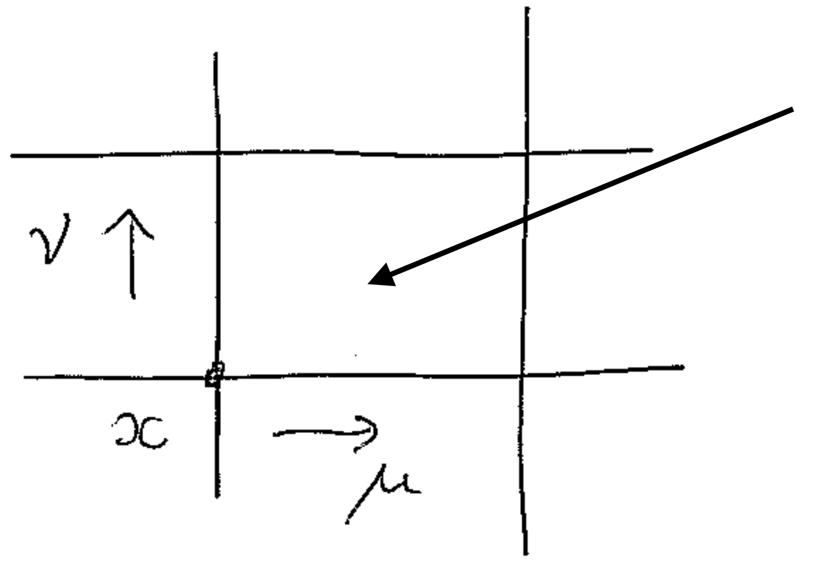
0-form symmetry (usual one, acting on local operators) has 1-form gauge field (link-based)

1-form symmetry has 2-form gauge field (plaquette based)



$$U_{x,\mu} \to z_\mu U_{x,\mu}$$
$$z_\mu = e^{i\frac{2\pi}{N} n_\mu}.$$

now, make z_μ x-dependent:



$$e^{iB_{\mu\nu}(x)}$$

plaquette based (2-form) $\mathbb{Z}_N$-valued

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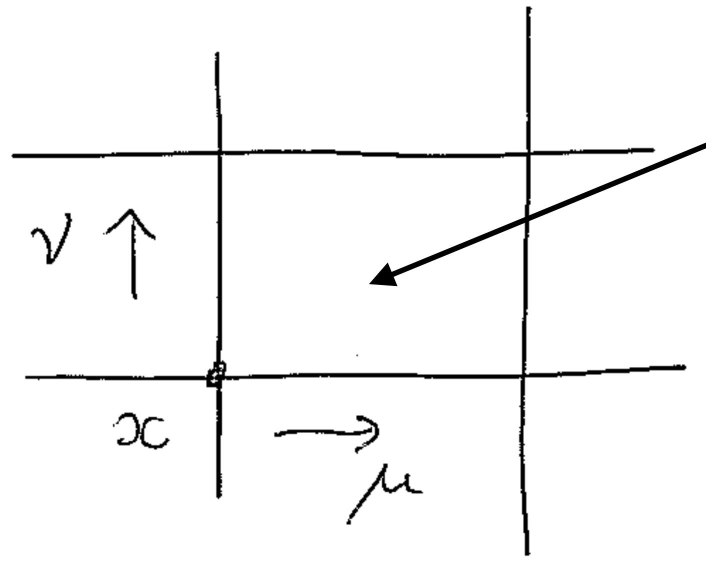
1-form symmetry has 2-form gauge field (plaquette based)

for 1-form gauge field, $\oint A_\mu dx^\mu$ is gauge invariant



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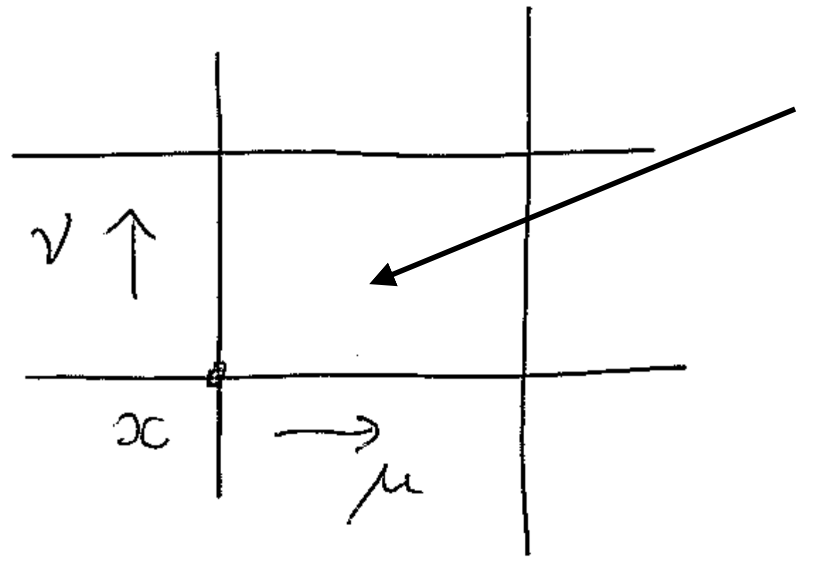
1-form symmetry has 2-form gauge field (plaquette based)

for 2-form abelian/$\mathbb{Z}_N$ gauge field, $\oint B_{\mu\nu} d^2\sigma^{\mu\nu}$
is gauge invariant; on $\mathbb{T}^3$ we can introduce curvature-free background for $\mathbb{Z}_N$ 2-form field



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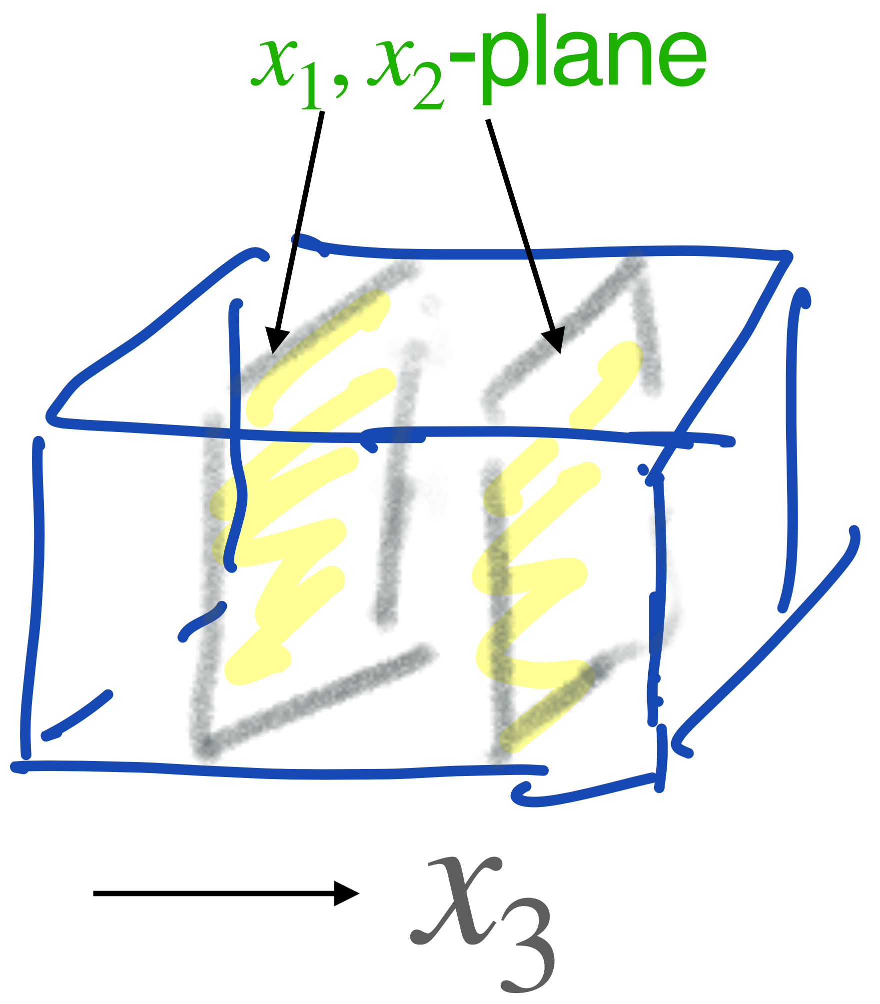
$$\oint dx^1 dx^2 B_{12} = \frac{2\pi m_3}{N} (\mathrm{mod} 2\pi)$$

$$\oint dx^2 dx^3 B_{23} = \frac{2\pi m_1}{N} (\mathrm{mod} 2\pi)$$

$$\oint dx^3 dx^1 B_{31} = \frac{2\pi m_2}{N} (\mathrm{mod} 2\pi)$$



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1.2 electric flux sectors in Hilbert space on $\mathbb{T}^3$

value of e_i is changed by one unit by acting with $\hat{W}_i$ on state:

1.3 magnetic fluxes on $\mathbb{T}^3$ (aka "twisted b.c."; "'t Hooft fluxes")

Hilbert space basis is: $| E, \vec{e} \rangle_{\vec{m}}$, with* $\hat{T}_i | E, \vec{e} \rangle_{\vec{m}} = | E, \vec{e} \rangle_{\vec{m}} \; e^{i \frac{2\pi}{N} e_i}$

in thermodynamic limit, usually only $\vec{e} = 0$ have finite energy while dependence on b.c., $\vec{m}$, is expected to be irrelevant, at least for gapped theories

(* at $\theta = 0$)

[check in TD limit, Teper, Stephenson; González-Arroyo… '80s-'90s] consider a unit "magnetic flux" (twist) in one plane (12, say) only: $\vec{m} = (0,0,1)$

Crucial observation ('t Hooft) - have to accept or ask later…

$\hat{T}_3$, the $Z_N^{(1)}$ generator in the direction orthogonal to the (12) plane of the twist

has winding number Q = $\frac{m_3}{N} (\text{mod } Z)$

[T_3 is a gauge transform, a map from torus to gauge group, so winding makes sense]

3d CS action, $S_{CS} = \int \text{tr}(AdA + \ldots)$, normalized to shift by unity under a unit-winding gauge transformation, so $e^{i2\pi S_{CS}}$ invariant

$e^{i2\pi S_{CS}}$

$$\hat{T}_3 \, e^{i2\pi \int_{T^3} \text{tr}(\hat{A}d\hat{A}+\ldots)} \, \hat{T}_3^{-1} = e^{i\frac{2\pi}{N}m_3} \, e^{i2\pi \int_{T^3} \text{tr}(\hat{A}d\hat{A}+\ldots)}$$

however, CS action shifts by $\frac{m_3}{N}$ under fractional winding

Crucial observation ('t Hooft)

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has winding number Q = $\frac{m_3}{N} (\mathrm{mod}\ Z)$

$$(*) \quad \hat{T}_3 \, e^{i2\pi \int_{T^3} \mathrm{tr}(\hat{A} d\hat{A} + \ldots)} \, \hat{T}_3^{-1} = e^{i\frac{2\pi}{N} m_3} \, e^{i2\pi \int_{T^3} \mathrm{tr}(\hat{A} d\hat{A} + \ldots)}$$

- fractional winding explained by 't Hooft ~ 1980
- as an equation in Hilbert space (*) appears first in unpublished Ch. 3 of van Baal's PhD thesis, 1984

at the time, (*) significance as an anomaly and implications for spectrum, incl. in TD limit, missed!

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- Eq. (*): Hilbert space expression of what GKKS ~2014 call θ -periodicity anomaly (GKKS study Euclidean path integral)

$$U_1 \Psi[A] = e^{i\theta}\, \Psi[A] \longrightarrow U_1(e^{i2\pi S_{CS}[A]}\Psi[A]) = e^{i(2\pi+\theta)}\, (e^{i2\pi S_{CS}[A]}\Psi[A])$$

  - hence $e^{i2\pi S_{CS}}$ is "operator shifting θ by 2π" (U_1 is operator of unit-winding gauge transform)
  - Eq. (*) says that when $m_3 \neq 0$ (mod N), shifting θ by 2π and center symmetry do not commute

we care because 2π shifts of θ can be part of physical symmetry (simplest: parity in pure-YM$_{\theta=\pi}$)

SU(N) with n_f adjoint Weyl quarks, for definiteness take SYM, $\underline{n_f = 1}$ below:

notation: $Q_{top.} = \frac{1}{32\pi^2}\int d^4x\, F^a_{\mu\nu}F^a_{\lambda\sigma}\epsilon^{\mu\nu\lambda\sigma} =: \int d^4x\, \partial_\mu K^\mu$ with $\int d^3x K^0 \equiv S_{CS}$

classical chiral U(1) $\lambda \to e^{i\alpha}\lambda$ "R-symmetry"

$\partial_\mu \hat{j}^\mu_f = \partial_\mu(\hat{\lambda}^{a\,\dagger}\bar{\sigma}^\mu\hat{\lambda}^a) = 2n_f N \partial_\mu \hat{K}^\mu$ ⟶ R-current not conserved

$\hat{J}^\mu_5 = \hat{j}^\mu_f - 2n_f N \hat{K}^\mu$

$\hat{Q}_5 = \int d^3x \hat{J}^0_5 = \int d^3x \hat{j}^0_f - 2n_f N \int d^3x \hat{K}^0$

⟶ $\hat{Q}_5$ conserved but not gauge invariant $(n_f = 1)$

$\hat{X}_{2N} = e^{i\frac{2\pi}{2N}\hat{Q}_5} = e^{i\frac{2\pi}{2N}\int d^3x \hat{j}^0_f}\, e^{-i2\pi\int d^3x \hat{K}_0}$ ⟶ gauge invariant operator of $Z^{(0)}_{2N}$ discrete R-symmetry

$\hat{T}_3\, e^{i2\pi\int_{T^3}\mathrm{tr}(\hat{A}d\hat{A}+\ldots)}\, \hat{T}_3^{-1} = e^{i\frac{2\pi}{N}}\, e^{i2\pi\int_{T^3}\mathrm{tr}(\hat{A}d\hat{A}+\ldots)} \Longrightarrow \hat{T}_3\, \hat{X}_{2N}\, \hat{T}_3^{-1} = e^{-i\frac{2\pi}{N}}\, \hat{X}_{2N}$

↑ $\int d^3x \hat{K}_0 = S_{CS}$

on $\mathbb{T}^3$ with $m_3 = 1$: <u>mixed 0-form/1-form anomaly</u>

SYM on twisted T^3 - invertible chiral/center anomaly

Hilbert space with spatial 't Hooft twist $n_{12} = m_3 = 1$; SYM has two global symmetries, $\hat{T}_3$ and $\hat{X}_{2N}$, 1-form and 0-form, invertible (=normal unitary operators on Hilbert space) commute with Hamiltonian, but not with each other:

$$\hat{T}_3\, \hat{X}_{2N}\, \hat{T}_3^{-1} = e^{-i\frac{2\pi}{N}}\, \hat{X}_{2N} \longrightarrow \hat{X}_{2N} | E, e_3 \rangle = | E, e_3 - 1 \rangle$$

action of chiral symmetry changes e_3 flux of state but not energy

all energy levels on the twisted T^3 are N-fold degenerate,

exact degeneracy at any finite volume, provided $n_{12} = m_3 = 1$!

[Cox, Wandler, EP 2106]

unusual in QFT! (TQFTs "living" on DW-instanton worldvolume responsible!)

different from topological order (e.g. $\mathbb{Z}_2$ in superconductors) where degeneracy only in "topological scaling limit"

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[Cox, Wandler, EP 2106]

as volume goes to infinity, if theory confines (center unbroken) clustering ground states are the lowest energy degenerate flux states, related by broken discrete chiral symmetry
here, a consequence of the mixed anomaly, not SUSY!

gaugino bilinear phase in different flux sectors: $\langle E, e_3 \, | \, \mathrm{tr}\lambda\lambda \, | \, E, e_3 \rangle = e^{i\frac{2\pi}{N}} \, \langle E, e_3 + 1 \, | \, \mathrm{tr}\lambda\lambda \, | \, E, e_3 + 1 \rangle$

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action of chiral symmetry changes e_3 flux of state but not energy

all energy levels on the twisted T^3 are N-fold degenerate,

<u>exact degeneracy at any finite volume,</u> provided $n_{12} = m_3 = 1$! [Cox, Wandler, EP 2106]

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gaugino bilinear phase in different flux sectors: $\langle E, e_3 | \mathrm{tr}\lambda\lambda | E, e_3 \rangle = e^{i\frac{2\pi}{N}} \langle E, e_3 + 1 | \mathrm{tr}\lambda\lambda | E, e_3 + 1 \rangle$

degeneracy does not require SUSY, similar degeneracies in non-SUSY QCD(adj) [Cox, Wandler, EP 2106]

exact degeneracies less severe if gauge group has smaller center… SP, Spin, E6, E7

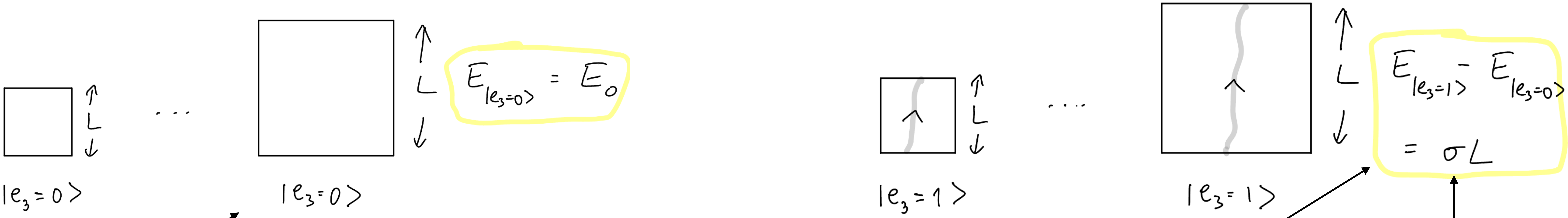
part 1 summary:

Studying a gauge theory on torus with twisted b.c.
(=in 2-form background fields for the 1-form symmetry)
is a powerful probe of the dynamics, especially in the presence of anomalies.

Mixed anomaly of invertible ~~or noninvertible~~ chiral symmetries with center symmetry implies exact degeneracy of flux sectors at any size torus.

Cartoon picture to remember:

A.) no anomaly: lowest energy in $e_3 \neq 0$ flux sector $\rightarrow \sigma L \rightarrow \infty$



torus with 2-form background (any),
upon increasing size to ∞

higher flux
sectors decouple at $L \rightarrow \infty$

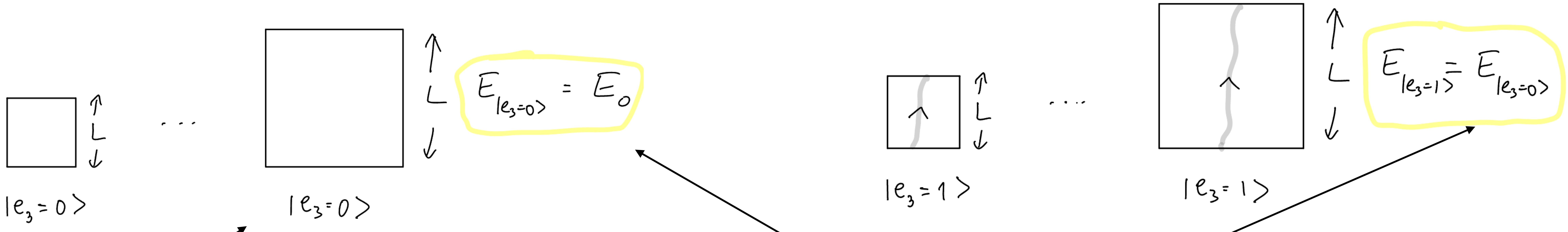
confining
string tension
(lattice!)

[Teper, Stephenson;
González-Arroyo,…1990s]

part 1 summary:

Cartoon picture to remember: for $\mathbb{Z}_2$ valued anomaly

B.) anomaly: lowest energies of $e_3 = 0$, $e_3 = 1$ flux sectors remain equal



torus with 2-form background (the one revealing anomaly!), upon increasing size to ∞

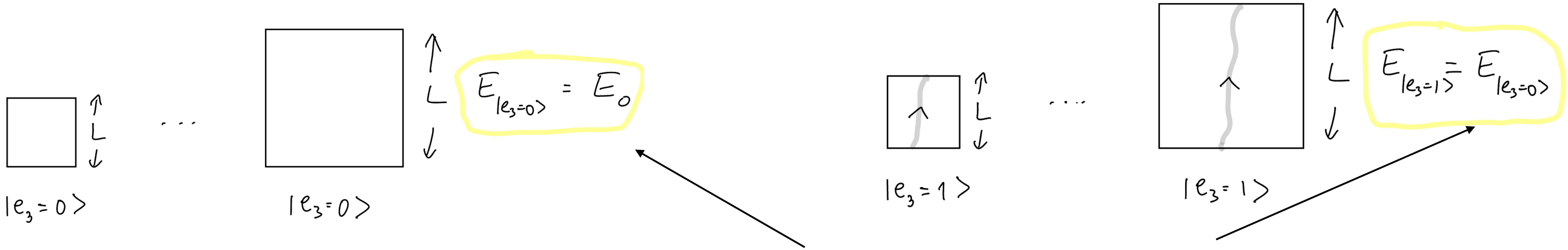
interchanged by chiral symmetry $\forall L < \infty$

(or parity, in pure YM at $\theta = \pi$)

part 1 summary:

Cartoon picture to remember: for $\mathbb{Z}_2$ valued anomaly

B.) anomaly: lowest energies of $e_3 = 0$, $e_3 = 1$ flux sectors remain equal



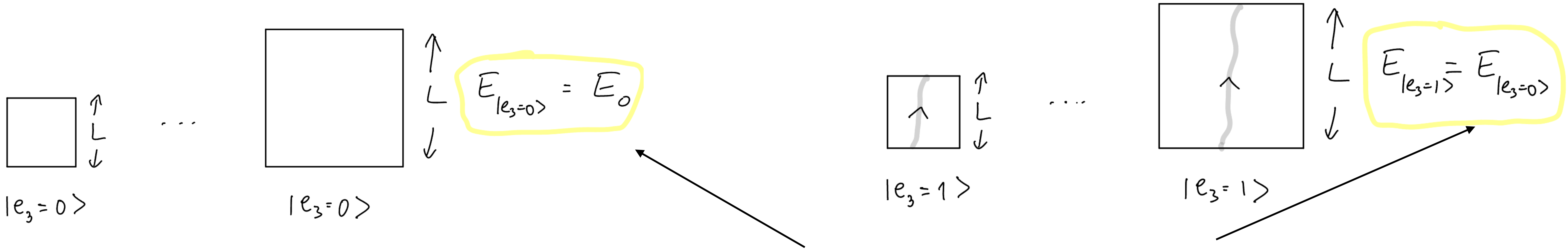
interchanged by chiral symmetry $\forall L < \infty$

infinite L, if center unbroken: these are the clustering vacua, chiral broken

part 1 summary:

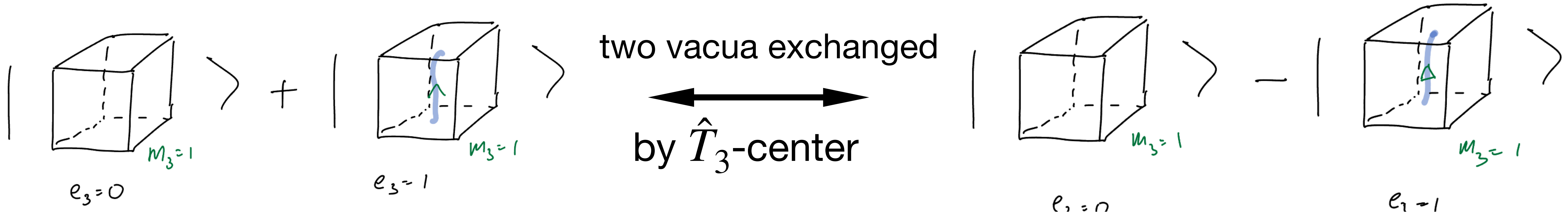
Cartoon picture to remember: for $\mathbb{Z}_2$ valued anomaly

B.) anomaly: lowest energies of $e_3 = 0$, $e_3 = 1$ flux sectors remain equal



interchanged by chiral symmetry $\forall L < \infty$

infinite L, if center broken:
deconfinement (Z_2 example)



1. HOW MIXED ANOMALIES BETWEEN CHIRAL (invertible or not) AND CENTER SYMMETRY (“1-form”) ARISE IN HILBERT SPACE OF GAUGE THEORY ON TORUS & WHAT THEY IMPLY

done with part 1

on to part 2

2. APPLICATION TO SYM: SEMICLASSICS ON $\mathbb{T}^4$ AND THE GAUGINO CONDENSATE *vs.* SEMICLASSICS ON $\mathbb{R}^4, \mathbb{R}^3 \times \mathbb{S}^1$

title of 2210.13568 is **The gaugino condensate from asymmetric four-torus with twists**

sounds like a mouthful & is 70 pages long!

$\mathcal{N}$=1 SYM: symmetries and nonrenormalization theorems

$$S_{SYM} = \frac{1}{g^2} \int_{\mathbb{T}^4} \mathrm{tr} \left[\frac{1}{2} F_{mn} F_{mn} + 2(\partial_n \bar{\lambda}_{\dot{\alpha}} + i[A_n, \bar{\lambda}_{\dot{\alpha}}]) \bar{\sigma}_n^{\dot{\alpha}\alpha} \lambda_\alpha \right]$$

G=SU(N)

chiral $U(1) : \lambda \to e^{i\alpha}\lambda$ by anomaly $\to Z_{2N}^{(0)}$

$\mathcal{N}$=1 SYM: symmetries and nonrenormalization theorems

$$S_{SYM} = \frac{1}{g^2}\int_{\mathbb{T}^4} \mathrm{tr}\left[\frac{1}{2}F_{mn}F_{mn} + 2(\partial_n\bar{\lambda}_{\dot{\alpha}} + i[A_n, \bar{\lambda}_{\dot{\alpha}}])\bar{\sigma}_n^{\dot{\alpha}\alpha}\lambda_\alpha\right]$$

G=SU(N)

(N=2 later)

chiral $U(1) : \lambda \to e^{i\alpha}\lambda$ by anomaly $\to Z_{2N}^{(0)}$

$Z_{2N}^{(0)} \to Z_2^{(0)}$ $\quad \langle \mathrm{tr}\,\lambda^2 \rangle \underset{(N=2)}{=} \pm\, 16\pi^2\Lambda^3$

the "mother" of all exact results in SUSY, no further corrections

$$\Lambda^3 = \mu^3 \frac{e^{-8\pi^2/Ng^2}}{g^2} \; \left(= \mu^3\, e^{-\frac{8\pi^2}{Ng_h^2(\mu)}} \right)$$ holomorphic scale

$\mathcal{N}$=1 SYM: symmetries and nonrenormalization theorems

1983-1999: Novikov, Shifman, Vainshtein, Zakharov; Amati, Konishi, Rossi, Veneziano; Affleck, Dine, Seiberg; Cordes; Finnell, Pouliot (SQCD —> SYM on R^4); Davies, Hollowood, Khoze, Mattis;… 2014 Anber, Teeple, EP (SYM on $R^3 \times S^1$ —> SYM on R^4)

two weakly-coupled calculations of $\langle \lambda^2 \rangle$

$$Z_{2N}^{(0)} \to Z_2^{(0)} \qquad \langle \mathrm{tr}\, \lambda^2 \rangle_{(N=2)} = \pm\, 16\pi^2 \Lambda^3$$

all history…?

$$\Lambda^3 = \mu^3 \frac{e^{-8\pi^2/Ng^2}}{g^2} \left(= \mu^3\, e^{-\frac{8\pi^2}{N g_h^2(\mu)}} \right)$$ holomorphic scale

1. weak coupling
confinement… "adiabatic continuity"?

$\mathbb{R}^3 \times \mathbb{S}^1$ (monopole-instantons, Ünsal et al, 2007+)
$\mathbb{R}^2 \times \mathbb{T}^2$ (center vortices, Tanizaki-Ünsal, 2022+)

2. generalized symmetries,
backgrounds, new anomalies

Gaiotto, Kapustin, Komargodski,Seiberg: 2014-…

one of two weakly-coupled calculations
of $\langle \lambda^2 \rangle$: continuous connection to R^4

$$\langle \lambda^2 \rangle = 16\pi^2 \Lambda^3$$

↔

using this new and deeper knowledge,
revisit old (1984!) calculations of $\langle \lambda^2 \rangle$ on T^4

$$\langle \lambda^2 \rangle = c\ 16\pi^2 \Lambda^3$$

Cohen, Gómez '84;
Shifman, Vainshtein '86

how well do we understand semiclassics in the femtouniverse?

is there continuity to infinite volume limit?

- test for condensate, in SYM, where some exact results are known

what fluctuations contribute to the gaugino condensate?

one of two weakly-coupled calculations of $\langle\lambda^2\rangle$: continuous connection to R^4 ⟷ using this new and deeper knowledge, revisit old (1984!) calculations of $\langle\lambda^2\rangle$ on T^4

$$\langle\lambda^2\rangle = 16\pi^2\Lambda^3$$

$$\langle\lambda^2\rangle = c\ 16\pi^2\Lambda^3$$

Cohen, Gómez '84; Shifman, Vainshtein '86 from Davies, Hollowood, Khoze, Mattis, ~ last time $\mathbb{T}^4$ gaugino condensate mentioned in literature:

[1] Our approach is different from the toron calculations of Ref. [28] where all four dimensions were compactified on a torus. The advantage of our method compared to that of [28] is that we do not have to fine-tune the compactification parameters for the finite-action configurations to exist. We also note that the value gluino condensate extracted from the toron approach of [28] in the finite-volume torus with the fine-tuned periods is difficult to interpret in the infinite volume and its numerical value agrees neither with the WCI (1.1b) nor with the SCI (1.1a) results. In the alternative toron set-up advocated in [29], the fine-tuning problem was avoided at the cost of introducing *singular* toron-like configurations with a branch cut and an IR regulator.

footnote in hep-th/9905015:

resolved → García Pérez, González-Arroyo, Pena, 2000

… advantage of our (i.e. their $\mathbb{R}^3 \times \mathbb{S}^1$) approach… we do not have to fine tune sides of torus…

… the value of the gaugino condensate from toron (i.e. $\mathbb{T}^4$) approach is difficult to interpret in infinite volume limit … numerical value disagrees …

??? … since never computed!

one of two weakly-coupled calculations of $\langle\lambda^2\rangle$: continuous connection to R^4 ⟷ using this new and deeper knowledge, revisit old (1984!) calculations of $\langle\lambda^2\rangle$ on T^4

$$\langle\lambda^2\rangle = 16\pi^2\Lambda^3$$

$$\langle\lambda^2\rangle = c\ 16\pi^2\Lambda^3$$

Cohen, Gómez ‘84; Shifman, Vainshtein ‘86

armed with Hilbert space story, consider condensate, $\lambda^2 \equiv \mathrm{tr}\lambda^2$:

T_3 eigenvalue

inserts n_{34}=1 twist

$$\langle\lambda^2\rangle_{n_{12},n_{34}} = \mathrm{Tr}_{\mathscr{H}_{n_{12}}}\left[e^{-\beta H}(-1)^F\ \hat{T}_3\ \lambda^2\right] = \sum_{E;e_3=0,1} (-)^F e^{-\beta E}\ \underline{(-1)^{e_3}\ \langle E, e_3|\lambda^2|E, e_3\rangle_{(n_{12})}}$$

$\overrightarrow{m} = (0,0,n_{12}),\ x_4 = x_4 + \beta$

hence this product is same for e_3=0,1

$\hat{X}|E,0\rangle_{(n_{12})} \sim |E,1\rangle_{(n_{12})}$ and $\hat{X}\,\lambda^2\,\hat{X}^\dagger = -\lambda^2$ imply that λ^2 has opposite signs in degenerate flux states

armed with Hilbert space story, consider condensate, $\lambda^2 \equiv \mathrm{tr}\lambda^2$:

inserts n_{34}=1 twist

T_3 eigenvalue

$$\langle \lambda^2 \rangle_{n_{12},n_{34}} = \mathrm{Tr}_{\mathscr{H}_{n_{12}}} \left[e^{-\beta H} (-1)^F \, \hat{T}_3 \, \lambda^2 \right] = \sum_{E; e_3 = 0,1} (-)^F e^{-\beta E} \, \underline{(-1)^{e_3} \, \langle E, e_3 | \lambda^2 | E, e_3 \rangle_{(n_{12})}}$$

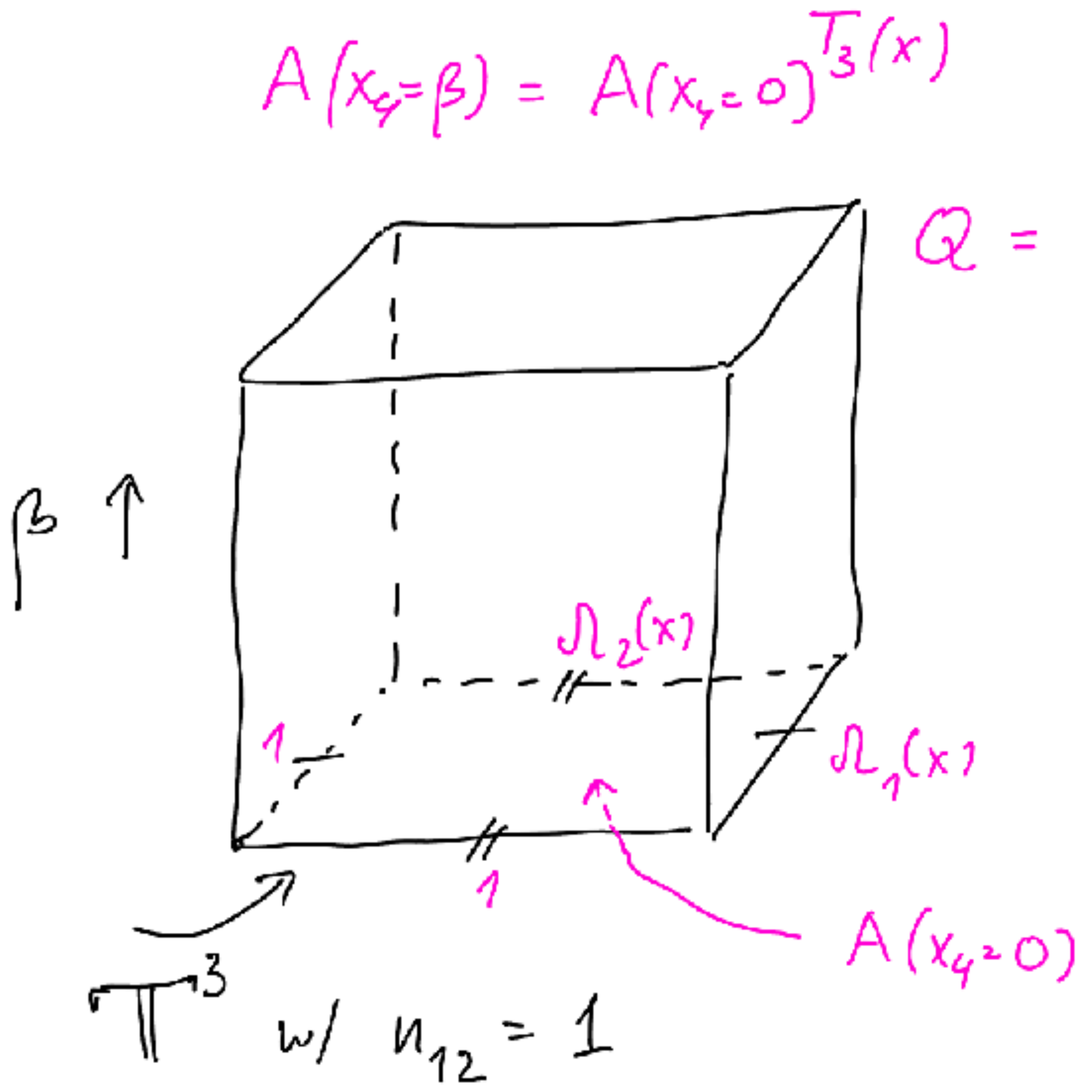
hence this product is same for e_3=0,1

e.g., sum over one flux sector x 2

$$\langle \lambda^2 \rangle_{n_{12},n_{34}} = 2 \sum_E (-)^F e^{-\beta E} \, \langle E,0 | \lambda^2 | E,0 \rangle_{(n_{12})}$$

normalize by path integral without λ^2 and $\hat{T}_3$ (i.e. no n_{34} twist, only n_{12}), i.e. Witten index

$$\langle 1 \rangle_{n_{12},0} = \mathrm{Tr}_{\mathscr{H}_{n_{12}}} \, e^{-\beta H} (-1)^F = \sum_{E; e_3 = 0,1} (-)^F e^{-\beta E} \, \langle E, e_3 | E, e_3 \rangle_{(n_{12})} = 2$$



$Q \in Z + 1/2$

$$\frac{\mathrm{Tr}_{\mathscr{H}_{n_{12}}}\, e^{-\beta H}(-1)^F \hat{T}_3\, \lambda^2}{\mathrm{Tr}_{\mathscr{H}_{n_{12}}}\, e^{-\beta H}(-1)^F} = \frac{\int_{n_{12}=1, n_{34}=1} \mathscr{D}A \mathscr{D}\lambda\, e^{-S}\, \lambda^2}{\int_{n_{12}=1, n_{34}=0} \mathscr{D}A \mathscr{D}\lambda\, e^{-S}}$$

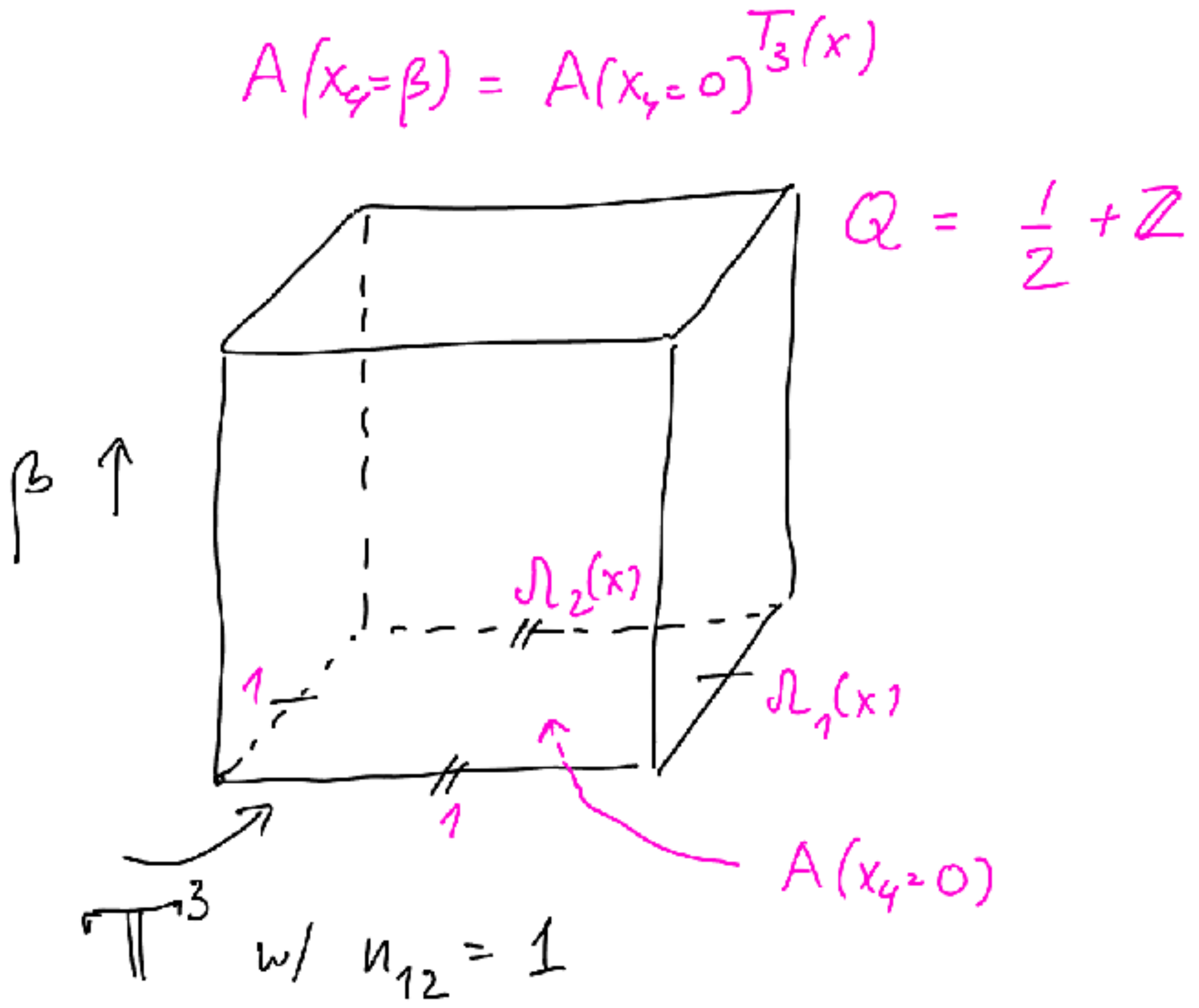
$Q \in Z$

$$\equiv \langle \lambda^2 \rangle = \sum_E (-)^F e^{-\beta E} \langle E, 0 | \lambda^2 | E, 0 \rangle_{(n_{12})}$$

semiclassical expansion expected to hold at small $\mathbb{T}^4$

$Q = \frac{1}{2}$, the leading contribution to numerator, will have two undotted λ zero modes

we shall discuss this calculation… but first the big picture



$Q \in Z + 1/2$

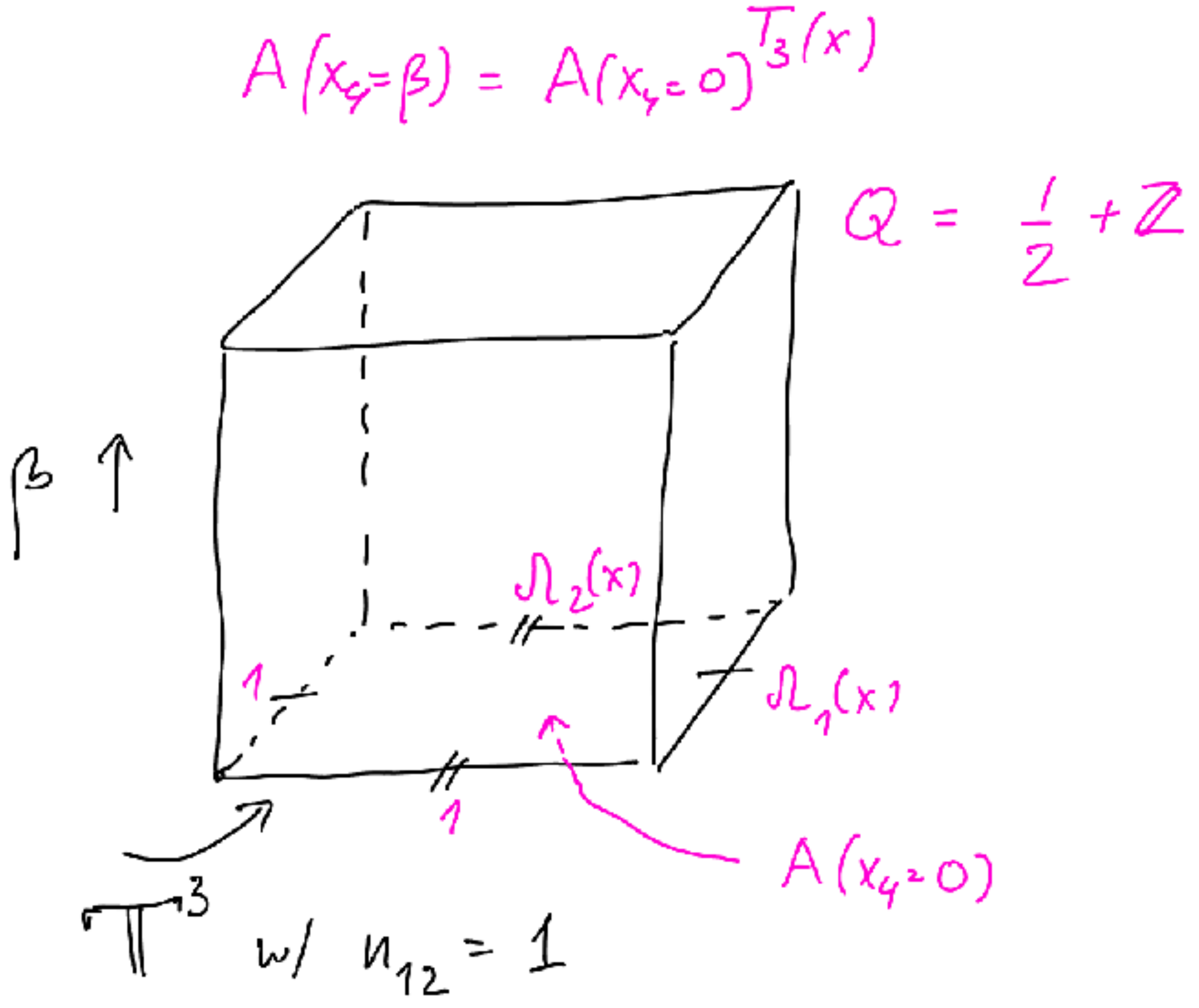
$$\frac{\mathrm{Tr}_{\mathscr{H}_{n_{12}}} e^{-\beta H}(-1)^F \hat{T}_3\, \lambda^2}{\mathrm{Tr}_{\mathscr{H}_{n_{12}}} e^{-\beta H}(-1)^F} = \frac{\int_{n_{12}=1,n_{34}=1} \mathscr{D}A\mathscr{D}\lambda\, e^{-S}\, \lambda^2}{\int_{n_{12}=1,n_{34}=0} \mathscr{D}A\mathscr{D}\lambda\, e^{-S}}$$

$Q \in Z$

$$\equiv \langle \lambda^2 \rangle = \sum_E (-)^F e^{-\beta E} \langle E,0 \,|\, \lambda^2 \,|\, E,0 \rangle_{(n_{12})}$$

take β infinite: only E=0

take $L_{1,2,3}$ infinite:
R^4 gaugino condensate in one of the vacua



$Q \in Z + 1/2$

$$\frac{\mathrm{Tr}_{\mathcal{H}_{n_{12}}} e^{-\beta H}(-1)^F \hat{T}_3\, \lambda^2}{\mathrm{Tr}_{\mathcal{H}_{n_{12}}} e^{-\beta H}(-1)^F} = \frac{\int_{n_{12}=1, n_{34}=1} \mathcal{D}A \mathcal{D}\lambda\, e^{-S}\, \lambda^2}{\int_{n_{12}=1, n_{34}=0} \mathcal{D}A \mathcal{D}\lambda\, e^{-S}}$$

$Q \in Z$

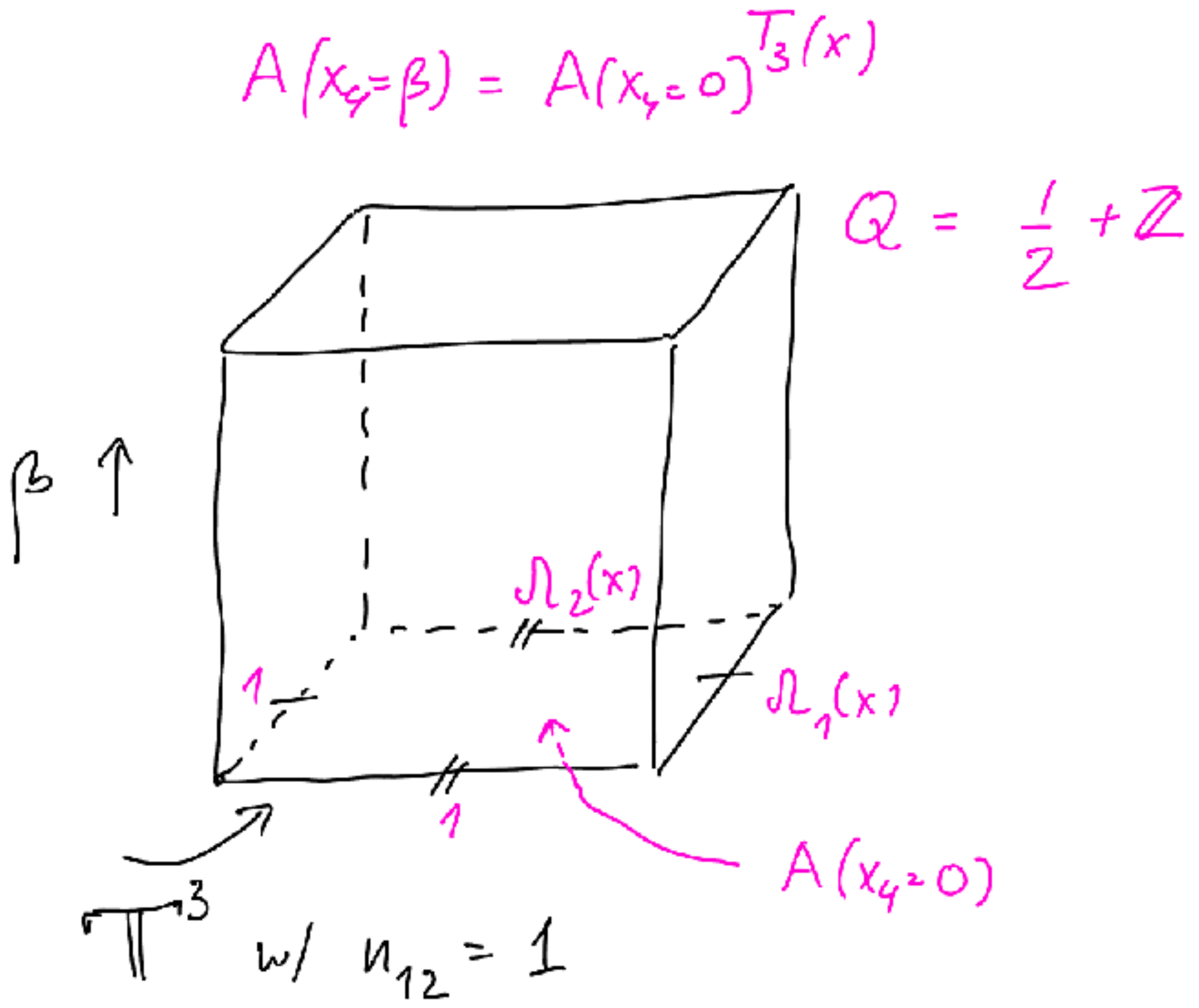
$$\equiv \langle \lambda^2 \rangle = \sum_E (-)^F e^{-\beta E} \langle E, 0 | \lambda^2 | E, 0 \rangle_{(n_{12})}$$

take β infinite: only E=0

take $L_{1,2,3}$ infinite:
R^4 gaugino condensate in one of the vacua

semiclassical calculation in small $\mathbb{T}^4$ limit
- made assumptions, stated later!

+ argue that result is L_μ, g_{YM}-independent
based on holomorphy - no f($L|\Lambda|$) allowed!



$$\frac{\mathrm{Tr}_{\mathscr{H}_{n_{12}}}\; e^{-\beta H}(-1)^F \hat{T}_3\, \lambda^2}{\mathrm{Tr}_{\mathscr{H}_{n_{12}}}\; e^{-\beta H}(-1)^F} = \frac{\int_{n_{12}=1, n_{34}=1} \mathscr{D}A\mathscr{D}\lambda\; e^{-S}\, \lambda^2}{\int_{n_{12}=1, n_{34}=0} \mathscr{D}A\mathscr{D}\lambda\; e^{-S}}$$

$Q \in Z + 1/2$ (numerator)

$Q \in Z$ (denominator)

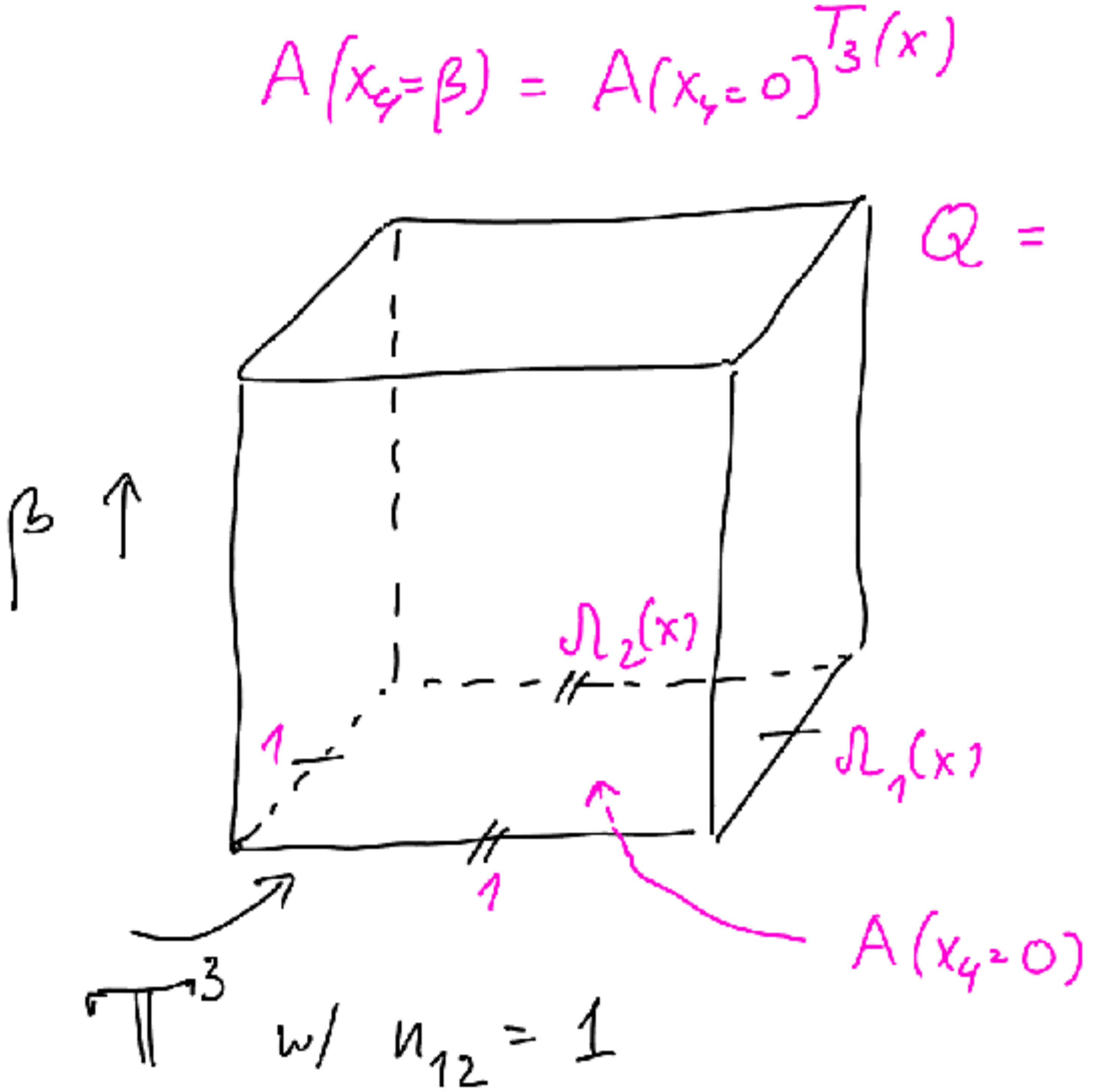
$$\equiv \langle \lambda^2 \rangle = \sum_E (-)^F e^{-\beta E}\, \langle E,0 \,|\, \lambda^2 \,|\, E,0 \rangle_{(n_{12})}$$

Holomorphy on $\mathbb{T}^4$?

usual argument on $\mathbb{R}^4$

$$\Lambda^* \frac{d}{d\Lambda^*} \langle \lambda^2 \rangle \sim \langle \lambda^2 F^* \rangle \sim \langle \lambda^2 \bar{Q}_{\dot\alpha} \bar\psi^{\dot\alpha} + \lambda^2 \bar\psi^{\dot\alpha} \bar{Q}_{\dot\alpha} \rangle \sim \langle \bar{Q}_{\dot\alpha} \lambda^2 \bar\psi^{\dot\alpha} + \lambda^2 \bar\psi^{\dot\alpha} \bar{Q}_{\dot\alpha} \rangle = 0$$

but on $\mathbb{T}^3$, for each E, e_3, $\sum_{\text{over states w/ given } E,\, e_3} (-)^F \langle E \,|\, X_{\dot 2} \bar{Q}_{\dot 1} + \bar{Q}_{\dot 1} X_{\dot 2} \,|\, E \rangle = 0$, as states $\in$ reps. of $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = \delta_{\alpha\dot\beta} E$



$Q \in Z + 1/2$

$$\frac{\mathrm{Tr}_{\mathcal{H}_{n_{12}}}\, e^{-\beta H}(-1)^F \hat{T}_3\, \lambda^2}{\mathrm{Tr}_{\mathcal{H}_{n_{12}}}\, e^{-\beta H}(-1)^F} = \frac{\int_{n_{12}=1,n_{34}=1} \mathcal{D}A\mathcal{D}\lambda\, e^{-S}\, \lambda^2}{\int_{n_{12}=1,n_{34}=0} \mathcal{D}A\mathcal{D}\lambda\, e^{-S}}$$

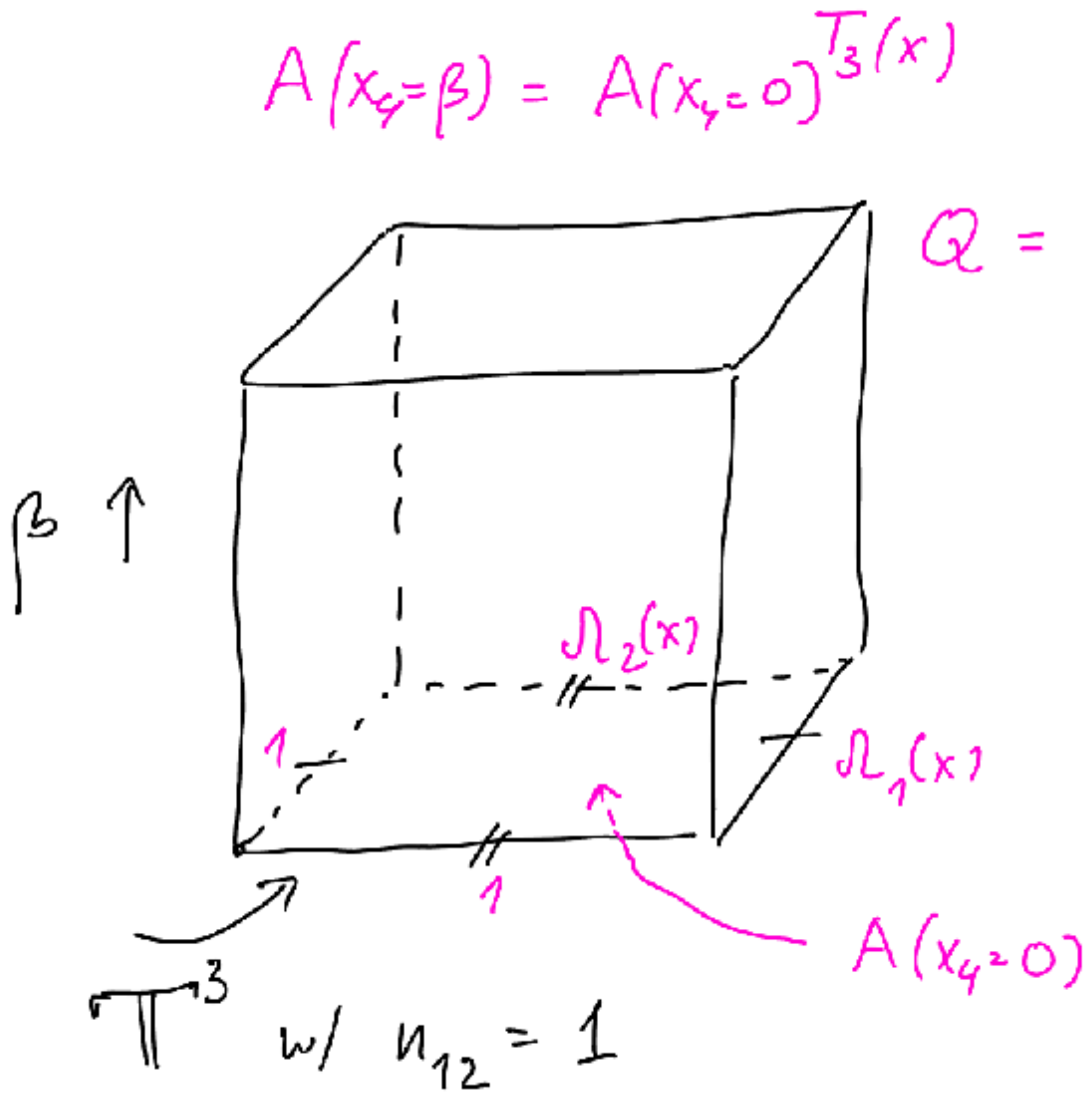
$Q \in Z$

$$\equiv \langle \lambda^2 \rangle = \sum_E (-)^F e^{-\beta E} \langle E,0 | \lambda^2 | E,0 \rangle_{(n_{12})}$$

<u>Holomorphy on $\mathbb{T}^4$?</u> holomorphy argument appears known/obvious to Shifman & Vainshtein, in their 1986 “Solution of anomaly puzzle…”

$$\Lambda^* \frac{d}{d\Lambda^*}\langle \lambda^2 \rangle \sim \langle \lambda^2 F^* \rangle \sim \langle \lambda^2 \bar{Q}_{\dot\alpha} \bar\psi^{\dot\alpha} + \lambda^2 \bar\psi^{\dot\alpha} \bar{Q}_{\dot\alpha} \rangle \sim \langle \bar{Q}_{\dot\alpha} \lambda^2 \bar\psi^{\dot\alpha} + \lambda^2 \bar\psi^{\dot\alpha} \bar{Q}_{\dot\alpha} \rangle = 0$$

—> holomoprphy on $\mathbb{T}^4$ as well, $\langle \lambda^2 \rangle = c\Lambda^3$, holomorphy -> no $L|\Lambda|$-dependence



$Q \in Z + 1/2$

$$\frac{\mathrm{Tr}_{\mathscr{H}_{n_{12}}} \, e^{-\beta H}(-1)^F \hat{T}_3 \, \lambda^2}{\mathrm{Tr}_{\mathscr{H}_{n_{12}}} \, e^{-\beta H}(-1)^F} = \frac{\int_{n_{12}=1, n_{34}=1} \mathscr{D}A \mathscr{D}\lambda \, e^{-S} \, \lambda^2}{\int_{n_{12}=1, n_{34}=0} \mathscr{D}A \mathscr{D}\lambda \, e^{-S}}$$

$Q \in Z$

$$\equiv \langle \lambda^2 \rangle = \sum_E (-)^F e^{-\beta E} \, \langle E, 0 \,|\, \lambda^2 \,|\, E, 0 \rangle_{(n_{12})}$$

$Q = \frac{1}{2}$, the leading semiclassical contribution to numerator, w/ two undotted λ zero modes.

what are these instantons?

skipping details García Perez, González-Arroyo, Pena introduced an analytic $\Delta = \frac{L_3 L_4 - L_1 L_2}{\sqrt{V}}$ expansion in allowing to construct the Q=1/2 instantons from the constant flux solution of 't Hooft (skip: issues with number of zero modes etc.)

following this Anber, EP 2210.13568 and thus deforming the symmetric T^4, we find

- only 2 λ (no $\bar{\lambda}$) zero modes
- four translational moduli z_n

→ *explicit expressions to* $O(\Delta)$

all orders Δ-independence of action

- *measure Δ-independent to all orders*
- *condensate Δ-independent to all orders*

argument assumes convergence (+ uses SUSY)

$$\int d(B \,\&\, F \text{ moduli}) \, \mathrm{tr}\lambda\lambda \Big|_{Q=\frac{1}{2} \text{ instanton}} \sim \int d^4x \, \mathrm{tr}F^2 = \frac{1}{2} S_{BPST} \quad \left(= \frac{4\pi^2}{g^2}\right)$$

SUSY

thus: most important is range of moduli?

range of moduli?

pure YM, Hamiltonian argument:

$$\langle W_1 \rangle_{n_{12},n_{34}} = \mathrm{Tr}_{\mathscr{H}_{n_{12}}} e^{-\beta H_\theta} \hat{T}_3 W_1 = 0, \text{ as } \langle E, \vec{e} \,|\, W_1 \,|\, E, \vec{e} \rangle = 0$$

- to find range of z_n moduli, require $\langle W_\mu \rangle = 0$ in pure-YM theory in small $\mathbb{T}^4$ with twists
- use uniqueness - numerical evidence strong!

$$e^{-\frac{4\pi^2}{g^2} - i\frac{\theta}{2}} \frac{V}{g^4} \int_M \prod_{k=1}^{4} dz_k \; W(x, z, C_{n_1,n_2,n_3,n_4}) + \text{h.c.} = 0 \;(\forall x, \theta) \quad \text{iff } z_k \in (0, 4\pi)$$

winding loop in Q=1/2
self-dual background

- the value of gaugino condensate ~ volume of moduli space

range of moduli?

- to find range of z_n moduli, require $\langle W_\mu \rangle = 0$ in pure-YM theory in small $\mathbb{T}^4$ with twists
- use uniqueness - numerical evidence strong!

iff $z_k \in (0,4\pi)$

- range of moduli found by demanding vanishing of Wilson loop vevs in pure-YM, is equivalent to that found by demanding that there exist gauge invariants, evaluated in solution background, that differentiate between all points $(0,4\pi)$ - i.e., we are not integrating over gauge equivalent values of moduli

Remark: Range of z_n moduli $(0,4\pi)$ means that instanton wraps twice around each direction of torus.
Local gauge invariants identify $z \sim z + 2\pi$, but ones dressed by Wilson loops see difference.

(also supported by numerics: F.D. Wandler, 2024 to appear)

Recall what we compute

$$\langle \lambda^2 \rangle = \sum_E (-)^F e^{-L_4 E} \langle E, e_3 = 0 | \lambda^2 | E, e_3 = 0 \rangle \Big|_{n_{12}=1,\ V_3 = L_1 L_2 L_3,\ \frac{L_3 L_4 - L_1 L_2}{\sqrt{L_1 L_2 L_3 L_4}} \ll 1,\ L_i \Lambda \ll 1}$$

all qualifications stated!

Collecting everything, we find

$$\langle \lambda^2 \rangle = 32\pi^2 \Lambda^3 = 2 \times \underline{\underline{16\pi^2 \Lambda^3}}$$

two times the $R^4, R^3 \times S^1$ result of weak-coupling calculations, all use same def. of scale $\Lambda^3 = \frac{M_{PV}^3}{g^2} e^{-\frac{4\pi^2}{g^2}}$

(reminder: factor of 2 from Witten index already divided out, so value in one vacuum only) thus, we seem to have a problem…

- we made an algebraic mistake (all factors spelled out in glory detail in paper)
- there is a loophole in L_i-independence argument? TD limit with flux more subtle?
- misidentified moduli space? (missed some global identification? **need rationale?**)
- other backgrounds contribute? (what? numerics supports uniqueness of Q=1/2 instanton)

- to boot, using one (no numeric study of uniqueness here!) of 't Hooft SU(N) solutions (+ Δ…) we find

$$\langle \lambda^2 \rangle = N \times 16\pi^2 \Lambda^3$$

N times the $R^4, R^3 \times S^1$ weak coupling instanton result,
in the usual normalization (N-fold degeneracy divided out, as in SU(2))

part 2 summary:

one of two weakly-coupled calculations of $\langle\lambda^2\rangle$: continuous connection to R^4 ⟷ using this new and deeper knowledge, revisit old (1984!) calculations of $\langle\lambda^2\rangle$ on T^4

$\langle\lambda^2\rangle_{R^4}$ $\langle\lambda^2\rangle_{T^4}$

$$\langle\lambda^2\rangle_{T^4} = 2 \times \langle\lambda^2\rangle_{R^4} \text{ for SU(2)}$$

why?

important for pushing & checking 'adiabatic continuity' program qualitatively

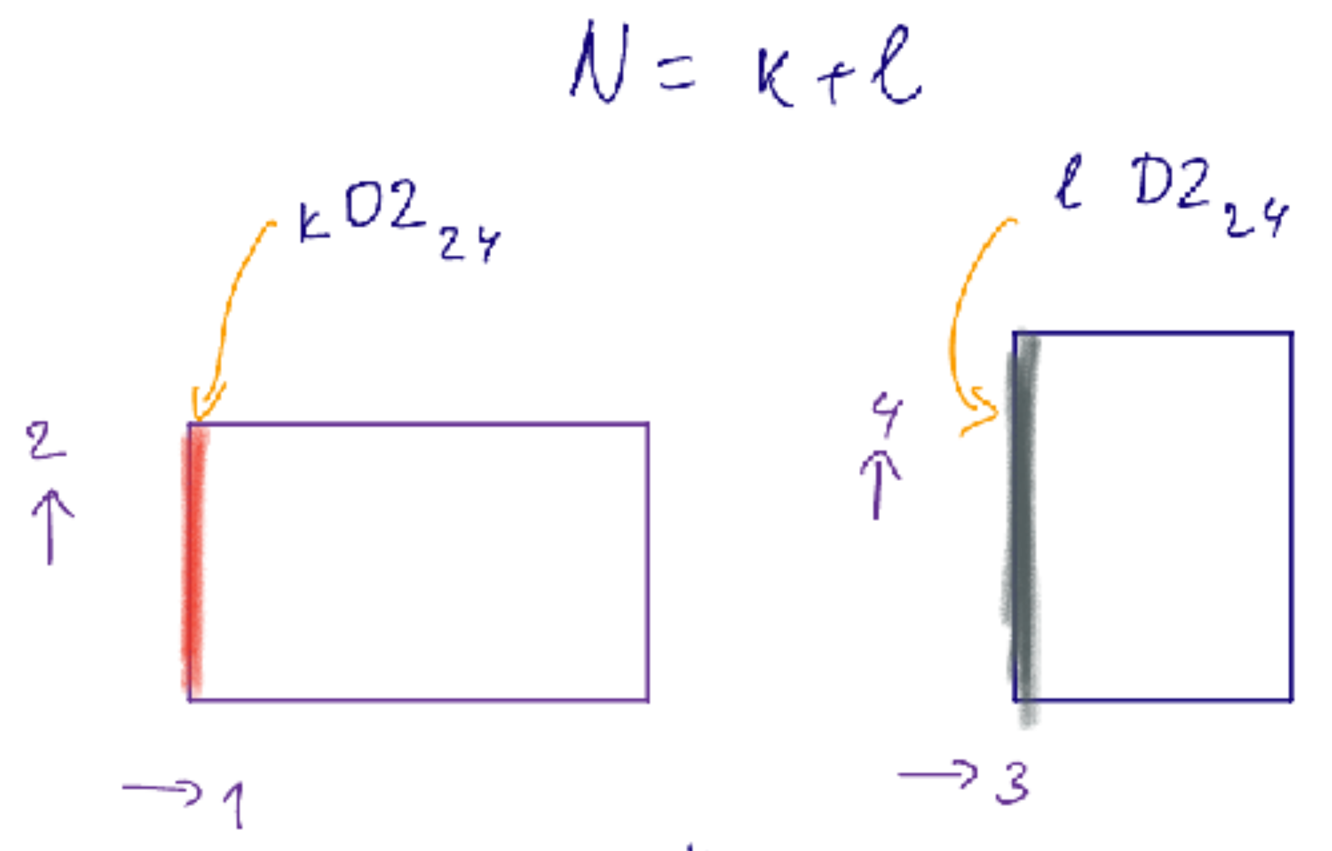
wish for better understanding of fractional charge instantons, semiclassics, and their role in gauge dynamics (for which some evidence has accumulated)

input from math-phys/string? (re. moduli space of fractional instantons) *part 2…*

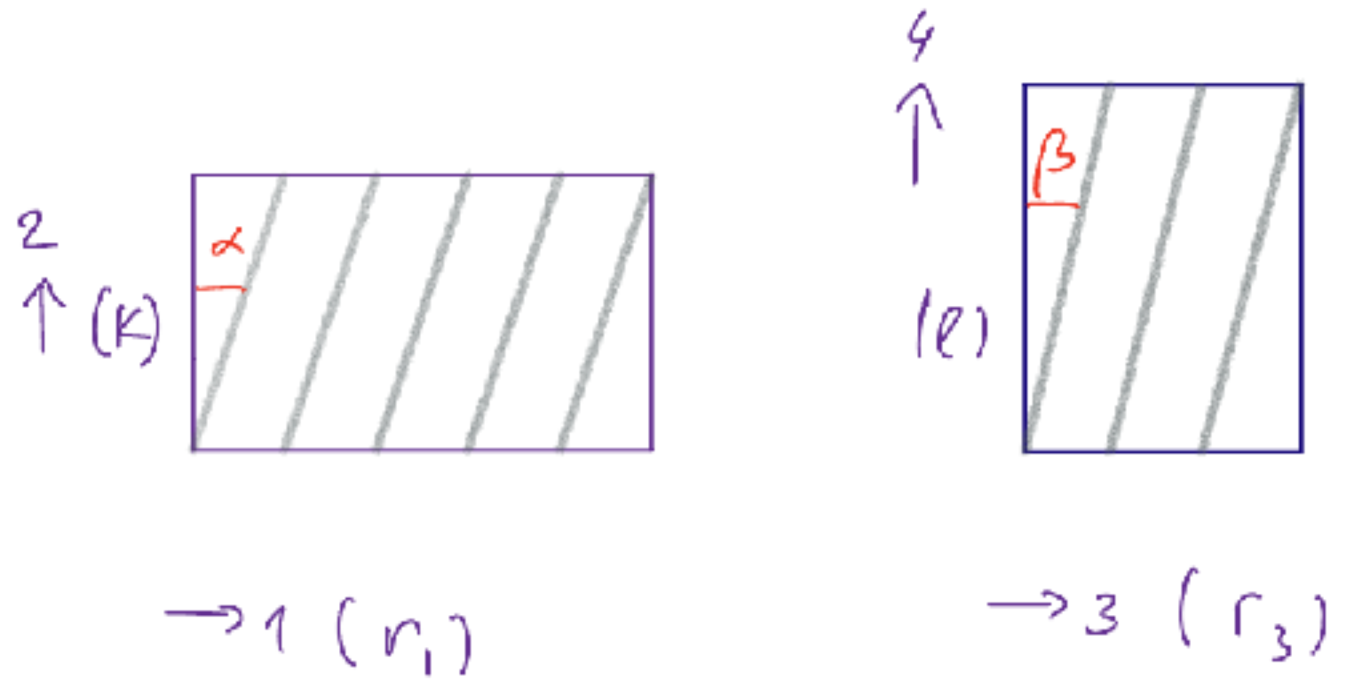
input from math-phys/string?

- moduli space of fractional instantons, motivated by D-branes vs ADHM

N D2-branes wrap 24 plane in $\mathbb{T}^4$ ~ T-dual in 13 -> N D4 on $\tilde{\mathbb{T}}^4$



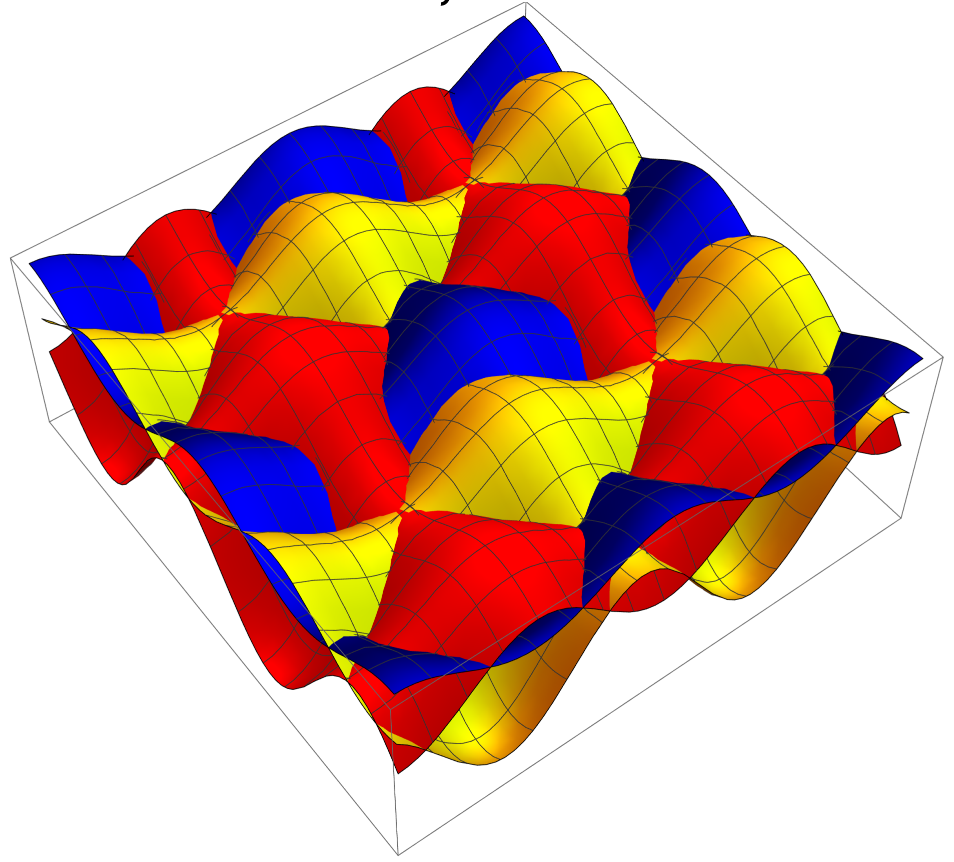
rotate two stacks ⇓ into 2-1 and 4-3, respectively



shown: $k = 5,\ r_1 = 1;\ l = 3,\ r_2 = 1$

part 2…

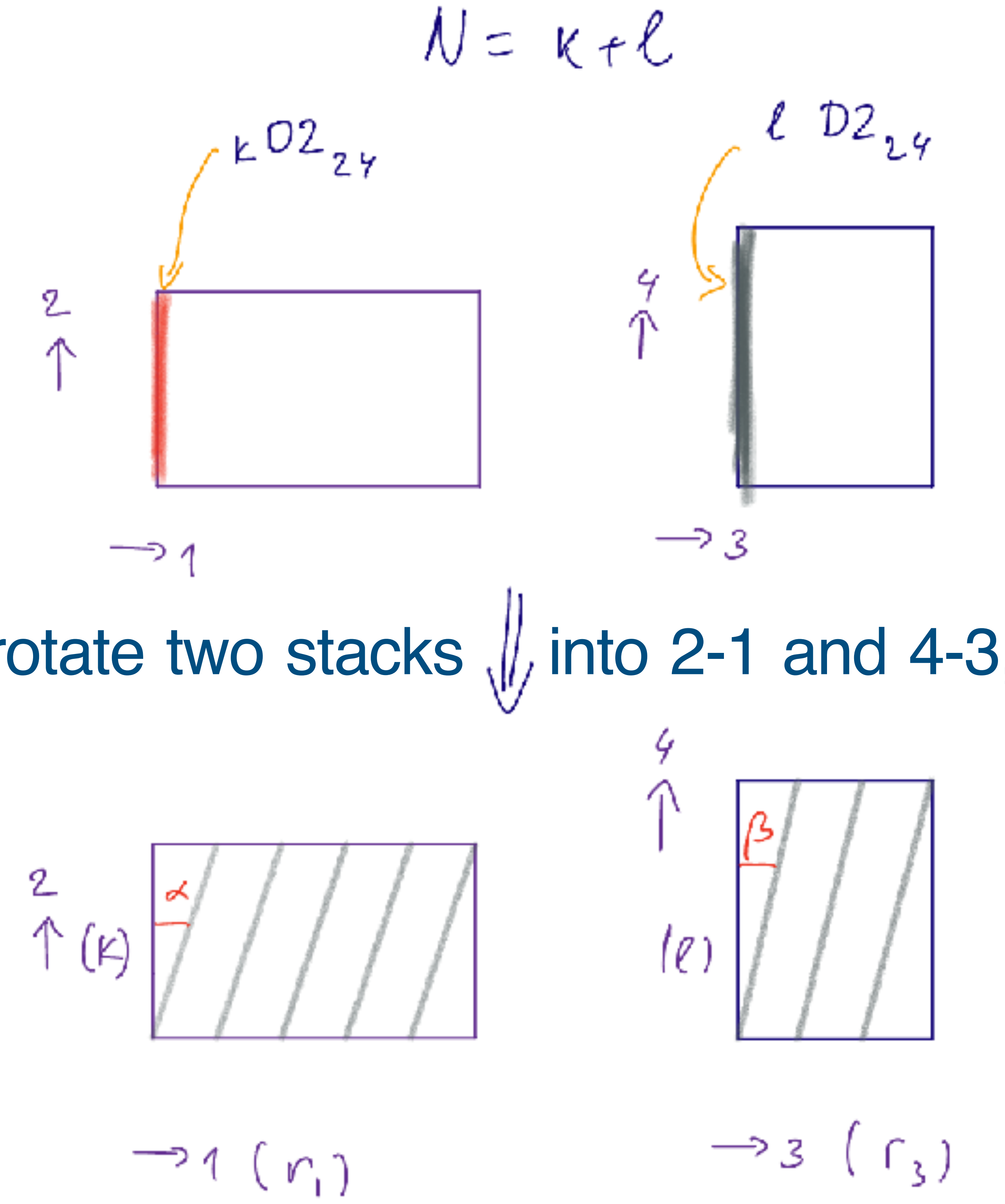
pic from Anber, EP 2307.04795



input from math-phys/string?

- moduli space of fractional instantons, motivated by D-branes vs ADHM

N D2-branes wrap 24 plane in $\mathbb{T}^4$ ~ T-dual in 13 -> N D4 on $\tilde{\mathbb{T}}^4$



shown: $k=5,\ r_1=1;\ l=3,\ r_3=1$ (BPS: $\alpha=-\beta$)

U(N) on $\tilde{\mathbb{T}}^4$ with $c_1^{(12)}=r_1,\ c_1^{(34)}=r_3,\ c_1^{(12)}=-\oint\frac{\mathrm{trF}_{12}}{2\pi}dx^1dx^2$

$ch_2=0=Q(U(N))$ but $Q(SU(N))=-\frac{r_1r_3}{N}$

$\frac{r_1}{kL_1L_2}=-\frac{r_3}{lL_3L_4}$ => SU(N) SD (U(1) not) 't Hooft's constant F on $\tilde{\mathbb{T}}^4$

detuning $\frac{r_1}{kL_1L_2}\neq-\frac{r_3}{lL_3L_4}$ => "lumpy" fractional instanton,

as per $\Delta-$expansion, numerics

… string?… too complex? (tachyon condensation)

(much structure hidden: monopole-instantons etc!)

global summary:

1. HOW MIXED ANOMALIES BETWEEN CHIRAL (invertible or not) AND CENTER SYMMETRY (“1-form”) ARISE IN HILBERT SPACE OF GAUGE THEORY ON TORUS & WHAT THEY IMPLY

=> degeneracies at finite volume

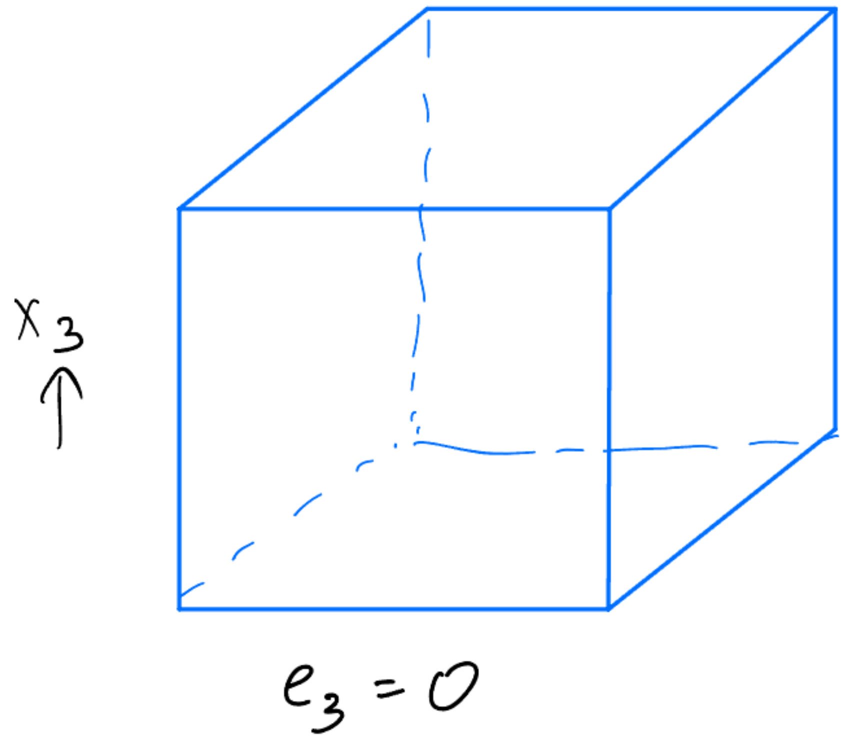
2. APPLICATION TO SYM: SEMICLASSICS ON $\mathbb{T}^4$ AND THE GAUGINO CONDENSATE *vs.* SEMICLASSICS ON $\mathbb{R}^4, \mathbb{R}^3 \times \mathbb{S}^1$

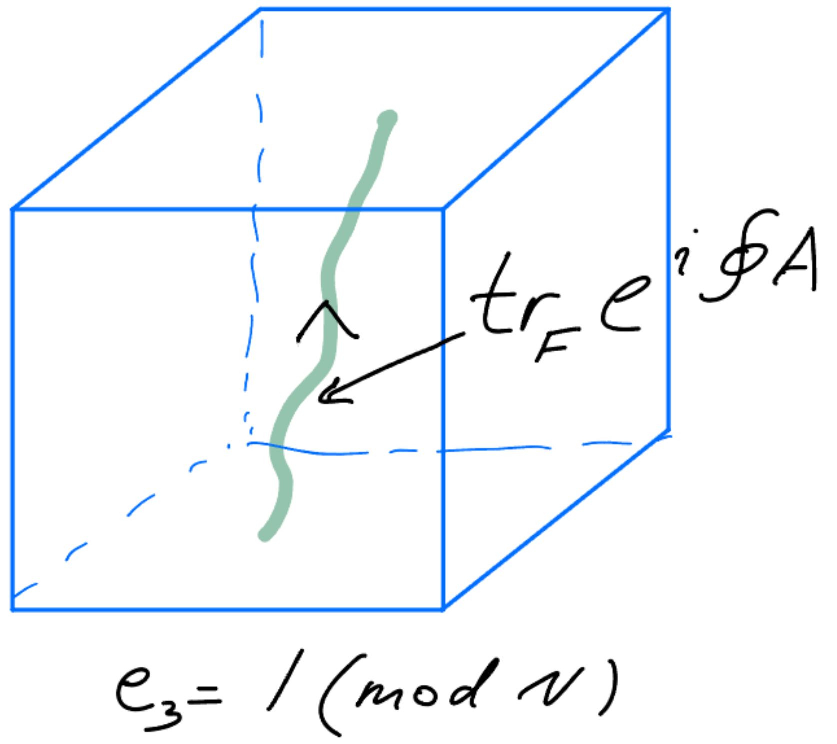
=> some puzzles re. “adiabatic continuity”

or: **“please, help me out with the factor of 2!”**

some old slides/backup

remarks on infinite vs. finite volume in ’t Hooft flux $n_{12} = 1$ background





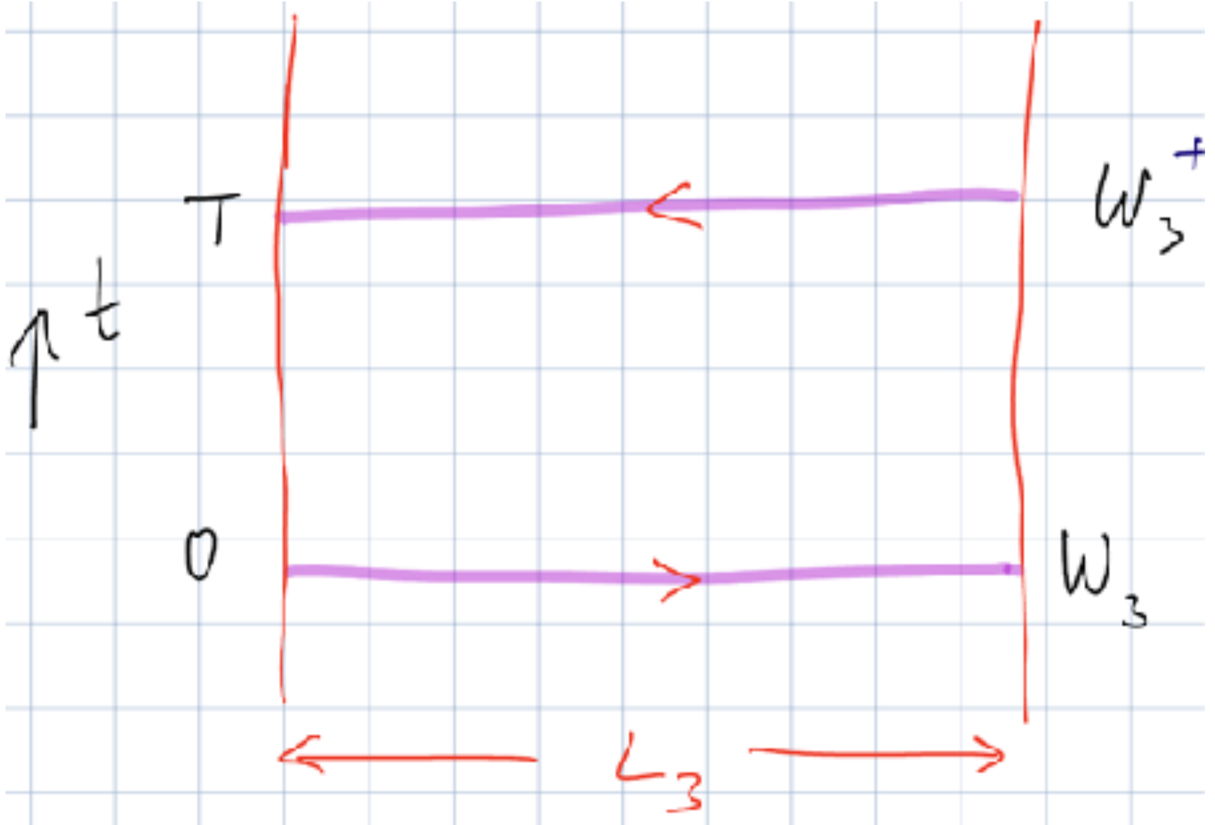
$= W_3$

Assuming confinement (unbroken center) -> broken chiral

$$|E=0, e_3 = 1\rangle_{(n_{12})}$$
$$|E=0, e_3 = 0\rangle_{(n_{12})}$$

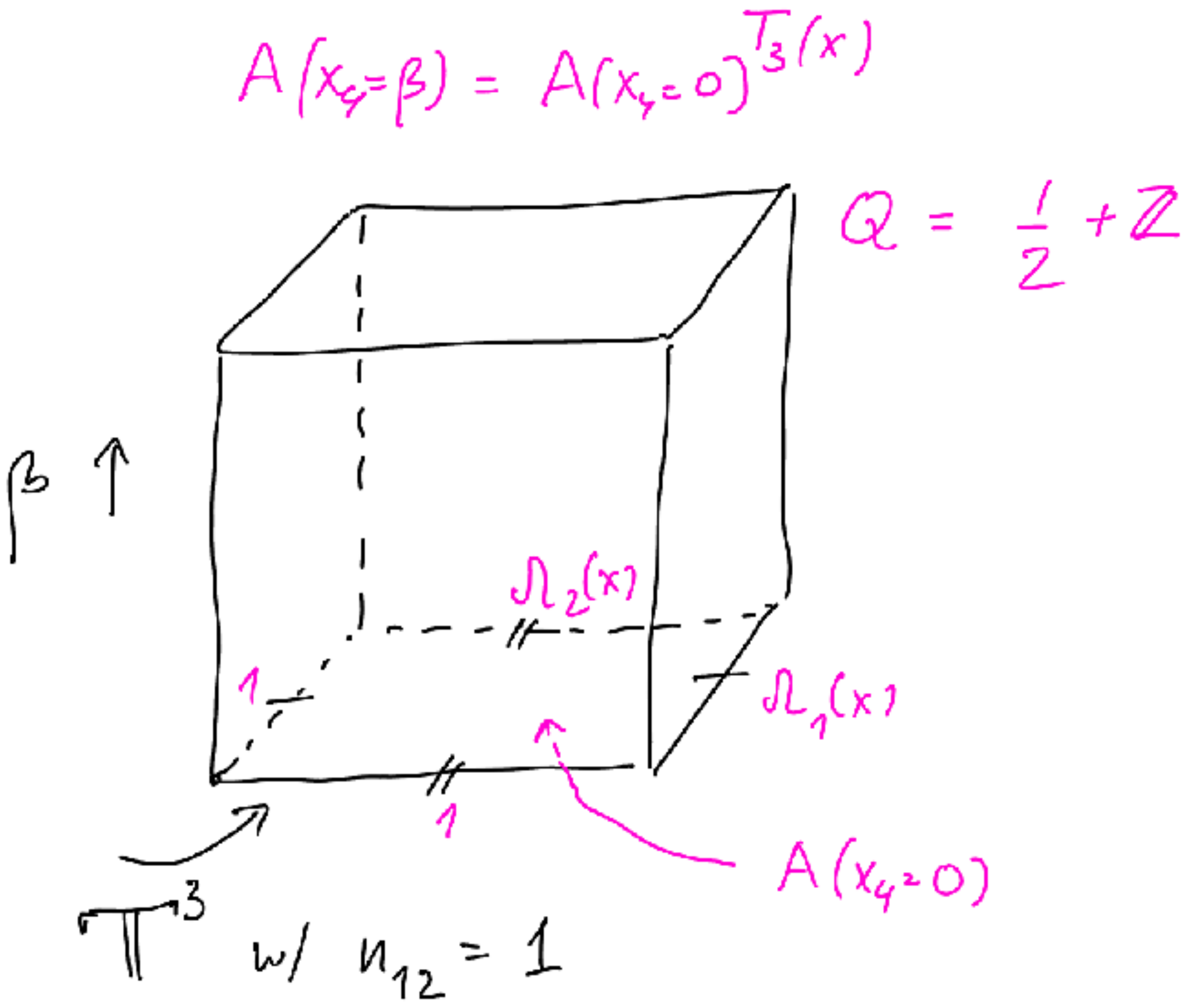
two clustering vacua in infinite volume limit

$${}_{(n_{12})}\langle 0, e_3 | W_3^\dagger(\vec{x}_{12}, T)\ W_3(\vec{x}_{12}, 0) | 0, e_3\rangle_{(n_{12})}\Big|_{T\to\infty} = \text{exact} = |{}_{(n_{12})}\langle 0, e_3 | W_3(\vec{x}_{12}, 0) | 0, e_3 + 1\rangle_{(n_{12})}|^2$$



$\neq 0$ for $L_{1,2,3} < \infty$ **(“perimeter,” “broken” T_3)**

for $L_{1,2} \to \infty$ m-x element expected to $\to 0$
by clustering $(W_3(\vec{x}_{12}, 0)$ local, at $L_3 < \infty)$ **(area law, unbroken T_3)**



$Q \in Z + 1/2$

$$\frac{\mathrm{Tr}_{\mathscr{H}_{n_{12}}}\, e^{-\beta H}(-1)^F \hat{T}_3\, \lambda^2}{\mathrm{Tr}_{\mathscr{H}_{n_{12}}}\, e^{-\beta H}(-1)^F} = \frac{\int_{n_{12}=1, n_{34}=1} \mathscr{D}A \mathscr{D}\lambda\, e^{-S}\, \lambda^2}{\int_{n_{12}=1, n_{34}=0} \mathscr{D}A \mathscr{D}\lambda\, e^{-S}}$$

$Q \in Z$

$$\equiv \langle \lambda^2 \rangle = \sum_E (-)^F e^{-\beta E} \langle E, 0 | \lambda^2 | E, 0 \rangle_{(n_{12})}$$

$Q = \frac{1}{2}$, the leading semiclassical contribution to numerator, w/ two undotted λ zero modes.

what are these instantons?

't Hooft, 1981, $Q = \frac{1}{2}$ constant flux background

BPS if symmetric T^4: $L_1 L_2 = L_3 L_4$

$$\bar{A}_n(x,z) = \bar{A}_n^3(x,z)\frac{\tau^3}{2} : \quad \begin{aligned} \bar{A}_1^3 &= \frac{2\pi x_2}{L_1 L_2} + \frac{z_1}{L_1}, \\ \bar{A}_2^3 &= \frac{z_2}{L_2}, \\ \bar{A}_3^3 &= \frac{2\pi x_4}{L_3 L_4} + \frac{z_3}{L_3}, \\ \bar{A}_4^3 &= \frac{z_4}{L_4}. \end{aligned}$$

moduli

$$F_{mn}^{(0)} = \frac{\tau^3}{2} \begin{pmatrix} 0 & -\frac{2\pi}{L_1 L_2} & 0 & 0 \\ \frac{2\pi}{L_1 L_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2\pi}{L_3 L_4} \\ 0 & 0 & \frac{2\pi}{L_3 L_4} & 0 \end{pmatrix}$$

Commun. Math. Phys. 81, 267–275 (1981)

Some Twisted Self-Dual Solutions for the Yang-Mills Equations on a Hypertorus*

such an action. All our solutions will be represented in a suitably chosen gauge that makes them look essentially translationally invariant and Abelian. However, considering the difficulty we had in finding them it looked worth-while to publish the result.

… SU(N) generalizations

’t Hooft, 1981, $Q = \frac{1}{2}$ constant flux background

BPS if symmetric T^4: $L_1 L_2 = L_3 L_4$

BPS - minimum action for given Q

- preserves 1/2 SUSY

(SYM: B/F det’s of nonzero modes cancel, up to power of PV regulator mass)

attempting symmetric T^4 … all looks bad!

- find 4 λ and 2 $\bar{\lambda}$ zero modes

(explicit, 2210.13568)

- these source gauge field EOM… lifted? how?

(we don’t know!)

- $L_1 L_2 = L_3 L_4$ does not allow taking some interesting limits, e.g., $R^2 \times T^2_{n_{12}}$

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Cohen, Gomez 1984 gave an expression using this solution ("toron") ***unaware (?) of subtleties mentioned, or of coefficient.***

In any case, since Hilbert space at finite $T^3_{n_{12}}$ was not understood at the time, interpretation would have been difficult.

’t Hooft, 1981, $Q = \frac{1}{2}$ constant flux background

BPS if symmetric T^4: $L_1 L_2 = L_3 L_4$

González-Arroyo, Pérez, Pena 2000

attempting symmetric T^4 … all looks bad! → deform the symmetric T^4, impose BPS:

- find 4 λ and 2 $\bar{\lambda}$ zero modes

  (explicit, 2210.13568)
- these source gauge field EOM… lifted? how?

  (we don't know!)
- $L_1 L_2 = L_3 L_4$ does not allow taking some interesting limits, e.g., $R^2 \times T^2_{n_{12}}$

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- only 2 λ zero modes
- no source term in YM field EOM
- $L_1 L_2 \neq L_3 L_4$, so can take limits

Sounds fantastic!?

There is "bad news," too: deformed-T^4 analytic BPS solution is only known to leading order in

$$\Delta = \frac{L_3 L_4 - L_1 L_2}{\sqrt{V}}$$

for SU(2), there is numerical evidence for uniqueness and convergence upon comparing to "exact" (=numerical) solution for $\Delta \leq 0.08\ldots$ so, for now, we stick with SU(2)

Remark:

If there were general statements known about the moduli space of $Q = \frac{r}{N}$ instantons on T^4, one could do certain calculations in SYM only using this knowledge (not explicit form of solutions) as integrals for some correlators reduce to those over bosonic and fermionic moduli.

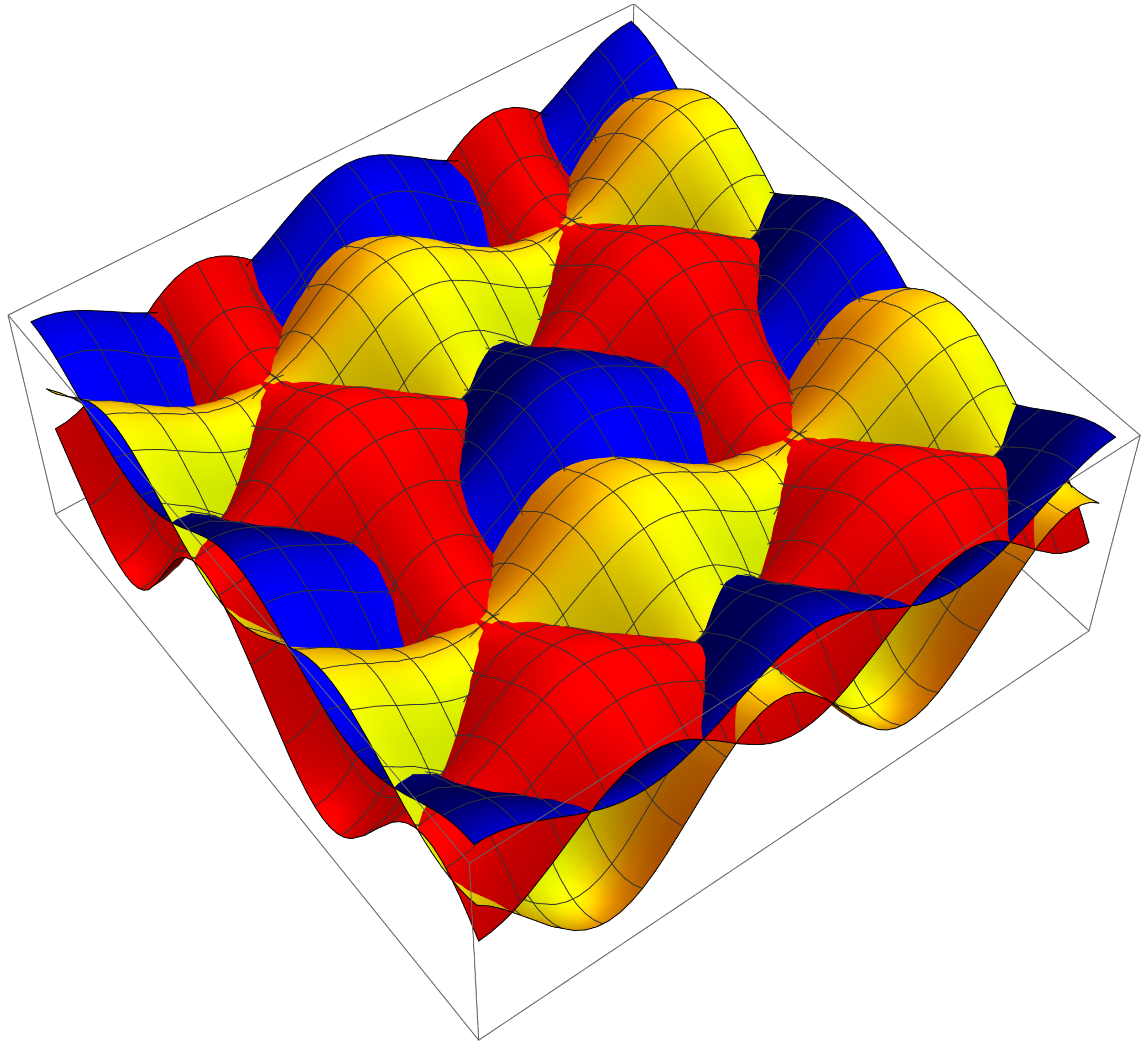
Alas…not known!

hence, we proceed by "trial and error" (consistency)

(as I'll discuss, our results may be taken to suggest that it is here where we likely need help!)

As an aside

at order Δ^1, gauge invariant densities (constant at Δ^0) acquire x-dependence



this is Q=3/N, in SU(N>3), 12 moduli are positions of 3 lumps
(yellow, red, blue; 2-torus shown doubled in size)

see Anber, EP 2307.04975

pure YM, Hamiltonian argument:

$$\langle W_1 \rangle_{n_{12}, n_{34}} = \mathrm{Tr}_{\mathscr{H}_{n_{12}}} \, e^{-\beta H_\theta} \, \hat{T}_3 \, W_1 = 0, \text{ as } \langle E, \vec{e} \, | \, W_1 \, | \, E, \vec{e} \rangle = 0$$

Most importantly: range of moduli?

- to find range of z_n moduli, require $\langle W_\mu \rangle = 0$ in pure-YM theory in femtouniverse with twists (use uniqueness):

$$e^{-\frac{4\pi^2}{g^2} - i\frac{\theta}{2}} \frac{V}{g^4} \int_M \prod_{k=1}^{4} dz_k \, W(x, z, C_{n_1, n_2, n_3, n_4}) + \text{h.c.} = 0 \; (\forall x, \theta)$$ iff $z_k \in (0, 4\pi)$

winding loop in Q=1/2
self-dual background

$$\begin{aligned} & W(x, C_{n_1,n_2,n_3,n_4}) \\ & = 2\cos\left[\frac{1}{2}\left(n_1(z_1 + \frac{2\pi x_2}{L_2}) + n_2(z_2 - \frac{2\pi x_1}{L_1}) + n_3(z_3 + \frac{2\pi x_4}{L_4}) + n_4(z_4 - \frac{2\pi x_3}{L_2})\right)\right] . \\ & \times \left[1 + \Delta\mathcal{F}(x, z)\right] . \end{aligned} \tag{5.5}$$

f-n of $z_1 + \dfrac{2\pi x_2}{L_2}$, etc., 2π periodic