"Advanced classical mechanics"

required { Landau & Lifshitz, "Mechanics" – classic of subject

(for more advanced: V.I. Arnold, "Mathematical methods of classical mechanics")

particle → electron, ball, chair, car, plane, Earth, Sun ... } so long as motion as a whole ~~of~~ interest – * no significant change of internal structure, deformation, etc ...

* essentially POINTLIKE

particle has position $\vec{r} = (x, y, z)$

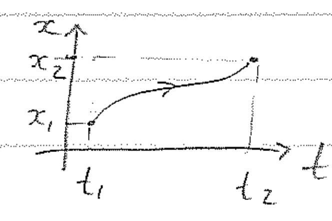
at time t

3d: Space: $(x, y, z) = \vec{r}$; $\underbrace{|\vec{r}_1 - \vec{r}_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}_{\text{distance between } \vec{r}_1 \text{ \& } \vec{r}_2}$

1d: time: t

(two points in space)

so $\vec{r}(t) \equiv$ trajectory



$$\vec{v}(t) \equiv \dot{\vec{r}}(t) \equiv \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \Rightarrow \text{velocity}$$

$$\vec{a}(t) \equiv \ddot{\vec{r}}(t) = \dot{\vec{v}}(t) = \left(\frac{d^2x}{dt^2}, \dots \right) \Rightarrow \text{acceleration}$$

$$\boxed{m\vec{a} = \vec{f}}$$

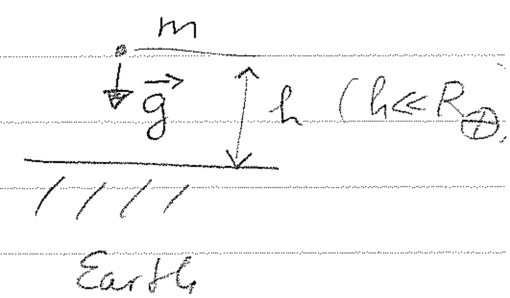
in inertial frame $\Leftrightarrow$ space homogeneous & isotropic & time homogeneous $\Rightarrow$ if no forces, if at rest - always at rest in inertial frame

$$m\ddot{\vec{r}}(t) = \vec{f}(\vec{r}(t), \dot{\vec{r}}(t), t)$$

mass of particle

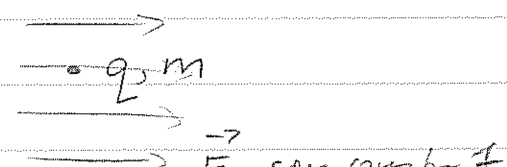
force ---

- gravitational



$\vec{f} = m\vec{g}$, constant

- electrostatic



$$\vec{f} = q\vec{E}$$

- Lorentz : $\vec{E}, \vec{B}$

$$\vec{f} = q\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right)$$

$$\underbrace{\left(+ \text{ weak, strong } \dots \right)}_{\text{all there is}}$$

main
problem of mechanics.

given $\vec{r}(t_1)$, $\dot{\vec{r}}(t_1)$ $\Longrightarrow$

$\Rightarrow$ find $\vec{r}(t_2)$, $\dot{\vec{r}}(t_2)$ for every $t_2 > t_1$.

principle of determinism $\Longleftrightarrow$ position & velocity @ t_1 determine all position & velocity at t_2.

— why so? $\longleftrightarrow$ experiment !

many particles? say, N particles (e.g. gas)

$3N$ coordinates $\equiv$ # degrees of freedom

$3N$ d.o.f. $q_1 \ldots q_s$, $s = 3N$, could be Cartesian coordinates of all the N particles, but don't have to be (e.g. center of mass could be of use — later)

we will call any $3N$ numbers $q_1 \ldots q_s$ that define the state of the system "generalized coordinates" -- for now think of Cartesian, examples will come later

$\{q_i\}$ - set of 3N generalized coord.

$\{\dot{q}_i\}$ - set of 3N generalized velocities

$\Downarrow$

if known at t_1, determine $\{q_i, \dot{q}_i\}$

at any later time uniquely.

for 3N particles, if $\{q_i\}$ are the Cartesian coordinates

e.g. $\{q_i\} = (x_1, y_1, z_1, x_2, y_2, z_2 \; --- \; x_N, y_N, z_N)$

we have 3N equations

$$m_i \ddot{q}_i = f_i$$

let's say all masses equal

In your general physics class, you simply solved equations like

$$m \ddot{\vec{r}} = \vec{f} \quad \text{for various } \vec{f}$$

to find $\vec{r}(t)$ $\longrightarrow$ here a more general view — "principle of least action"

("or "Hamilton's principle")

("or "Lagrangian mechanics")

to elucidate, consider a particle moving in 1d, in a potential $U(x)$:

$$f(x) = -\frac{dU(x)}{dx} = -U'(x)$$

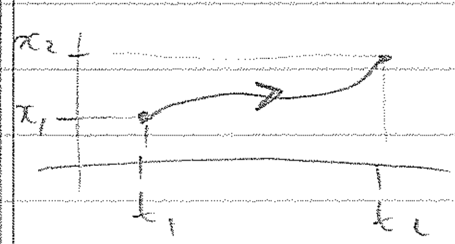
then $m\ddot{x} = -U'(x)$ expresses Newton's equ. of motion

we know that: $U(x) \equiv$ potential energy

$T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2 \equiv$ kinetic energy

then total ENERGY $\Rightarrow$ $E = T + U$ (conserved, if U is time-independent)

if we solve equ. of motion can find trajectory



Another quantity of interest is the

LAGRANGIAN __ L ___ (or $\mathcal{L}$)

in this case

$$\mathcal{L} = T - U = \mathcal{L}(x, \dot{x})$$

and the <u>action $S[x]$</u> is

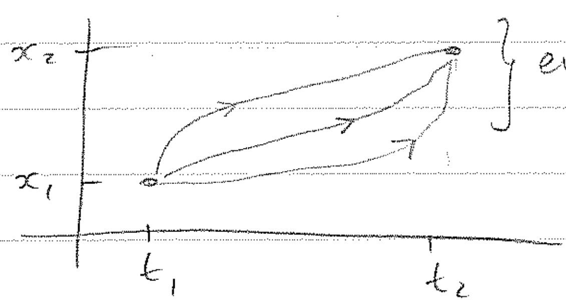
$$S[x] \equiv \int_{t_1}^{t_2} dt\, \mathcal{L}(x(t), \dot{x}(t))$$

or in more detail

$$= \int_{t_1}^{t_2} dt \left(\frac{m\dot{x}^2(t)}{2} - U(x(t)) \right)$$

$\rightarrow S[x]$ is a <u>functional</u> $\Longleftrightarrow$ <u>depends on trajectory $x(t)$</u>

consider



} every curve $\equiv$ trajectory

s.t. for given $t \in (t_1, t_2)$ unique $x(t)$!!

pick trajectory $x(t)$

$\rightarrow$ calculate $S[x]$ $\Longrightarrow$ $\forall$ trajectory $\rightarrow$ its own $S[x]$

start @ (x_1, t_1) finish at (x_2, t_2)

function : point (number) $\longrightarrow$ point (number)

functional trajectory (function) $\longrightarrow$ (point (number))

(*) quite common $\Longrightarrow$ ex: $f(x) \to f(0)$ (function $\to$ #) $\Rightarrow$ δ-function

e.g. trajectory $x(t)$ is a function: $\underbrace{t \to x}_{\forall t \to x(t)}$

while $S[x]$ is a functional of

trajectory $\{x(t)\} \longrightarrow S[x] = \int_{t_1}^{t_2} dt\, \mathcal{L}(x, \dot{x})$

$(x_1 t_1);(x_2 t_2)$

Why is such an awkward thing (action) useful?

-- principle of least action $\equiv$

$\equiv$ if at t_1 @ x_1 and at t_2 @ x_2

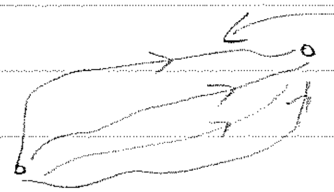
the particle moves on a trajectory $x(t)$

such that $S[x]$ takes the least possible value.

before deriving this result, let's try to see what it says $\longrightarrow$

for a given $\mathcal{L}(x, \dot{x})$ (for us: $U(x)$, since m is given)

there are potentially many trajectories between $x_1\, t_1$ & $x_2\, t_2$



basically, every function $x(t)$ s.t. $x(t_1) = x_1$, $x(t_2) = x_2$

→ there's an ∞ number of such function (an ∞ # of possible trajectories)

the claim is that the particle "chooses" the trajectory for which $S[x]$ is minimal - - - -

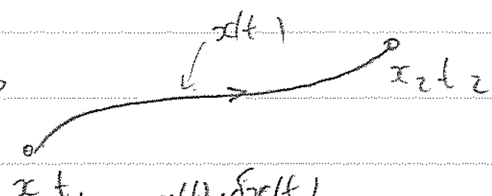
[Euler-Lagrange] - —

How do we know this is true?

$\equiv$ how would we decide if a given $S[x]$ is minimal?

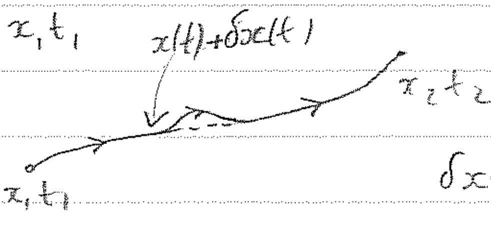
in pictures:

take a trajectory →



: $S[x]$

change slightly, by $\delta x(t)$ →



$x_2 t_2$: $S[x + \delta x]$

$\delta x(t)$: $\delta x(t_1) = 0$, $\delta x(t_2) = 0$

we say that

$x(t)$ is an <u>extremum of the functional $S[x]$</u>

if for arbitrary small (smooth) changes of $x(t)$,

called "variations" $-\ \delta x(t)$ $-$

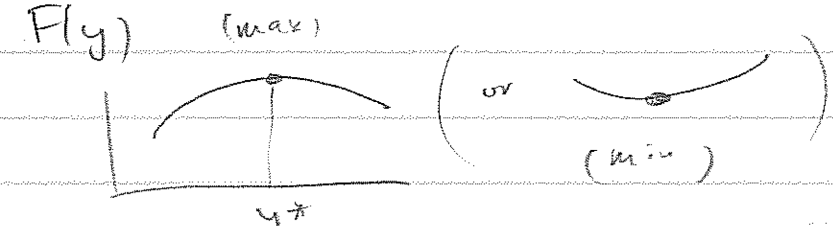
the action is <u>stationary</u>, i.e.

(*) $\quad S[x+\delta x] - S[x] = 0 + \text{terms of order } (\delta x)^2$

recall that

for a function $F(y)$ we have $F(y_* + \delta) - F(y_*) = \underline{O}(\delta^2)$

if y_* is an extremum



in other words

$$\left.\frac{dF(y)}{dy}\right|_{y=y^*} = 0 \iff \text{extremum}$$

Similarly, here, $\displaystyle \left.\frac{\delta S[x]}{\delta x(t)}\right|_{x=x(t)} = \lim_{\delta x(t)\to 0} \frac{S[x+\delta x] - S[x]}{\delta x(t)} = 0$

$\Updownarrow$

$x(t)$ is extremum.

"first variational derivative of $S[x]$"

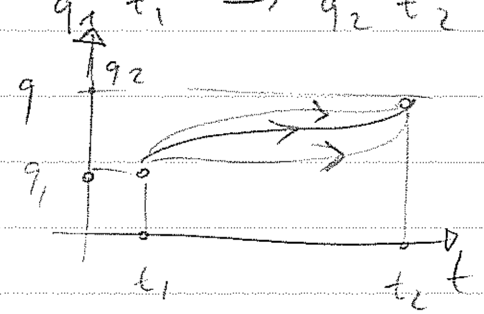
equ. (*) expresses same thing as

$$\left.\frac{\delta S}{\delta x}\right|_{x=x(t)} = 0$$

<u>So let's now show that for $L = T - U$,</u>
<u>least action principle gives Newton's equations.</u>

to agree w/ book, lets rename $x(t) \to q(t)$

e.g. $q_1\ t_1 \to q_2\ t_2$



let $\quad S[q] = \int_{t_1}^{t_2} dt\, L(q, \dot q, t)$

this is more general than our example

of $L(q,\dot q, t) = T - U = \frac{m\dot q^2}{2} - U(q)$

here L only depends on q & $\dot q$,
<u>not</u> explicitly on t

consider a trajectory $q(t)$ & a variation $\delta q(t)$ —

— arbitrary, but $\delta q(t_1) = \delta q(t_2) = 0$ —— so that we sample all trajectories from $q_1\, t_1$ to $q_2\, t_2$.

then $S[q+\delta q] - S[q] =$

$$= \int_{t_1}^{t_2} dt \left(L(q+\delta q, \dot{q} + \dot{(\delta q)}, t) - L(q, \dot{q}, t) \right)$$

for small δq, $\dot{(\delta q)}$ $\left(\dot{(\delta q)} \equiv \frac{d}{dt}\delta(q) \right)$

$=$ all smooth enough

$$\cancel{= \int_{t_1}^{t_2} dt \left(\frac{\partial L(q,\dot{q},t)}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt}(\delta q) \right)}$$ sorry

we have

$$L(q+\delta q, \dot{q}+\dot{(\delta q)}, t) =$$

$$= L(q,\dot{q},t) + \frac{\partial L(q,\dot{q},t)}{\partial q}\delta q + \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}} \frac{d}{dt}\delta q + O((\delta q)^2)$$

$$\left(f(x+\delta x, y+\delta y) = f(x,y) + \frac{\partial f(x,y)}{\partial x}\delta x + \frac{\partial f(x,y)}{\partial y}\delta y + \ldots \right)$$

hence

$$\underbrace{S[q+\delta q] - S[q]}_{\text{also called often } \delta S[q]} = \int_{t_1}^{t_2} dt \left(\frac{\partial L(q,\dot{q},t)}{\partial q}\delta q + \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}} \frac{d}{dt}(\delta q) \right) + O((\delta q)^2).$$

look @ 2nd term: $\displaystyle \int_{t_1}^{t_2} dt \left(\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}} \frac{d}{dt}\delta q \right) =$

$$= \int_{t_1}^{t_2} dt \left(\frac{d}{dt}\left(\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}} \delta q \right) - \frac{d}{dt}\left(\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}} \right) \cdot \delta q \right)$$

$$= \underbrace{\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}} \delta q(t) \Big|_{t_1}^{t_2}} - \int_{t_1}^{t_2} dt\ \delta q \frac{d}{dt}\left(\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}} \right)$$

since $\delta q(t) = 0$
@ $t = t_1$ or t_2
this term $= 0$.

remember for looking trajectory extremizing action so $\delta S = 0$.

hence we have that for an <u>extremal $q(t)$</u>:

$$0 = S[q + \delta q] - S[q] = \int_{t_1}^{t_2} dt\ \delta q(t) \left(\frac{\partial L(q,\dot{q},t)}{\partial q} - \frac{d}{dt} \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}} \right)$$

$$+ \underline{O}((\delta q)^2)$$

\# but NOW recall $\delta q(t)$ is <u>ARBITRARY</u>

& we have $\int_{t_1}^{t_2} dt\ \delta q(t)\ f(t) = 0$

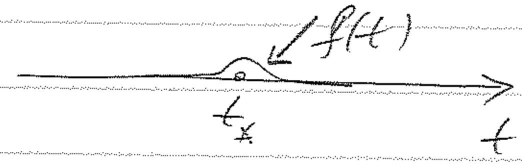
& both $f(t)$ & $\delta q(t)$ are smooth

→ <u>claim:</u> $f(t) \equiv 0$. →

this is called "the main theorem of variational calculus" but is very intuitive!

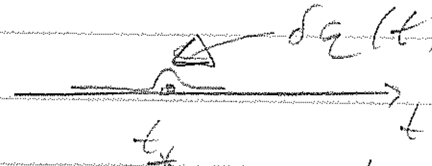
Proof: S'ppose $f(t_*) > 0$ for some t_* between t_1 & t_2

but then since f is smooth it must be

that 

i.e. $f(t)$ is also $\neq 0$ near t_*

But now I can take $\delta q(t)$ to be



$\delta q(t)$ also $\neq 0$ near t_* (& >0)

$\rightarrow$ then $\int_{t_1}^{t_2} \delta q(t)\, f(t)\, dt > 0$

but this contradicts $\int_{t_1}^{t_2} \delta q(t)\, f(t)\, dt = 0$ condition ■

So we have that a trajectory $q(t)$ that extremizes action is such that

$$\boxed{\frac{\partial L(q,\dot{q},t)}{\partial q} = \frac{d}{dt}\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}}}$$

Euler-Lagrange equation