Midterm Exam, PHY 354S, Advanced Classical Mechanics

Thursday, March 4, 2010, 6-8 pm, duration: 2hrs, one aid-sheet allowed (8x11 inches, double-sided)

1. As you remember from Homework 1, the Lagrangian for a particle of mass m and charge q moving in an external electromagnetic field ($\vec{E}(\vec{r},t) = -\vec{\nabla}\phi(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}$, $\vec{B}(\vec{r},t) = \vec{\nabla} \times \vec{A}(\vec{r},t)$) is:

$$L(\vec{r},\vec{v},t) = \frac{1}{2}m\vec{v}^2 + q\ \vec{v}\cdot\vec{A}(\vec{r},t) - q\phi(\vec{r},t)\ , \tag{1}$$

where $\vec{v} = \dot{\vec{r}} = d\vec{r}/dt$.

1. Use the general expression for the energy $\mathcal{E}$ of any system with Lagrangian $L(\{\dot{q}\},\{q\},t)$ found in class: $\mathcal{E} = \sum_i p_i\dot{q}_i - L(\{\dot{q}\},\{q\},t)$ (the sum is over the various coordinates $\{q\}$ of the system and p_i is the generalized momentum corresponding to the i-th coordinate q_i) to show that the energy of a particle in an electromagnetic field is $\mathcal{E} = \frac{1}{2}m\vec{v}^2 + q\phi(\vec{r},t)$. What is the physical interpretation of the two terms in $\mathcal{E}$?

   8 points

2. Show explicitly, using the equation of motion $m\vec{a} = q(\vec{E} + \vec{v}\times\vec{B})$, that for a static electromagnetic field, when ϕ and $\vec{A}$ have no explicit dependence on t, the energy $\mathcal{E}$ is conserved. Would you have expected this result without calculation?

   6 points

3. Use the general relation $\frac{d\mathcal{E}}{dt} = -\frac{\partial L}{\partial t}$ to find the rate of change of the energy of a particle in $\vec{E}$ and $\vec{B}$ fields changing in time. Show that this relation implies that $\frac{d}{dt}\left(\frac{1}{2}m\vec{v}^2\right) = q\vec{v}\cdot\vec{E}$.

   7 points

4. Give a physical interpretation of the relation obtained in 3. above (e.g., explain what does it mean and why is the r.h.s. independent of $\vec{B}$?)

   6 points

2. A particle of mass m is constrained to move on the surface of a sphere of radius a, without the influence of any other external fields.

1. Write down the Lagrangian and find the Euler-Lagrange equations.

   8 points

2. Enumerate the conserved quantities.

   7 points

3. Find the trajectories of the particle in suitably chosen coordinates.

   8 points

3. A system of $n+1$ particles consists of one particle of mass M and n particles of equal masses m. Find the form of the Lagrangian in the center of mass frame, showing that the problem reduces to that of the motion involving n particles.

15 points

4. Consider the bound trajectories in a spherically symmetric potential $\sim r^{-\frac{3}{2}}$. Find the ratio of times a particle of a given mass spends between the corresponding points on similar trajectories. Express this ratio of times as a function of the ratio of energies associated with the trajectories.

15 points

Total number of points: 27+23+15+15=90