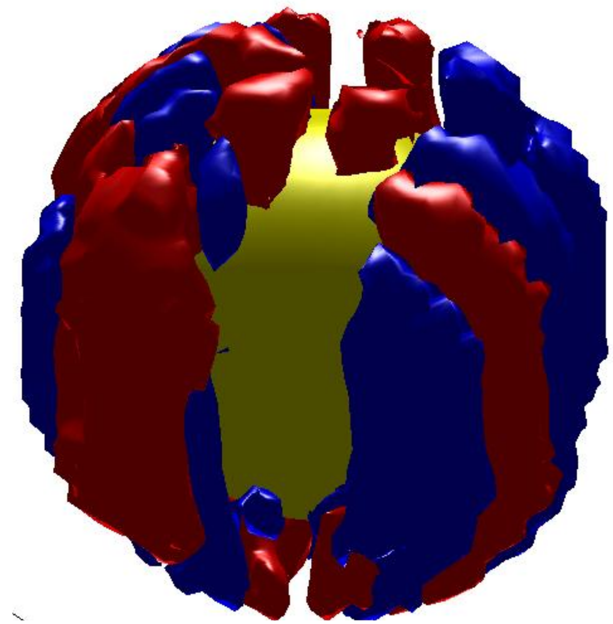
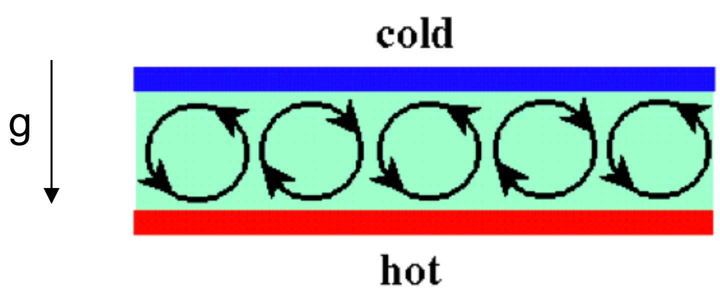
Fluid Dynamics of Planetary Cores



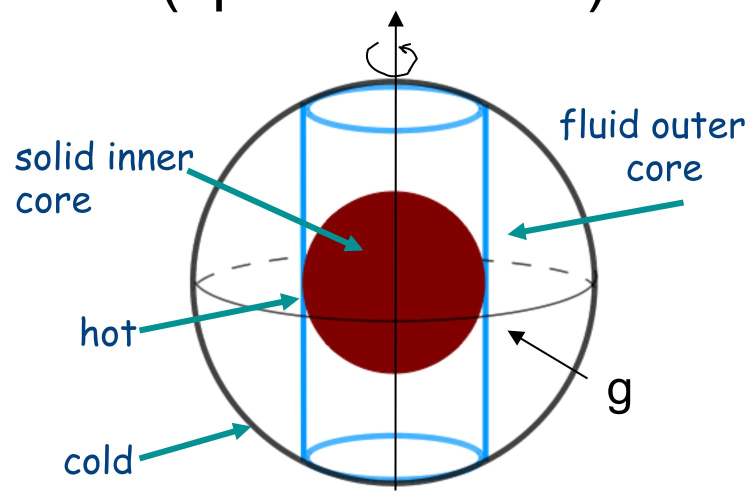
PHY1530: Fluid Dynamics

Geometry

Rayleigh Benard Convection: (planar)



Core Convection: (spherical shell)



other differences:
- spherical shell is rotating
- gravity in radial direction
- magnetic fields!

How Planets Make Magnetic Fields

Dynamo action:

complex motions + electrically conducting fluid + presence of a magnetic field $\Longrightarrow$ maintain magnetic field against Ohmic decay

Magnetic Induction Equation:

(can derive from Maxwell's eqn's & Ohm's law)

$$\underbrace{\frac{\partial \vec{B}}{\partial t}}_{\text{change}} = \underbrace{\nabla \times (\vec{v} \times \vec{B})}_{\text{creation}} + \underbrace{\frac{1}{\sigma\mu} \nabla^2 \vec{B}}_{\text{destruction}}$$

Necessary condition for dynamo action:

Magnetic Reynolds number: $$R_m = \frac{\nabla \times (v \times B)}{\frac{1}{\sigma\mu} \nabla^2 B} \approx \sigma\mu VL$$

- For dynamo action in a fluid, we need R_m>10-100
- For Earth's core: R_m~500, for a sphere of copper: R_m~6, for a typical star: R_m~10^9

Summary of Planetary Dynamos

Present Dynamos

Earth
Jupiter
Saturn

Uranus
Neptune

Ganymede

Past Dynamos

Mars
Moon
Planetesimals

Current dynamo?

Magneto-convective dynamo?

Jury Still Out

Mercury

Venus

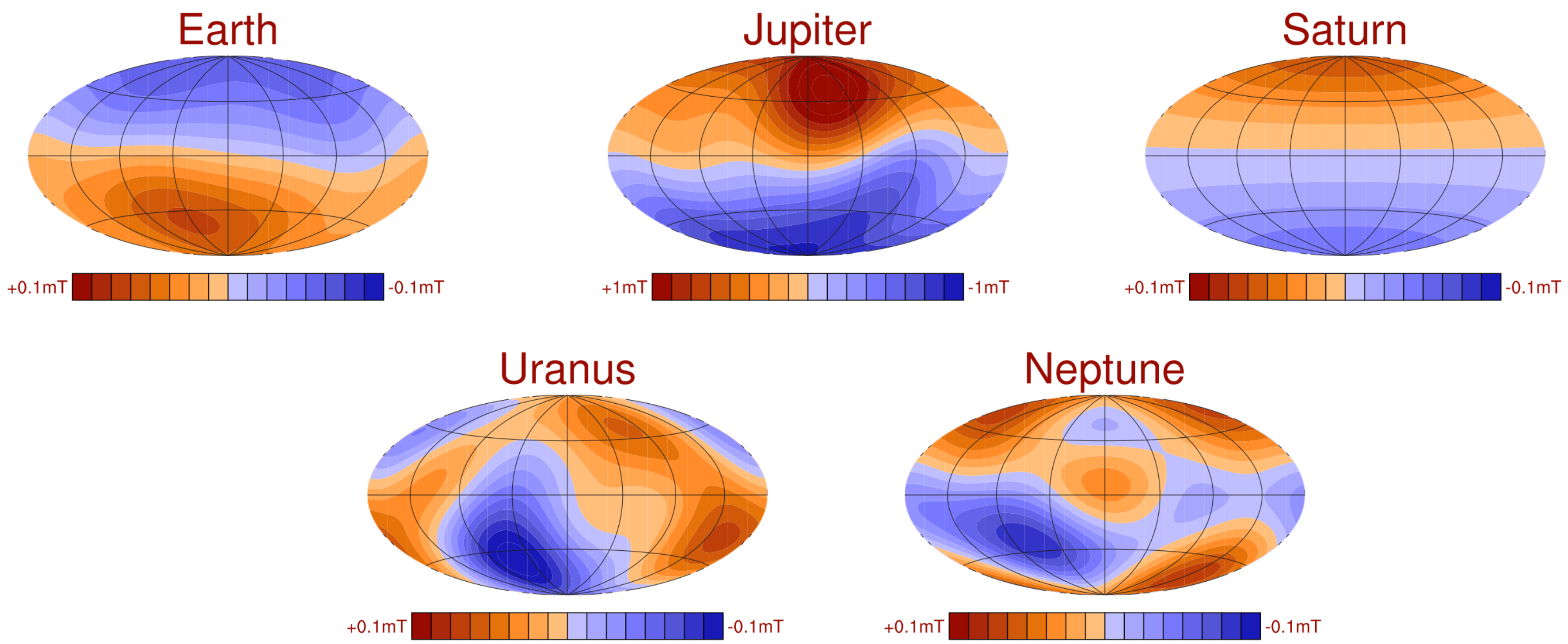
Io

Past dynamo

Past dynamo

Induced field?

Comparing Planetary Magnetic Fields



Dimensional dynamo equations
(using Boussinesq approximation)

N-S: $$\left(\frac{\partial}{\partial t}+\vec{v}\bullet\nabla\right)\vec{v}+2\vec{\Omega}\times\vec{v}=-\nabla p+\frac{\Delta\rho}{\rho}\vec{g}+\nu\nabla^2\vec{v}+\frac{1}{\rho\mu}\left(\nabla\times\vec{B}\right)\times\vec{B}$$

Continuity: $$\nabla\bullet\vec{v}=0$$

Magnetic Induction: $$\left(\frac{\partial}{\partial t}-\eta\nabla^2\right)\vec{B}=\nabla\times(\vec{v}\times\vec{B})$$

Energy: $$\left(\frac{\partial}{\partial t}-\kappa\nabla^2\right)T=-\vec{v}\bullet\nabla T$$

$\vec{v}$: velocity
$\vec{\Omega}$: rotation vector
p : modified pressure
ρ : density
$\Delta\rho$: density perturbation

$\vec{g}$: gravity
ν : viscosity
$\vec{J}$: current density
$\vec{B}$: magnetic field
η : magnetic diffusivity = $(\sigma\mu_0)^{-1}$

σ : electrical conductivity
μ_0: magnetic permeability
κ: thermal diffusivity
T: temperature

Differences with RBC equations

Lorentz force

N-S: $$\left(\frac{\partial}{\partial t}+\vec{v}\bullet\nabla\right)\vec{v}+\boxed{2\vec{\Omega}\times\vec{v}}=-\nabla p+\frac{\Delta\rho}{\rho}\vec{g}+\nu\nabla^2\vec{v}+\boxed{\frac{1}{\rho\mu}\left(\nabla\times\vec{B}\right)\times\vec{B}}$$

Coriolis force

Continuity: $$\nabla\bullet\vec{v}=0$$

advection

Magnetic Induction: $$\boxed{\left(\frac{\partial}{\partial t}-\eta\nabla^2\right)\vec{B}=\nabla\times(\vec{v}\times\vec{B})}$$

needed new equation for new variable 'B' in N-S

diffusion

Energy: $$\left(\frac{\partial}{\partial t}-\kappa\nabla^2\right)T=-\vec{v}\bullet\nabla T$$

Non-dimensionalize:

Length: $r = r_0 r'$ $\quad r_0$: core radius

Time $t = \tau_\eta t' = \frac{r_0^2}{\eta} t'$ $\quad \tau_\eta$: magnetic diffusion time

Velocity $v = \frac{r_0}{\tau_\eta} v' = \frac{\eta}{r_0} v'$

Temperature $T = \Delta T T'$ $\quad h_T$: heat flux at inner boundary

Magnetic field $B = B_{\Lambda=1} B' = \sqrt{2\Omega\rho\mu\eta} B'$ $\quad B_{\Lambda=1}$: Elsasser number=1 scale

pressure $P = 2\Omega\eta\rho_o P'$

and use equation of state (dimensional): $\Delta\rho = -\rho_0 \alpha (T - T_0)$
and a linear approximation for gravity (dimensional): $\vec{g} = -g_0 \vec{r} / r_0$

Plug into the equations, simplify and remove the primes gives….

Non-dimensional equations

N-S: $$Ro_M\left(\frac{\partial}{\partial t}+\vec{v}\bullet\nabla\right)\vec{v}+\hat{z}\times\vec{v}=-\nabla p+Ra(T-T_0)\vec{r}+E\nabla^2\vec{v}+\vec{J}\times\vec{B}$$

inertia, Coriolis, pressure gradient, buoyancy, viscous, Lorentz

MIE: $$\frac{\partial\vec{B}}{\partial t}=\nabla\times(\vec{v}\times\vec{B})+\nabla^2\vec{B}$$

Energy: $$\left(\frac{\partial}{\partial t}-q_\kappa\nabla^2\right)T=-\vec{v}\bullet T$$

Continuity: $$\nabla\bullet\vec{v}=0$$

Notice: 4 non-dimensional numbers:
Ro_M, Ra, E, q_k
(ie 4 independent control parameters)

Non-dimensional numbers

(Input)		Earth's core	
Magnetic Rossby #:	$Ro_M = \frac{\eta}{2\Omega L^2}$	10^{-9}	
Ekman #:	$E = \frac{\nu}{2\Omega L^2}$	10^{-15}	
Modified Rayleigh #:	$Ra = \frac{\alpha_T g_0 h_T r_0^2}{2\Omega\eta}$	?	buoyancy/coriolis since rotation is main hindrance to convection
Roberts #:	$q_\kappa = \frac{\kappa}{\eta}$	10^{-6}	
⇓			
Prandtl #:	$\Pr = \frac{\nu}{\kappa}$	$10^{-1} \to 10^{-2}$	(dependent on other 4)

(Output)			
Magnetic Reynolds #:	$\mathrm{Re}_M = \frac{UL}{\eta}$	750	Using velocities near CMB
Kinetic Reynolds #:	$\mathrm{Re} = \frac{UL}{\nu}$	1.5×10^9	Using velocities near CMB
Elsasser #:	$\Lambda = \frac{B^2}{2\Omega\rho\mu_0\eta}$	$O(1)$	Using magnetic field near CMB
Rossby #:	$Ro = \frac{U}{2\Omega L}$	10^{-6}	Using velocities near CMB

Comparing planets

	Earth	Ganymede	Jupiter	Uranus
Ω, rotation rate (s^{-1})	7×10^{-5}	1×10^{-5}	2×10^{-4}	1.4×10^{-5}
Density (kg/m^3)	1.1×10^4	6×10^3	1×10^3	1×10^3
Size of dynamo region (m)	3×10^6	7×10^5	3×10^7	$\sim1\times10^7$
H_T, temperature scale height (m)	1×10^7	4×10^7	1×10^8	1×10^7
Conductive heat flow along adiabat (W/m^2)	1.5×10^{-2}	1×10^{-3}	$<10^{-1}$	$<10^{-2}$
Nominal convective heat flow (W/m^2)	1×10^{-2}	1×10^{-3}	3	$\sim10^{-1}$
Magnetic diffusivity (m^2/s)	2	4	30	~100
R_m based on v_{ml}	3×10^3	70	3×10^4	700
R_m based on v_{mac}	50	5	400	25
Λ, Elsasser number at top of dynamo	0.3	0.3	0.3	$\sim0.01?$

$Ro_M = \dfrac{\eta}{2\Omega L^2}$	1.5×10^{-9}	2×10^{-7}	8×10^{-11}	3×10^{-8}
$E = \dfrac{\nu}{2\Omega L^2}$	7.6×10^{-16}	5×10^{-14}	2×10^{-18}	3×10^{-16}
$q_\kappa = \dfrac{\kappa}{\eta}$	2.5×10^{-6}	1.2×10^{-6}	1.6×10^{-7}	5×10^{-8}

Some Rotating MHD

$$Ro_M\left(\frac{\partial}{\partial t}+\vec{v}\bullet\nabla\right)\vec{v}+\hat{z}\times\vec{v}=-\nabla p+Ra(T-T_0)\vec{r}+E\nabla^2\vec{v}+\vec{J}\times\vec{B}$$

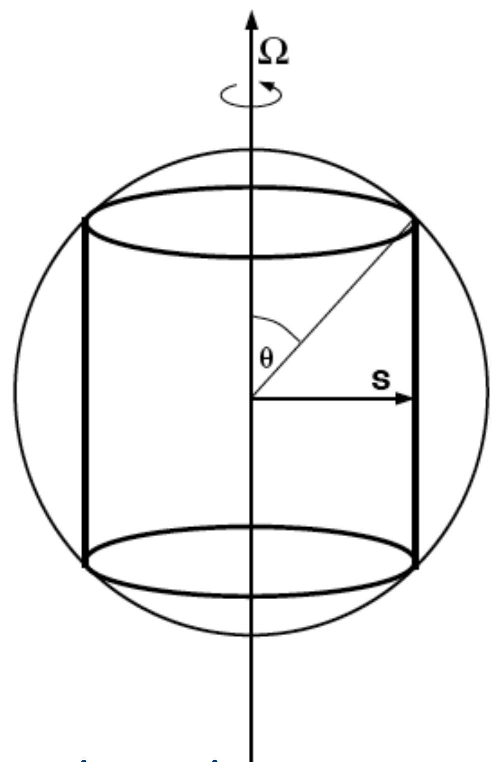
Start simple

- -no magnetic field, buoyancy
- -consider mainstream flow (inertia & viscosity small)
- -dominant force balance given by:

$$\hat{z}\times\vec{v}=-\nabla p$$

-take curl of equation to get:

$$\frac{\partial\vec{v}}{\partial z}=0$$ "Taylor Proudman Theorem"



- -main force balance requires z-independent motion
- -non-penetrative boundary conditions & continuity eq'n give: $\vec{v}=V_G(s)\hat{\phi}$

-i.e. motion constrained to cylinders coaxial with the rotation axis, called"geostrophic flow":

Some Rotating MHD

Add buoyancy: 2 effects:

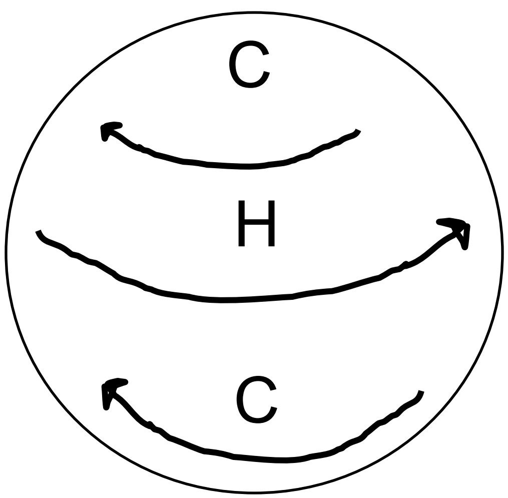
(1) Thermal Winds

$$2\rho\vec{\Omega}\times\vec{v} = -\nabla p - \rho\alpha\Theta\vec{g}$$

take curl:

$$\frac{\partial\vec{v}}{\partial z} = \frac{\alpha}{2\Omega}(\vec{g}\times\nabla\Theta)$$

so if temp. varies with latitude or longitude, it can cause variations of velocity in the z direction, but not in "r" direction (b/c g in r direction)



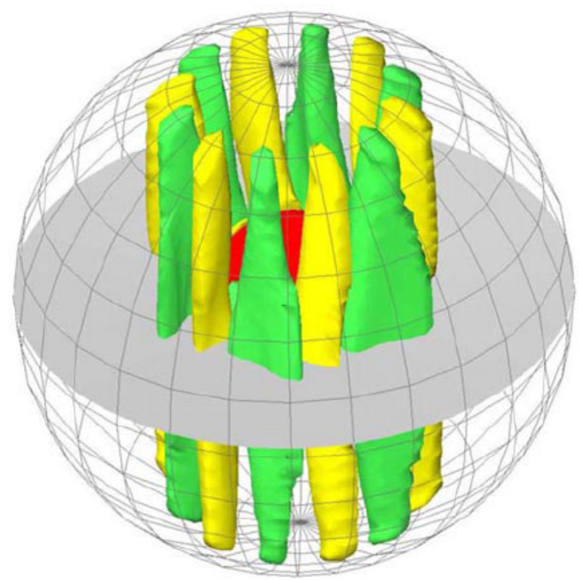
(2) Convection

-geostrophic flows & thermal winds can't transport heat from interior to CMB (no radial motion)

-convective motions require presence of other forces (viscous, inertia, magnetic) to offset effect of rotation (Taylor-Proudman Theorem)

-however, convection still satisfies Taylor-Proudman theorem to leading order (derivatives in z much smaller than other directions)

-convection at onset:



Some Rotating MHD

Now lets add a magnetic field:
- -either Lorentz force is part of dominant force balance
- -or its not

If it is: $2\rho\vec{\Omega}\times\vec{v} = -\nabla p + \vec{J}\times\vec{B} - \rho\alpha\Theta\vec{g}$ "magnetostrophic balance"

Can determine necessary B magnitude for this balance

$$\left|2\rho\vec{\Omega}\times\vec{v}\right| \sim \left|\vec{J}\times\vec{B}\right| \Rightarrow B \sim \sqrt{2\Omega\rho\mu\eta}$$

Weak & Strong Dynamos

$$Ro_M\left(\frac{\partial}{\partial t}+\vec{v}\bullet\nabla\right)\vec{v}+\hat{z}\times\vec{v}=-\nabla p+Ra(T-T_0)\vec{r}+E\nabla^2\vec{v}+\vec{J}\times\vec{B}$$

If Lorentz force part of dominant balance:

-"strong field dynamo", magnetic field is stronger

- convection is 'easier' (i.e. lower critical Ra #) b/c magnetic field balances Coriolis force, easing rotational constraint
- so planets want to be in this regime

If it isn't:

- "weak field dynamo", magnetic field generally weaker
- rotational constraint makes convection difficult
- magnetic field doesn't affect velocity field so much (Lorentz force is small)

Why not ignore viscosity and inertia?

-involved in balancing deviations from a Taylor state --> important dynamically
-without them:

$$\hat{\boldsymbol{z}} \times \boldsymbol{u} = -\nabla p + \boldsymbol{j} \times \boldsymbol{B} + R_a \Theta \hat{\boldsymbol{r}},$$

$$\Longrightarrow \int_\Sigma (\boldsymbol{j} \times \boldsymbol{B})_\phi \, dS = 0,$$

(Taylor State)

-in a Taylor state, magnetic torques on surface of cylinders in the core vanish
-this is a major constraint on the structure of the magnetic field

-with inertia and viscosity, can balance the Lorentz torque:

$$R_o(1-s^2)^{\frac{1}{2}} \frac{\partial u_\phi(s)}{\partial t} + E^{\frac{1}{2}}(1-s^2)^{-\frac{1}{4}} u_\phi(s) = \frac{1}{4\pi s} \int_\Sigma (\boldsymbol{j} \times \boldsymbol{B})_\phi \, dS.$$

inertia viscosity

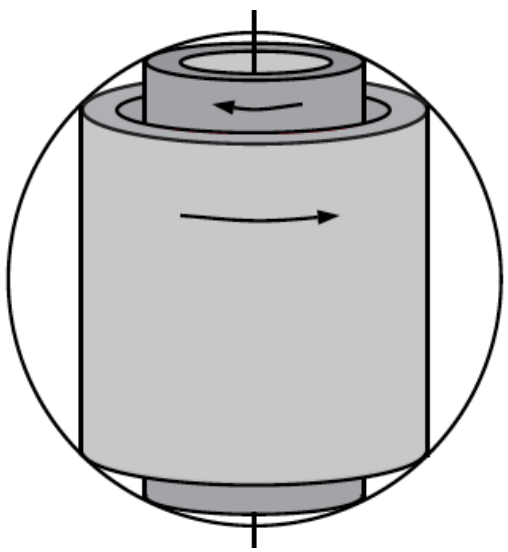
viscosity: lorentz torque scales with u, so any initial torque results in motion that acts to decrease torque. Eventually torque=0 and u=0 and you are in a Taylor state

inertia: lorentz torque scales with acceleration du/dt, so any initial torque results in acceleration. When torque=0, still have u, so you overshoot --> "Torsional oscillations"

Note: also can't ignore viscosity b/c of degree of equation

Evidence for torsional oscillations

-is there any evidence for torsional oscillations? Yes! observational evidence in length of day records and geomagnetic jerks



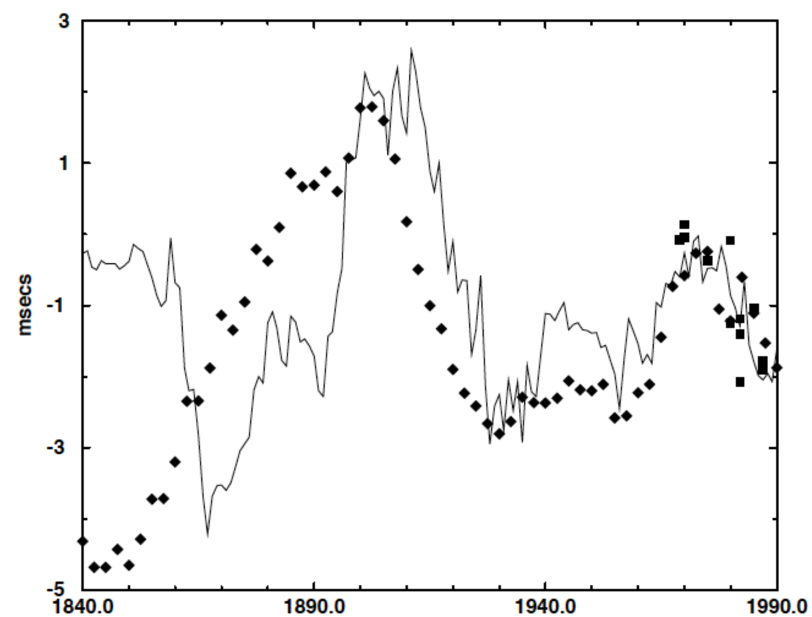
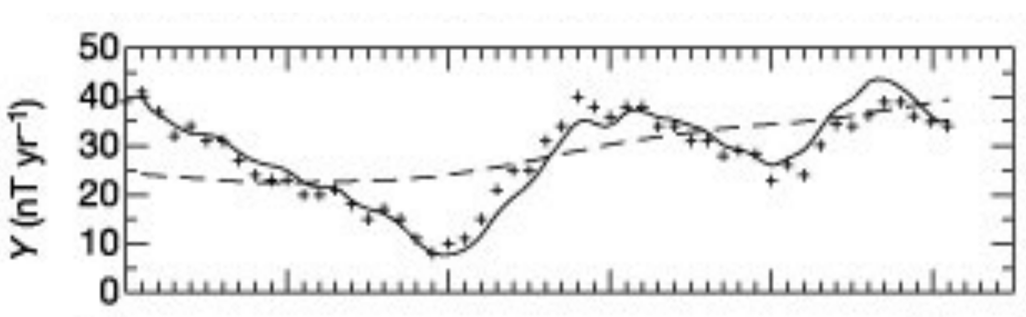
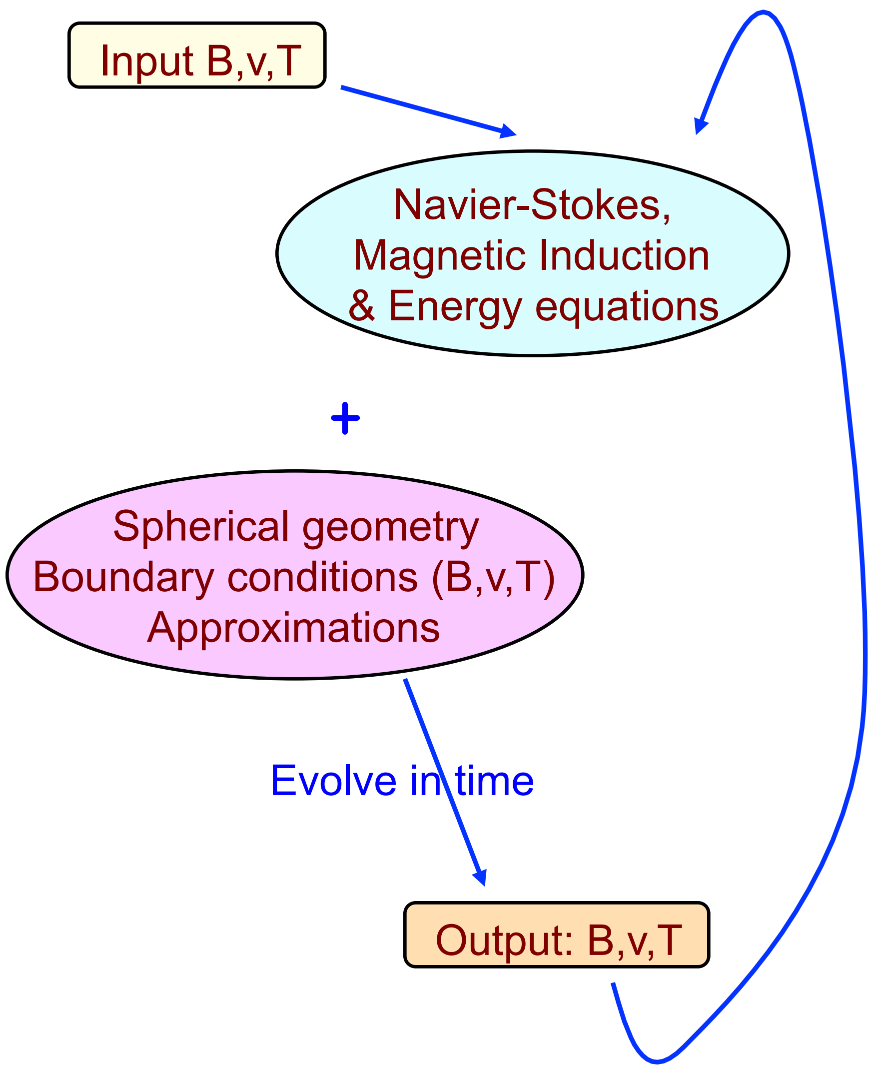


Figure 3   Change in the length of day derived directly from geodetic observations (*solid line*) and that are predicted from calculations of the core angular momentum derived from geomagnetic observations. The *square symbols* show the results of Jault et al (1988) and the *diamond symbols* the results of Jackson et al (1993).



Numerical Models:

The Process

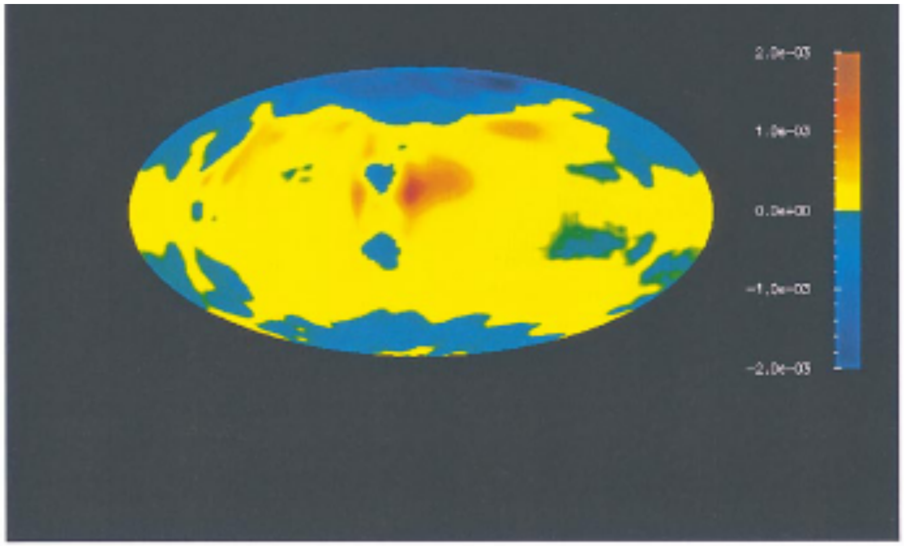


Characteristics

- geometry: $r_{io} = \dfrac{r_i}{r_o}$
- small viscosity: $E = \dfrac{\nu}{2\Omega r_o^2} << 1$
- small inertia: $Ro = \dfrac{\eta}{2\Omega r_o^2} << 1$
- strong convection: $Ra = \dfrac{\alpha g_o \Delta T r_o}{2\Omega\eta} >> Ra_c$

Results from dynamo models

(KB) weak field dynamo



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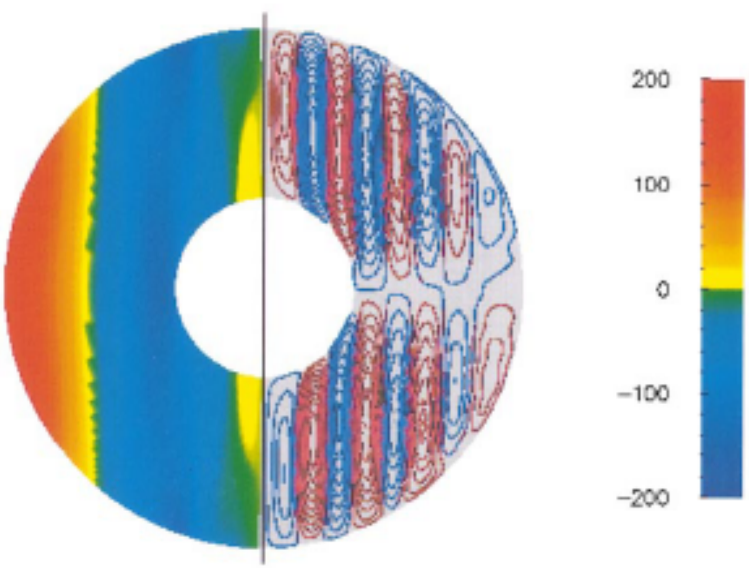

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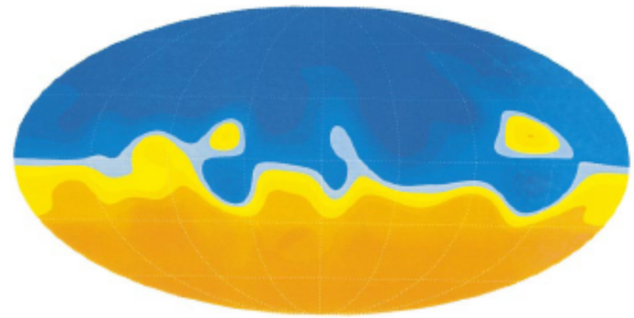
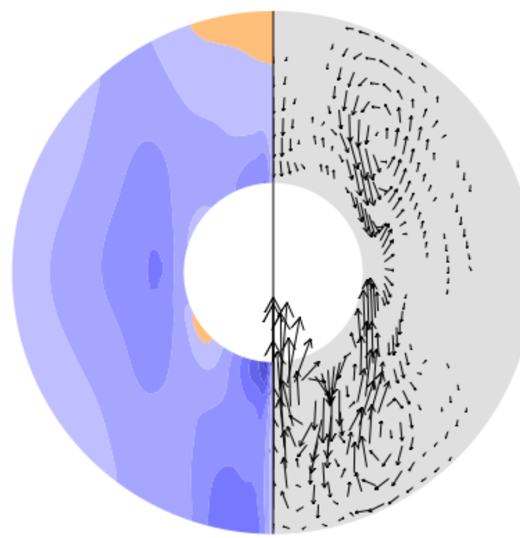
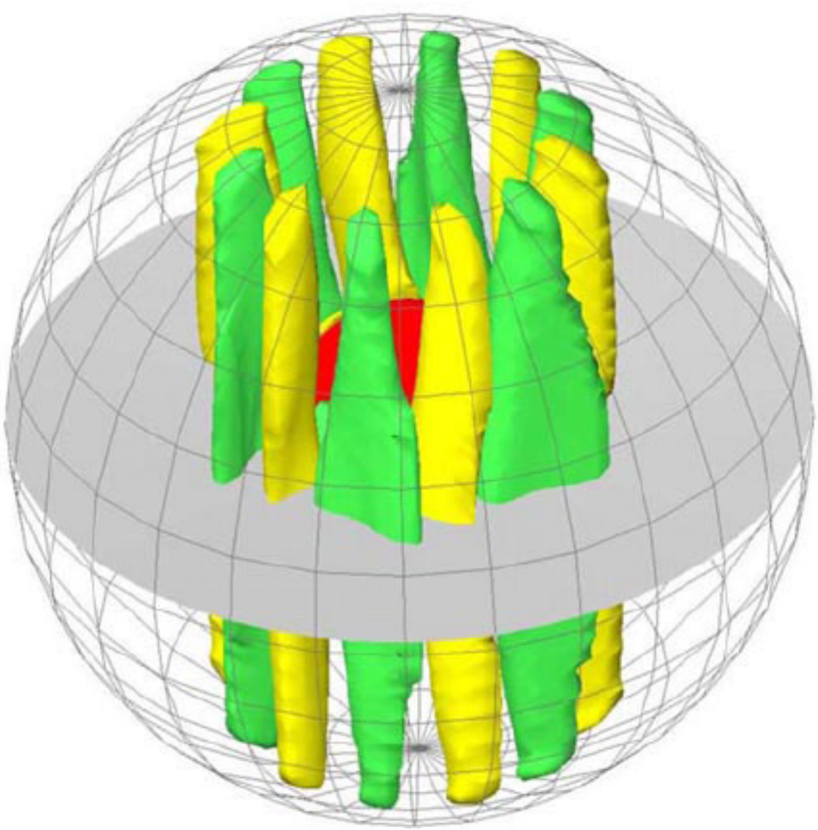
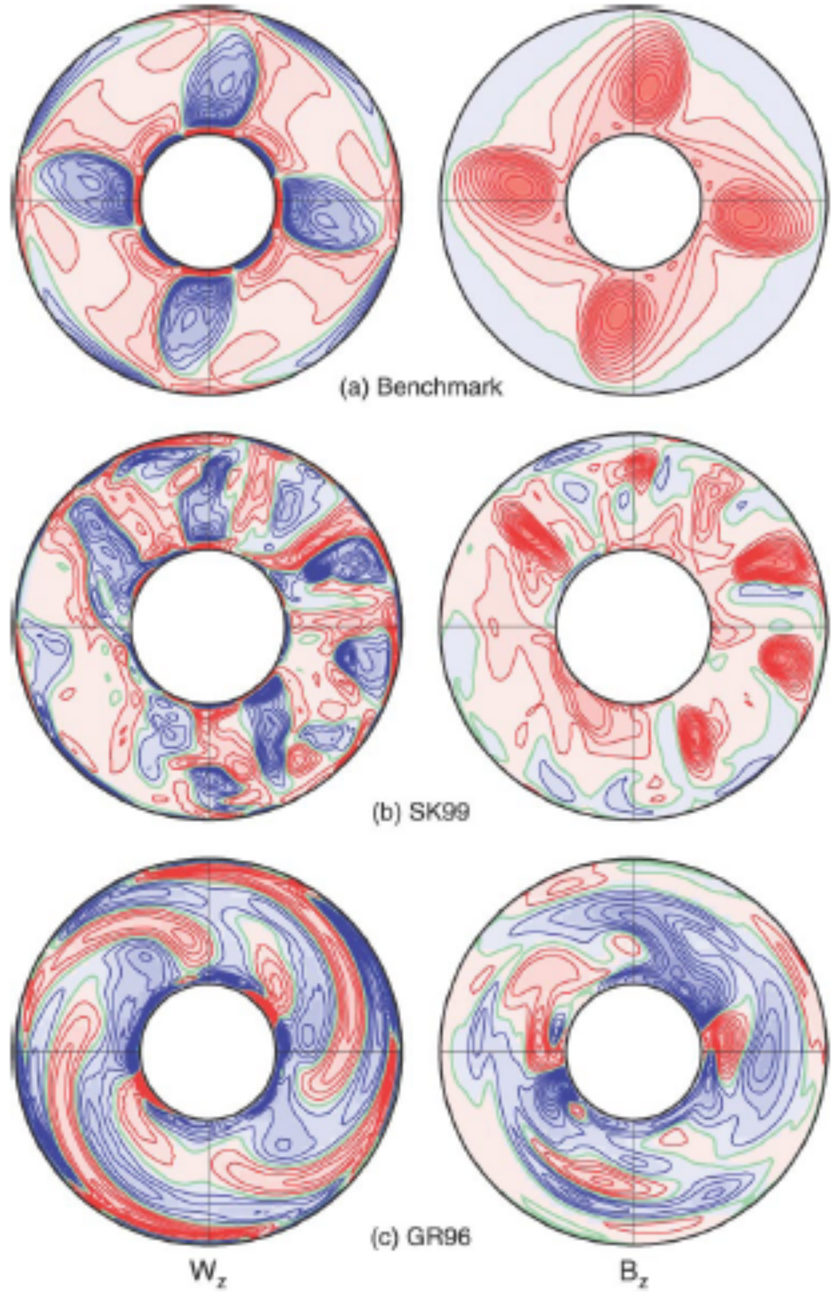
(KB) strong field dynamo



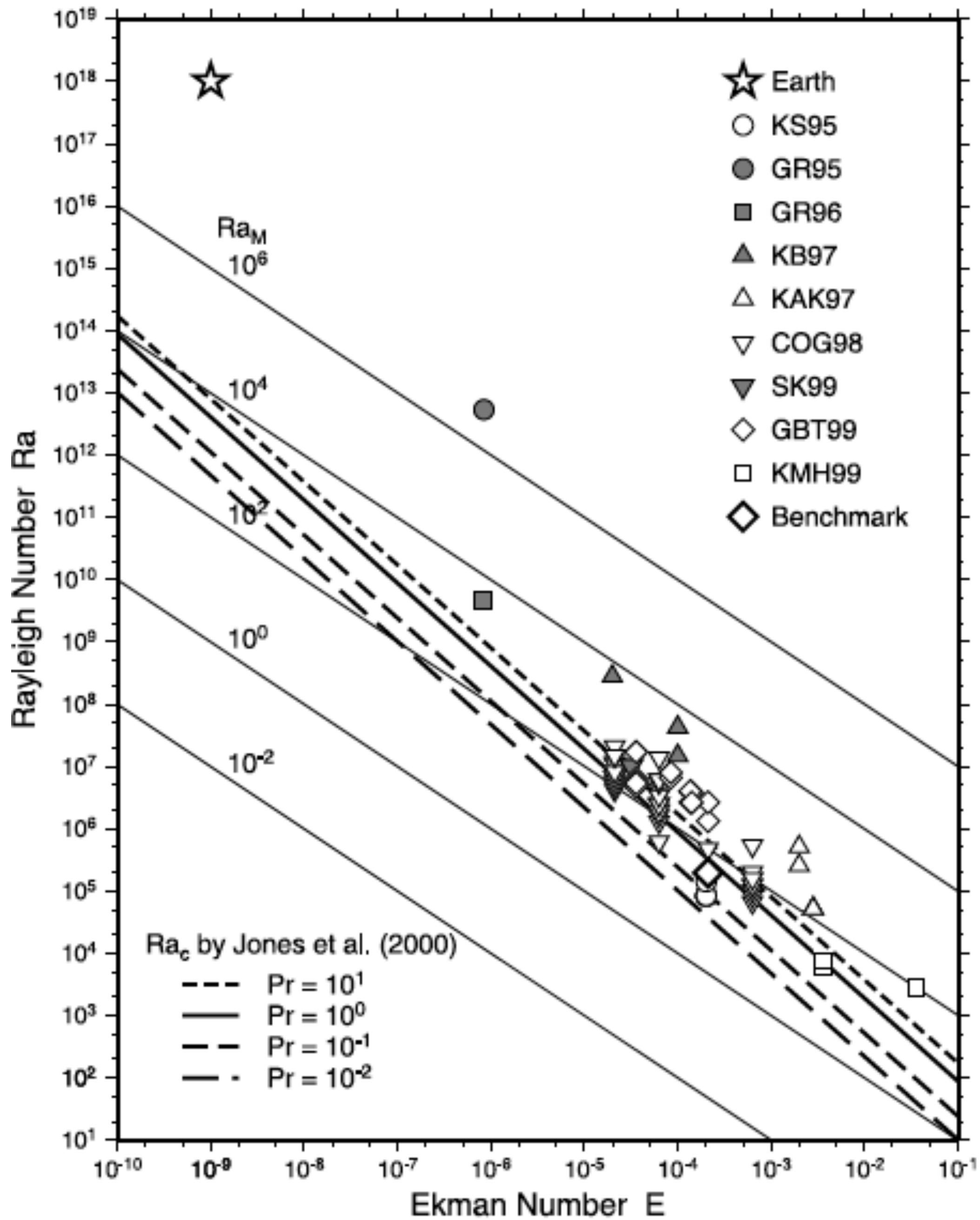


convection cells in dynamo models





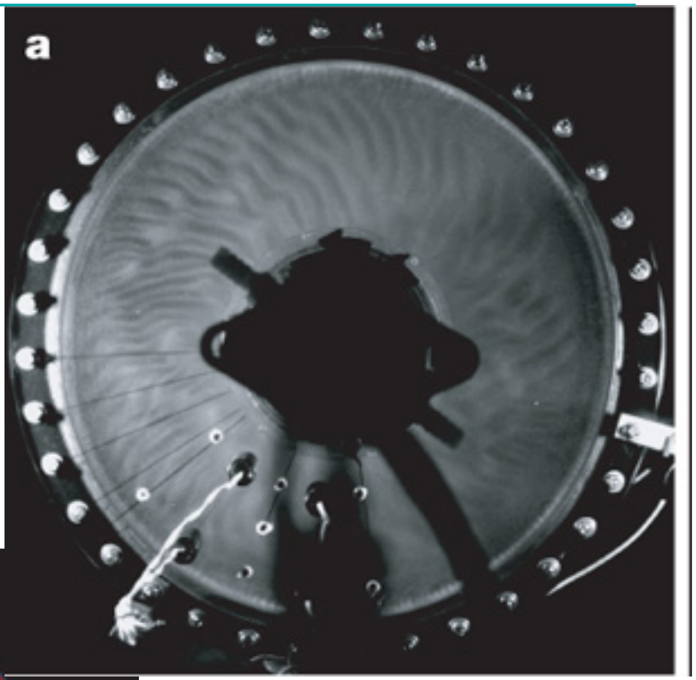
How far are we?



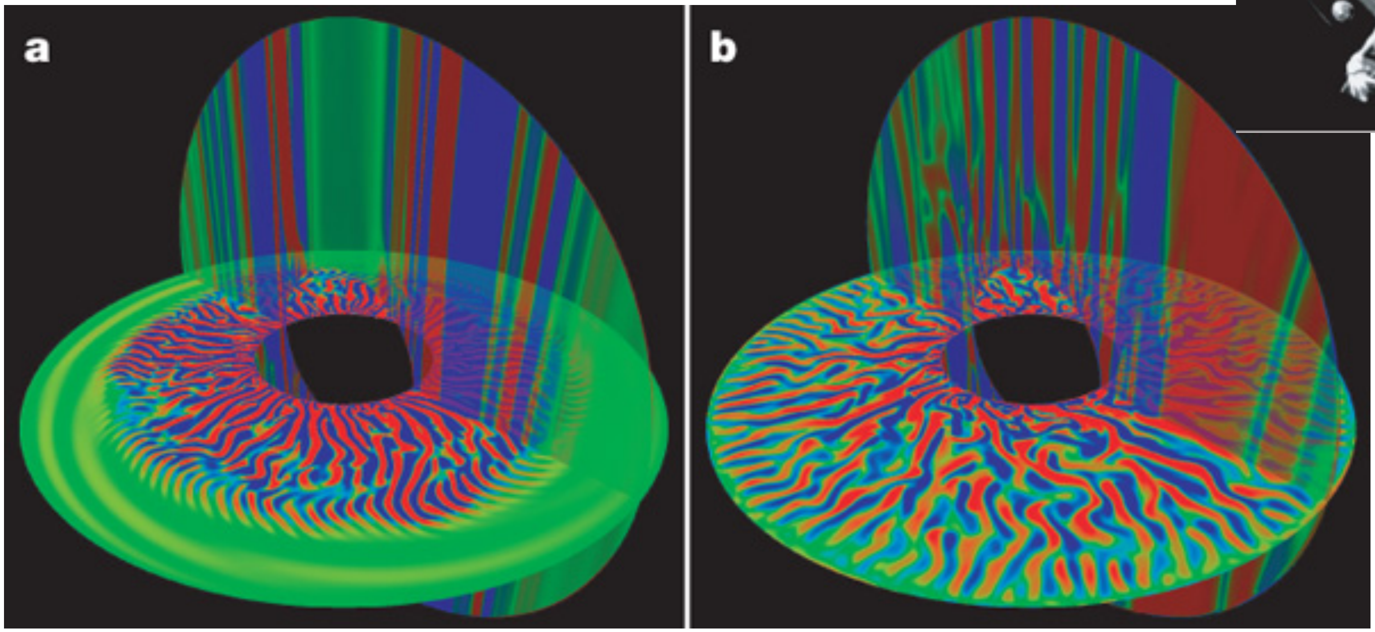
Convection cells in dynamo models

A Kageyama *et al.* *Nature* **2008**

Using the Earth Simulator:
(but not equilibrated yet,
magnetic field weak)



axial component of the vorticity

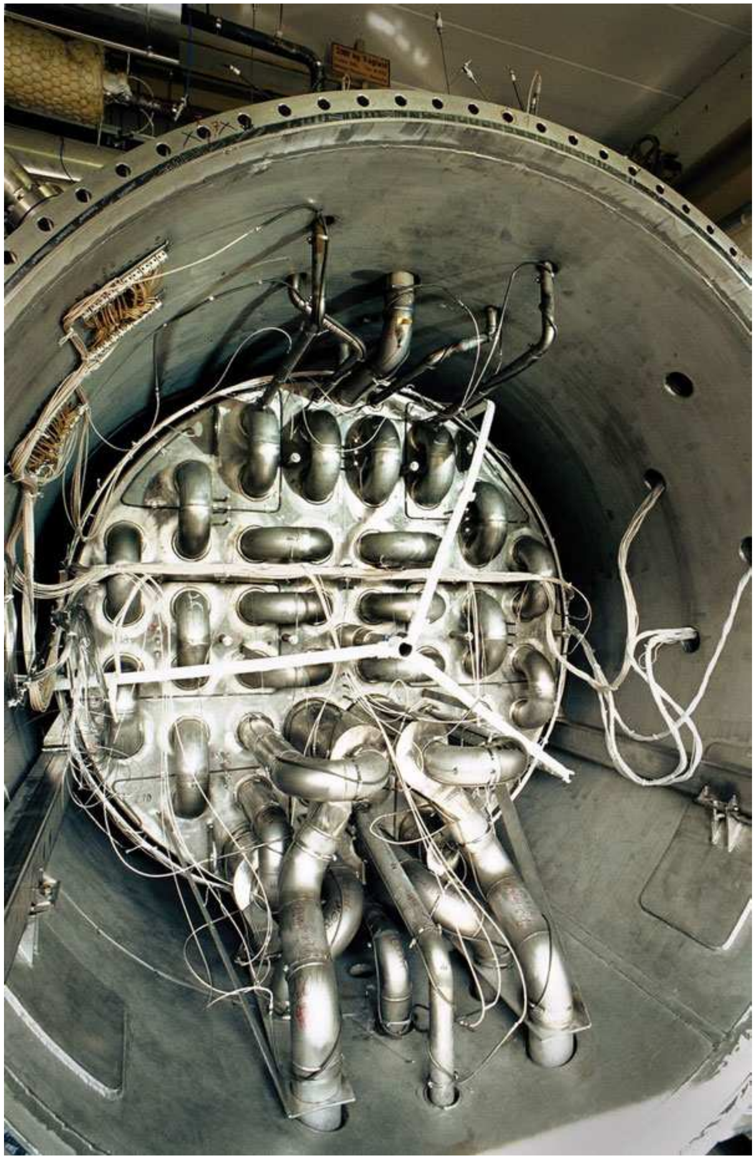


similar to experiments

E=2.3x10^{-7}: (lowest ever achieved), E=2.6x10^{-6}

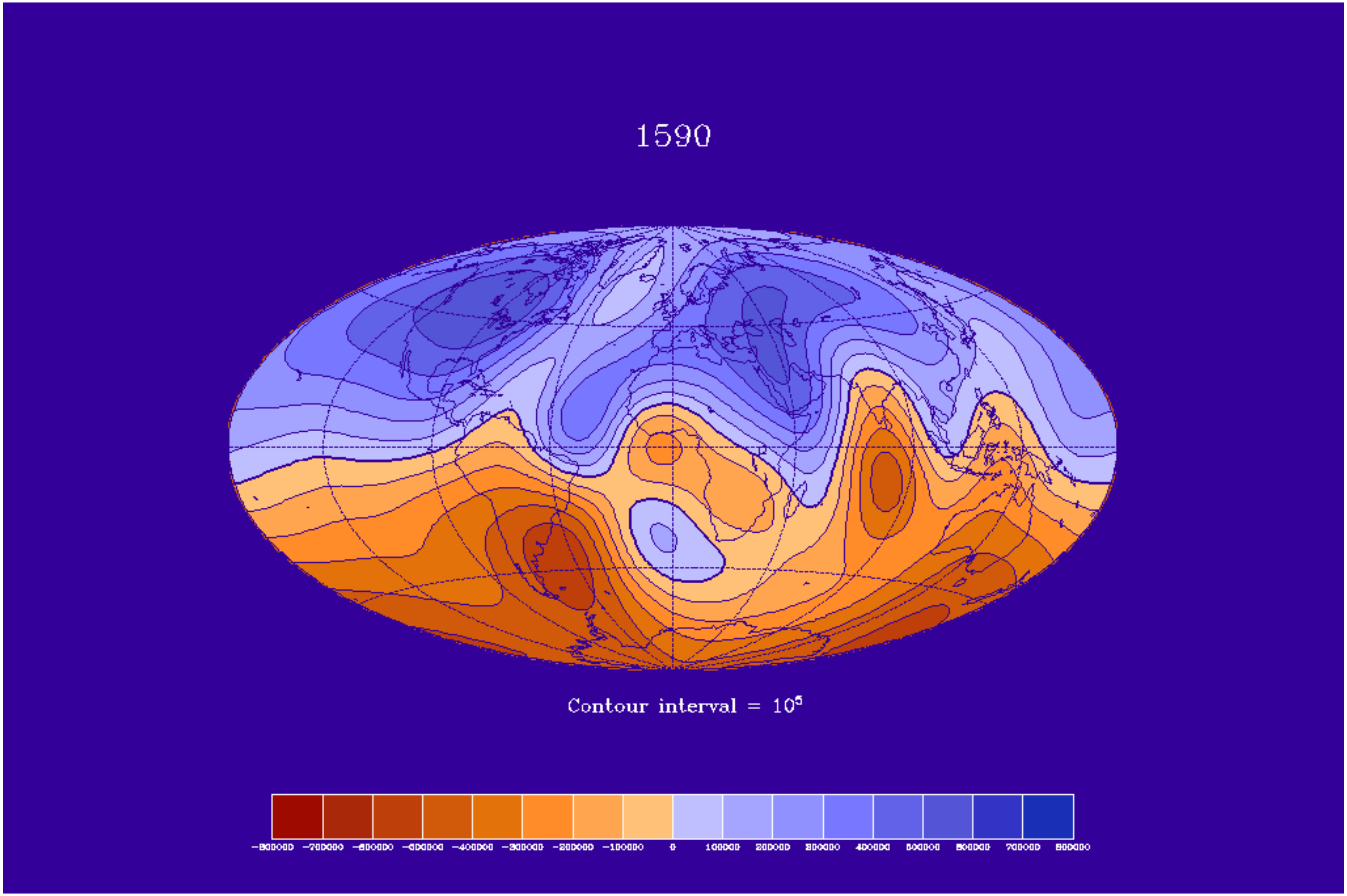
Experiments:

Karlsruhe dynamo



Dan Lathrop’s planetary dynamo

Case Study: Earth’s dynamo



Earth-like Numerical Model: