Fluid Dynamics of Planetary Cores

PHY1530: Fluid Dynamics
Rayleigh Benard Convection: (planar)

Core Convection: (spherical shell)

other differences:
- spherical shell is rotating
- gravity in radial direction
- magnetic fields!
How Planets Make Magnetic Fields

**Dynamo action:**

complex motions + electrically conducting fluid + presence of a magnetic field → maintain magnetic field against Ohmic decay

**Magnetic Induction Equation:**

\[
\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{1}{\sigma \mu} \nabla^2 \vec{B}
\]

(change creation destruction)

(can derive from Maxwell's eqn's & Ohm's law)

**Necessary condition for dynamo action:**

Magnetic Reynolds number: 

\[
R_m = \frac{\nabla \times (\vec{v} \times \vec{B})}{\frac{1}{\sigma \mu} \nabla^2 \vec{B}} \approx \sigma \mu V L
\]

• For dynamo action in a fluid, we need \( R_m > 10-100 \)

• For Earth’s core: \( R_m \sim 500 \), for a sphere of copper: \( R_m \sim 6 \), for a typical star: \( R_m \sim 10^9 \)
Summary of Planetary Dynamos

Present Dynamos
- Earth
- Jupiter
- Saturn
- Uranus
- Neptune
- Ganymede

Past Dynamos
- Mars
- Moon
- Planetesimals

Jury Still Out
- Mercury
- Venus
- Io

Current dynamo?
Magneto-convective dynamo?
Past dynamo
Past dynamo
Induced field?
Comparing Planetary Magnetic Fields
Dimensional dynamo equations
(using Boussinesq approximation)

\[ \text{N-S:} \left( \frac{\partial}{\partial t} + \bar{v} \cdot \nabla \right) \bar{v} + 2 \bar{\Omega} \times \bar{v} = -\nabla p + \frac{\Delta \rho}{\rho} \bar{g} + \nu \nabla^2 \bar{v} + \frac{1}{\rho \mu} (\nabla \times \bar{B}) \times \bar{B} \]

\[ \text{Continuity:} \quad \nabla \cdot \bar{v} = 0 \]

\[ \text{Magnetic Induction:} \quad \left( \frac{\partial}{\partial t} - \eta \nabla^2 \right) \bar{B} = \nabla \times \left( \bar{v} \times \bar{B} \right) \]

\[ \text{Energy:} \quad \left( \frac{\partial}{\partial t} - \kappa \nabla^2 \right) T = -\bar{v} \cdot \nabla T \]

\( \bar{v} \): velocity  \quad \bar{\Omega} \): rotation vector  \quad \bar{g} \): gravity
\( p \): modified pressure  \quad \nu \): viscosity  \quad \bar{J} \): current density  \quad \sigma \): electrical conductivity
\( \rho \): density  \quad \bar{B} \): magnetic field  \quad \mu_0 \): magnetic permeability  \quad \kappa \): thermal diffusivity
\( \Delta \rho \): density perturbation  \quad \eta \): magnetic diffusivity = \((\sigma \mu_0)^{-1}\)  \quad T \): temperature
Differences with RBC equations

**N-S:** \[
\left( \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \vec{v} + 2\Omega \times \vec{v} = -\nabla p + \frac{\Delta \rho}{\rho} \mathbf{g} + \nu \nabla^2 \vec{v} + \frac{1}{\rho \mu} \left( \nabla \times \vec{B} \right) \times \vec{B}
\]

**Continuity:** \[\nabla \cdot \vec{v} = 0\]

**Magnetic Induction:** \[
\left( \frac{\partial}{\partial t} - \eta \nabla^2 \right) \vec{B} = \nabla \times (\vec{v} \times \vec{B})
\]

**Energy:** \[
\left( \frac{\partial}{\partial t} - \kappa \nabla^2 \right) T = -\vec{v} \cdot \nabla T
\]

- **Lorentz force**
- **Coriolis force**
- **advection**
- **diffusion**
- **needed new equation for new variable 'B' in N-S**

Coriolis force needed new equation for new variable 'B' in N-S
Non-dimensionalize:

Length:
\[ r = r_0 r' \] \[ r_0 : \text{ core radius} \]

Time
\[ t = \tau_\eta t' = \frac{r_0^2}{\eta} t' \] \[ \tau_\eta : \text{ magnetic diffusion time} \]

Velocity
\[ v = \frac{r_0}{\tau_\eta} v' = \frac{\eta}{r_0} v' \]

Temperature
\[ T = \Delta T T' \] \[ h_T : \text{ heat flux at inner boundary} \]

Magnetic field
\[ B = B_{\Lambda=1} B' = \sqrt{2\Omega \mu \eta} B' \] \[ B_{\Lambda=1} : \text{ Elsasser number=1 scale} \]

Pressure
\[ P = 2\Omega \eta \rho_o P' \]

and use equation of state (dimensional):
\[ \Delta \rho = -\rho_0 \alpha (T - T_0) \]

and a linear approximation for gravity (dimensional):
\[ \vec{g} = -g_0 \vec{r} / r_0 \]

Plug into the equations, simplify and remove the primes gives….
Non-dimensional equations

N-S: \( \frac{Ro_M}{\partial t} \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \right) \vec{v} + \hat{z} \times \vec{v} = -\nabla p + Ra(T - T_0)\vec{r} + E\nabla^2\vec{v} + J \times \vec{B} \)

MIE: \( \frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \nabla^2 \vec{B} \)

Energy: \( \left( \frac{\partial}{\partial t} - q_k \nabla^2 \right) T = -\vec{v} \cdot T \)

Continuity: \( \nabla \cdot \vec{v} = 0 \)

Notice: 4 non-dimensional numbers:
- \( Ro_M \), \( Ra \), \( E \), \( q_k \)
- (ie 4 independent control parameters)
## Non-dimensional numbers

### (Input)

<table>
<thead>
<tr>
<th>Number</th>
<th>Formula</th>
<th>Earth's core</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic Rossby #:</td>
<td>$Ro_M = \frac{\eta}{2\Omega L^2}$</td>
<td>$10^{-9}$</td>
<td></td>
</tr>
<tr>
<td>Ekman #:</td>
<td>$E = \frac{v}{2\Omega L^2}$</td>
<td>$10^{-15}$</td>
<td></td>
</tr>
<tr>
<td>Modified Rayleigh #:</td>
<td>$Ra = \frac{\alpha_T g_{0} h_{T} r_{0}^2}{2\Omega \eta}$</td>
<td>?</td>
<td>buoyancy/coriolis since rotation is main hindrance to convection</td>
</tr>
<tr>
<td>Roberts #:</td>
<td>$q_{\kappa} = \frac{\kappa}{\eta}$</td>
<td>$10^{-6}$</td>
<td></td>
</tr>
<tr>
<td>Prandtl #:</td>
<td>$Pr = \frac{\nu}{\kappa}$</td>
<td>$10^{-1} \rightarrow 10^{-2}$</td>
<td>(dependent on other 4)</td>
</tr>
</tbody>
</table>

### (Output)

<table>
<thead>
<tr>
<th>Number</th>
<th>Formula</th>
<th>Value</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic Reynolds #:</td>
<td>$Re_M = \frac{UL}{\eta}$</td>
<td>750</td>
<td>Using velocities near CMB</td>
</tr>
<tr>
<td>Kinetic Reynolds #:</td>
<td>$Re = \frac{UL}{\nu}$</td>
<td>$1.5 \times 10^9$</td>
<td>Using velocities near CMB</td>
</tr>
<tr>
<td>Elsasser #:</td>
<td>$\Lambda = \frac{B^2}{2\Omega \rho \mu_0 \eta}$</td>
<td>$O(1)$</td>
<td>Using magnetic field near CMB</td>
</tr>
<tr>
<td>Rossby #:</td>
<td>$Ro = \frac{U}{2\Omega L}$</td>
<td>$10^{-6}$</td>
<td>Using velocities near CMB</td>
</tr>
</tbody>
</table>
Comparing planets

<table>
<thead>
<tr>
<th></th>
<th>Earth</th>
<th>Ganymede</th>
<th>Jupiter</th>
<th>Uranus</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega$, rotation rate (s$^{-1}$)</td>
<td>$7 \times 10^{-5}$</td>
<td>$1 \times 10^{-5}$</td>
<td>$2 \times 10^{-4}$</td>
<td>$1.4 \times 10^{-5}$</td>
</tr>
<tr>
<td>Density (kg/m$^3$)</td>
<td>$1.1 \times 10^4$</td>
<td>$6 \times 10^3$</td>
<td>$1 \times 10^3$</td>
<td>$1 \times 10^3$</td>
</tr>
<tr>
<td>Size of dynamo region (m)</td>
<td>$3 \times 10^6$</td>
<td>$7 \times 10^5$</td>
<td>$3 \times 10^7$</td>
<td>$\sim 1 \times 10^7$</td>
</tr>
<tr>
<td>$H_T$, temperature scale height (m)</td>
<td>$1 \times 10^7$</td>
<td>$4 \times 10^7$</td>
<td>$1 \times 10^8$</td>
<td>$1 \times 10^7$</td>
</tr>
<tr>
<td>Conductive heat flow along adiabat (W/m$^2$)</td>
<td>$1.5 \times 10^{-2}$</td>
<td>$1 \times 10^{-3}$</td>
<td>$&lt; 10^{-1}$</td>
<td>$&lt; 10^{-2}$</td>
</tr>
<tr>
<td>Nominal convective heat flow (W/m$^2$)</td>
<td>$1 \times 10^{-2}$</td>
<td>$1 \times 10^{-3}$</td>
<td>$3$</td>
<td>$\sim 10^{-1}$</td>
</tr>
<tr>
<td>Magnetic diffusivity (m$^2$/s)</td>
<td>2</td>
<td>4</td>
<td>30</td>
<td>$\sim 100$</td>
</tr>
<tr>
<td>$R_m$ based on $v_{ml}$</td>
<td>$3 \times 10^3$</td>
<td>70</td>
<td>$3 \times 10^4$</td>
<td>700</td>
</tr>
<tr>
<td>$R_m$ based on $v_{mac}$</td>
<td>50</td>
<td>5</td>
<td>400</td>
<td>25</td>
</tr>
<tr>
<td>$\Lambda$, Elsasser number at top of dynamo</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>$\sim 0.01$?</td>
</tr>
</tbody>
</table>

$$R_{OM} = \frac{\eta}{2\Omega L^2}$$

$$E = \frac{\nu}{2\Omega L^2}$$

$$q_r = \frac{\kappa}{\eta}$$
Some Rotating MHD

\[
Ro_M \left( \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \vec{v} + \hat{z} \times \vec{v} = -\nabla p + Ra(T - T_0) \vec{r} + E \nabla^2 \vec{v} + \vec{J} \times \vec{B}
\]

Start simple
- no magnetic field, buoyancy
- consider mainstream flow (inertia & viscosity small)
- dominant force balance given by:

\[
\hat{z} \times \vec{v} = -\nabla p
\]

- take curl of equation to get:

\[
\frac{\partial \vec{v}}{\partial z} = 0 \quad \text{“Taylor Proudman Theorem”}
\]

- main force balance requires z-independent motion
- non-penetrative boundary conditions & continuity
  eq'n give:

\[
\vec{v} = V_G(s) \hat{\phi}
\]

- i.e. motion constrained to cylinders coaxial with the rotation axis, called "geostrophic flow":

\[
\Omega
\]

\[
\theta
\]

\[
s
\]
Add buoyancy: 2 effects:

(1) Thermal Winds

\[ 2 \rho \Omega \times \vec{v} = -\nabla p - \rho \alpha \Theta \vec{g} \]

take curl:

\[ \frac{\partial \vec{v}}{\partial z} = \frac{\alpha}{2\Omega} (\vec{g} \times \nabla \Theta) \]

so if temp. varies with latitude or longitude, it can cause variations of velocity in the z direction, but not in “r” direction (b/c \( g \) in r direction)

(2) Convection

- geostrophic flows & thermal winds can’t transport heat from interior to CMB (no radial motion)
- convective motions require presence of other forces (viscous, inertia, magnetic) to offset effect of rotation (Taylor-Proudman Theorem)
- however, convection still satisfies Taylor-Proudman theorem to leading order (derivatives in z much smaller than other directions)
- convection at onset:
Some Rotating MHD

Now lets add a magnetic field:
- either Lorentz force is part of dominant force balance
- or its not

If it is: \[ 2\rho \Omega \times \vec{v} = -\nabla p + \vec{J} \times \vec{B} - \rho \alpha \Theta \vec{g} \]  “magnetostrophic balance”

Can determine necessary B magnitude for this balance

\[ |2\rho \Omega \times \vec{v}| \sim |\vec{J} \times \vec{B}| \Rightarrow B \sim \sqrt{2\Omega \mu \eta} \]
Weak & Strong Dynamos

\[ Ro_M \left( \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \vec{v} + \hat{z} \times \vec{v} = -\nabla p + Ra(T - T_0)\vec{r} + E\nabla^2 \vec{v} + \vec{J} \times \vec{B} \]

If Lorentz force part of dominant balance:
- “strong field dynamo”, magnetic field is stronger
- convection is ‘easier’ (i.e. lower critical Ra #) b/c magnetic field
  balances Coriolis force, easing rotational constraint
- so planets want to be in this regime

If it isn’t:
- “weak field dynamo”, magnetic field generally weaker
- rotational constraint makes convection difficult
- magnetic field doesn’t affect velocity field so much
  (Lorentz force is small)
Why not ignore viscosity and inertia?

-involved in balancing deviations from a Taylor state --> important dynamically

-without them:

\[
\ddot{z} \times u = -\nabla p + j \times B + R_\alpha \Theta \dot{r},
\]

\[
\int_\Sigma (j \times B)_\phi dS = 0.
\]

(Taylor State)

-in a Taylor state, magnetic torques on surface of cylinders in the core vanish

-this is a major constraint on the structure of the magnetic field

-with inertia and viscosity, can balance the Lorentz torque:

\[
R_\phi (1 - s^2)^{1/2} \frac{\partial u_\phi(s)}{\partial t} + E^{1/2} (1 - s^2)^{-1/2} u_\phi(s) = \frac{1}{4\pi s} \int_\Sigma (j \times B)_\phi dS.
\]

\[\text{inertia}\]

\[\text{viscosity}\]

viscosity: Lorentz torque scales with \(u\), so any initial torque results in motion that acts to decrease torque. Eventually torque=0 and \(u=0\) and you are in a Taylor state

inertia: Lorentz torque scales with acceleration \(\text{du/dt}\), so any initial torque results in acceleration. When torque=0, still have \(u\), so you overshoot --> “Torsional oscillations”

Note: also can’t ignore viscosity b/c of degree of equation
Evidence for torsional oscillations

-is there any evidence for torsional oscillations? Yes! Observational evidence in length of day records and geomagnetic jerks
Numerical Models:

The Process

- Input B,v,T
- Navier-Stokes, Magnetic Induction & Energy equations
- Spherical geometry
- Boundary conditions (B,v,T)
- Approximations

Evolve in time

Output: B,v,T

Characteristics

- geometry:
  \[ r_{io} = \frac{r_i}{r_o} \]
- small viscosity:
  \[ E = \frac{v}{2\Omega r_o^2} \ll 1 \]
- small inertia:
  \[ Ro = \frac{\eta}{2\Omega r_o^2} \ll 1 \]
- strong convection:
  \[ Ra = \frac{\alpha g_o \Delta T r_o}{2\Omega \eta} \gg Ra_c \]
Results from dynamo models

(KB) weak field dynamo

(KB) strong field dynamo
convection cells in dynamo models
How far are we?
Convection cells in dynamo models


Using the Earth Simulator:
(but not equilibrated yet, magnetic field weak)

axial component of the vorticity

E=2.3x10^{-7}: (lowest ever achieved),  E=2.6x10^{-6}
Experiments:

Karlsruhe dynamo

Dan Lathrop’s planetary dynamo
Case Study: Earth's dynamo

Field characteristics:
- Axial dipole dominance
- Chaotic reversals
- Secular variation
- High latitude flux spots

Contour interval = 10^5
Earth-like Numerical Model: