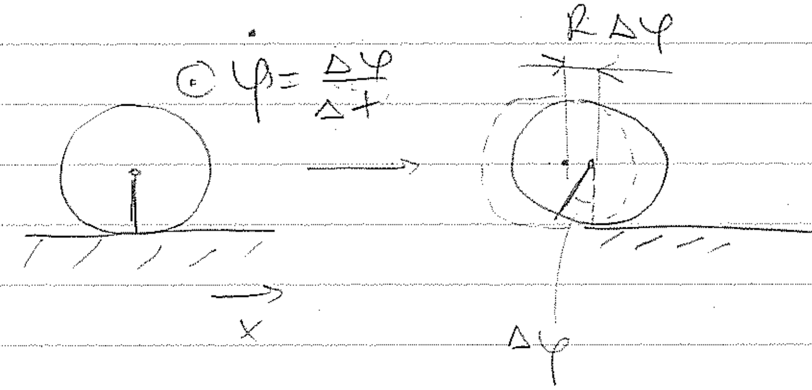
no slipping means that if it revolves once it's traveled $2\pi R$

so, if it's rotated on $\Delta\varphi$ wrt C.M,

$@\ \dot{\varphi} = \frac{\Delta\varphi}{\Delta t}$



CM moves on $R\Delta\varphi$ → hence $\Delta X_{CM} = R\,\Delta\varphi$

$$\& \quad V_{CM} = R\dot{\varphi} \qquad (\text{divide by } \Delta t)$$

$$\text{so} \quad T = \frac{1}{2}\mu R^2 \dot{\varphi}^2 + \frac{1}{2} I \dot{\varphi}^2$$

$$= \frac{1}{2}\left(\mu R^2 + I\right)\dot{\varphi}^2$$

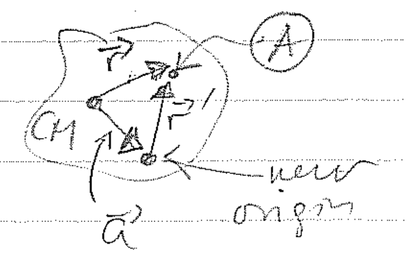
→ this is a pretty easy one -- but another point of view is useful as well as it helps in more complicated cases:

recall $\vec{v} = \vec{V}_{CM} + \vec{\Omega} \times \vec{r}$ → $\vec{r} = \vec{r}\,' + \vec{a}$

$(\vec{r}\,' = \vec{r} - \vec{a})$

$$\vec{v} = \underbrace{\left(\vec{V}_{CM} + \vec{\Omega} \times \vec{a}\right)}_{\text{velocity of new origin}} + \vec{\Omega} \times \vec{r}\,'$$

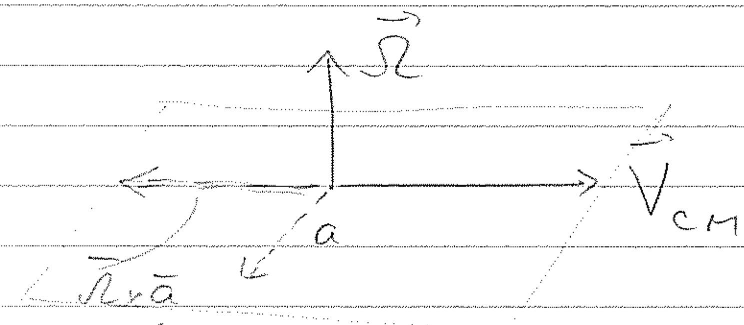
(velocity of arbitrary point A of body)



now, if $\bar{V}_{CM}$ & $\bar{\Omega}$ are $\perp$ at any instant,

one can always find $\vec{a}$ (i.e. an origin)

s.t. this origin is at rest at that moment:



– by assumption $\perp$, so $V_{CM} \in$ plane $\perp$ Ω

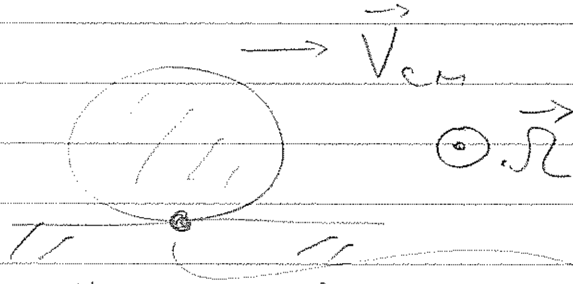
choose $\bar{a}$ s.t. $\bar{\Omega}\times\bar{a} = -\bar{V}_{CM}$

in this new frame, as we showed earlier (p 111)

angular velocity is the same; the axis of

rotation (parallel to $\bar{\Omega}$) thru new origin is called

the "instantaneous axis of rotation" $\Longleftrightarrow$ at that moment motion is pure rotation around this AXIS



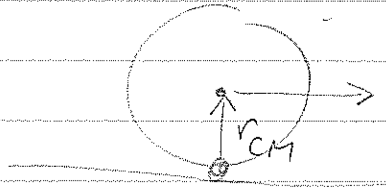
} always parallel

rolling cylinder

point where cylinder touches ground is instantaneously @ rest wrt ground ($\equiv (x,y,z)$ system) as there's no slipping –

– this is the instantaneous axis of rotation –

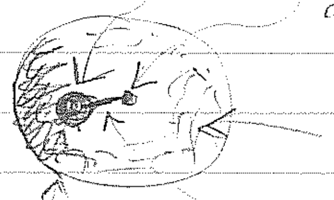
so one can think that at any instant cylinder is rotating w/ ω around the point where it touches the ground



$\vec{v}_{CM} = \vec{\Omega} \times \vec{r}_{CM}$ so $V_{CM} = R\dot{\varphi}$

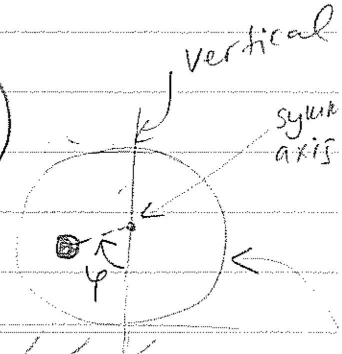
origin of $x'y'z'$ (of instantaneous axis of rotation, that is)

this concept (of instantaneous axis) is not very useful in this case, but consider a nonhomogeneous cylinder, s.t.:



CM is off axis

(let distance axis – CM $\equiv a$)



now we know $T = \frac{1}{2}\mu V_{CM}^2 + \frac{1}{2} I \dot{\varphi}$

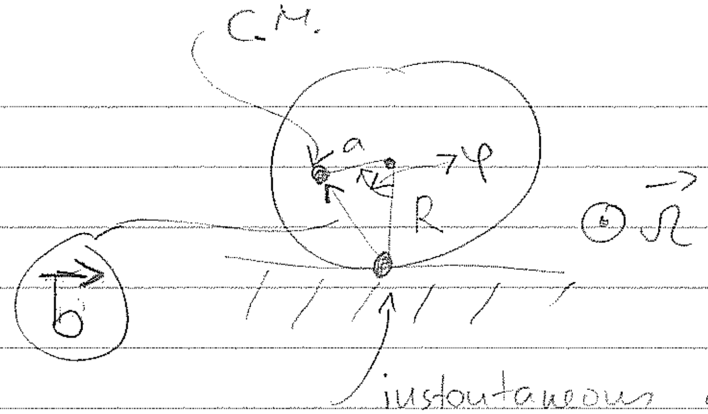
total mass

(moment of inertia in CM frame around axis // symmetry axis (e.g. $I_3\Omega_3$ only nonzero))

but getting V_{CM} (directly) is not so easy in this case —

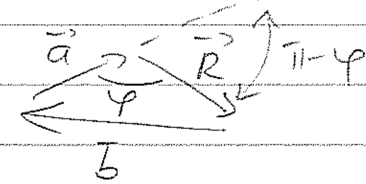
— **BUT**: instantaneous axis of rotation is same —

(instead of φ, can choose any other angle between $x'y'z'$ axes & fixed axes)



& so $\vec{V}_{CM} = \vec{\Omega} \times \vec{b}$, $|\vec{V}_{CM}| = \dot{\varphi}|\vec{b}|$

to find $|\vec{b}|$ use



$$\vec{b} = \vec{a} + \vec{R}$$

$$\vec{b}^2 = a^2 + R^2 + 2\vec{a}\cdot\vec{R}$$

$$= a^2 + R^2 + 2aR\cos(\pi-\varphi)$$

$$= a^2 + R^2 - 2aR\cos\varphi$$

so $|\vec{b}| = \sqrt{a^2 + R^2 - 2aR\cos\varphi}$

& $\vec{V}_{CM}^2 = \dot{\varphi}^2(a^2 + R^2 - 2aR\cos\varphi)$

so $T = \frac{1}{2}I\dot{\varphi}^2 + \frac{1}{2}\mu\dot{\varphi}^2(a^2 + R^2 - 2aR\cos\varphi)$

((NB: this is not enough to study the motion — must include gravity — which creates a torque, since CM is off-center ⇒ later))

* **MORAL**: used instantaneous axis to calculate V_{CM} i.t.o. $\dot{\varphi}$ ((also useful the other way if V_{CM} is easy to find))

$$L = T - U = \frac{1}{2}\dot{\varphi}^2\left(I + \mu(a^2 + R^2 - 2aR\cos\varphi)\right)$$

$$\underbrace{-\mu g(R - a\cos\varphi)}$$

(note: if $a=0$ doesn't affect motion, clear)

potential of body of mass $\mu \equiv$
$\equiv$ (height of CM above ground) $\times (\mu g)$

$$\frac{\partial L}{\partial \dot{\varphi}} = \dot{\varphi}\left(I + \mu(a^2 + R^2 - 2aR\cos\varphi)\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \ddot{\varphi}\left(I + \mu(a^2 + R^2 - 2aR\cos\varphi)\right) + (\dot{\varphi})^2 2\mu aR\sin\varphi$$

$$\frac{\partial L}{\partial \varphi} = +(\dot{\varphi})^2 \cancel{2}\mu aR\sin\varphi - \mu g a\sin\varphi$$

hence $\ddot{\varphi}\left(I + \mu(a^2 + R^2 - 2aR\cos\varphi)\right) = -\mu g a\sin\varphi \; \left(-\dot{\varphi}^2\mu aR\sin\varphi\right)$

for small φ: $\cos\varphi \approx 0$ $\sin\varphi \approx \varphi$; neglecting $\mu\dot{\varphi}^2\varphi$ term

$$\ddot{\varphi} = -\left(\frac{\mu g a}{I + \mu(a-R)^2}\right)\varphi$$

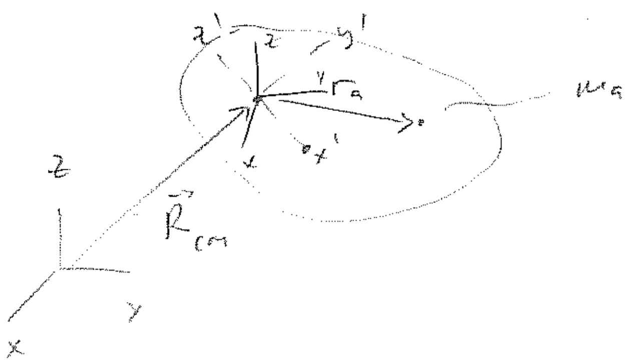
$$\ddot{\varphi} = -\omega^2\varphi \longleftrightarrow \omega = \sqrt{\frac{\mu g a}{I + \mu(a-R)^2}}$$

$$\varphi = \varphi_0\cos(\omega t + \alpha) \longleftarrow \text{harmonic oscillations}$$

(otherwise – nonlinear $\longrightarrow$ solve numerically)

$$L = T = \frac{1}{2}\mu \vec{V}_{cm}^{\,2} + \frac{1}{2}\sum_{i,j=1}^{3} I_{ij}\,\Omega_i\,\Omega_j$$

$$I_{ij} = \sum_a m_a\left(\vec{r}_a^{\,2}\,\delta_{ij} - r_{ai}\,r_{aj}\right)$$: CM moment of inertia



? are the dynamical variables (d.o.f)

* $\vec{R}_{cm}$
* three numbers describing orientations of x'y'z' wrt xyz

called "Euler angles" – soon!

- what are these?
- need to know in order to derive E.-L. equs.

<u>but before that some analogy</u>

Ω_i : angular velocity (vector) | $\dot{q}$ – velocity

$\frac{\partial L}{\partial \dot{q}} = p$ – momentum

$$\frac{\partial L}{\partial \Omega_i} = \,?\, = \frac{\partial}{\partial \Omega_i}\left(\sum_{kl}\frac{1}{2} I_{kl}\,\Omega_k\,\Omega_l\right) =$$

$$= \left(\frac{\partial \Omega_k}{\partial \Omega_i} = \delta_{ki}\right) = \sum_{k,l}\frac{1}{2}\left(I_{kl}\,\delta_{ki}\,\Omega_l + I_{kl}\,\Omega_k\,\delta_{li}\right) =$$

$I_{ik} = I_{ki}$

$$= \frac{1}{2}\sum_l I_{il}\,\Omega_l + \frac{1}{2}\sum_k I_{ki}\,\Omega_k = \frac{1}{2}\sum_l I_{il}\,\Omega_l + \frac{1}{2}\sum_k I_{ik}\,\Omega_k$$

= (rename $k \to l$ in 2nd)

$$= \frac{1}{2}\sum_l I_{il}\Omega_l + \frac{1}{2}\sum_l I_{il}\Omega_l = \sum_l I_{il}\Omega_l$$

$$= \frac{\partial L}{\partial \Omega_i}$$ ⟵ momentum associated ~~w~~/ "velocity" Ω_i

(just like $\frac{\partial L}{\partial \dot{q}_i} = p_i$)

→ natural to call "angular momentum" $M_i = \sum_l I_{il}\Omega_l$

$$H(p,q) = p\dot{q} - L(q,\dot{q})$$

use $\dot{q} \to \Omega_i = (I^{-1})_{il} M_l$

$p \to M_i$

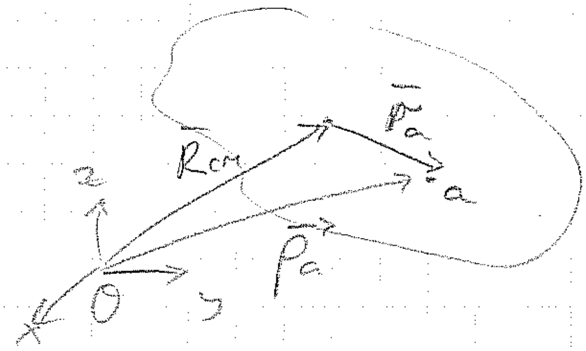
$$L = \frac{1}{2}\sum_{ij} I_{ij}\Omega_i\Omega_j$$

HW.

$$H = \frac{1}{2}\sum_{ij} M_i (I^{-1})_{ij} M_j$$

wrt O of (x,y,z)

$$\vec{M}_O = \sum_a m_a \vec{\rho}_a \times \vec{v}_a$$



$$\vec{\rho}_a = \vec{R}_{CM} + \vec{r}_a$$

$$\vec{v}_a = \vec{V}_{CM} + \vec{\Omega} \times \vec{r}_a$$

$$\vec{M}_O = \sum_a m_a \vec{\rho}_a \times \left(\vec{V}_{CM} + \vec{\Omega} \times \vec{r}_a \right)$$

$$= \left(\sum_a m_a \vec{\rho}_a \right) \times \vec{V}_{CM} + \sum_a m_a \vec{\rho}_a \times \left(\vec{\Omega} \times \vec{r}_a \right)$$

$$= \mu \vec{R}_{CM} \times \vec{V}_{CM} + \sum_a m_a \vec{R}_{CM} \times \left(\vec{\Omega} \times \vec{r}_a \right) + \sum_a m_a \vec{r}_a \times \left(\vec{\Omega} \times \vec{r}_a \right)$$

$$= \mu \vec{R}_{CM} \times \vec{V}_{CM} + \vec{R}_{CM} \times \left(\vec{\Omega} \times \underbrace{\cancel{\left(\sum_a m_a \vec{r}_a \right)}}_{0} \right) + \sum_a m_a \vec{r}_a \times \left(\vec{\Omega} \times \vec{r}_a \right)$$

$\left(\sum_a m_a \vec{r}_a = 0 \text{ since CM} \right)$

in <u>CM frame</u> we have — <u>we'll mean $\vec{M}$ to denote this:</u>

$$\vec{M} = \sum_a m_a \vec{r}_a \times \left(\vec{\Omega} \times \vec{r}_a \right)$$

$$M_i = \sum_a m_a \sum_{jk} \epsilon_{ijk}\, r_{a_j} \left(\vec{\Omega} \times \vec{r}_a \right)_k$$

$$= \sum_a m_a \sum_{jk} \epsilon_{ijk}\, r_{a_j} \sum_{\ell m} \epsilon_{k\ell m}\, \Omega_\ell\, r_{a_m} = (*)$$

---

totally antisymm

$(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$: $\sum_{j,k}$ understood, $\epsilon_{123} = +1$, $\epsilon_{213} = -1$, ...

$\epsilon_{ijk} = 0$ when any two indices coincide

$\epsilon_{123} = +1$, any permutation $\times -1$

$\epsilon_{213} = -1$

$\epsilon_{231} = 1$ $\qquad \sum_k \epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

$$(*) = \sum_a m_a \sum_{jlm} r_{a_j} \Omega_l r_{a_m} \underbrace{\sum_k \epsilon_{ijk}\epsilon_{klm}}_{\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}}$$

$$= \sum_a m_a \left(\Omega_i \vec{r}_a^{\,2} - r_{a_i} \vec{r}_a \vec{\Omega} \right)$$

$$= \sum_a m_a \sum_j \left(\Omega_i \delta_{ij} \vec{r}_a^{\,2} - \Omega_j r_{a_i} r_{a_j} \right)$$

$$= \sum_j \Omega_j \sum_a \left(\delta_{ij} m_a \vec{r}_a^{\,2} - r_{a_i} r_{a_j} m_a \right)$$

$$= \sum_j \Omega_j I_{ij} = \sum_j I_{ij} \Omega_j$$

$$\boxed{M_i = \sum_j I_{ij} \Omega_j}$$ angular m-m

$$\left.\begin{aligned} M_1 &= I_1 \Omega_1 \\ M_2 &= I_2 \Omega_2 \\ M_3 &= I_3 \Omega_3 \end{aligned}\right) \text{ if principal axes!}$$

In short, we have

$$\vec{M}_{CM} = \sum_a m_a \vec{r}_a \times (\vec{\Omega} \times \vec{r}_a)$$

all particles of body

(means $\vec{M}$ around origin = C.M.)

becomes, in body-fixed frame centered on CM

$$M_i = \sum_j I_{ij} \Omega_j$$

components of $\vec{M}$ & $\vec{\Omega}$ in said frame

or, if we take $(\hat{1}, \hat{2}, \hat{3} = \hat{i})$ = principal axes, we have $I_{ij} = I_i \delta_{ij}$

$$M_i = I_i \Omega_i \ , \ i = 1, 2, 3$$

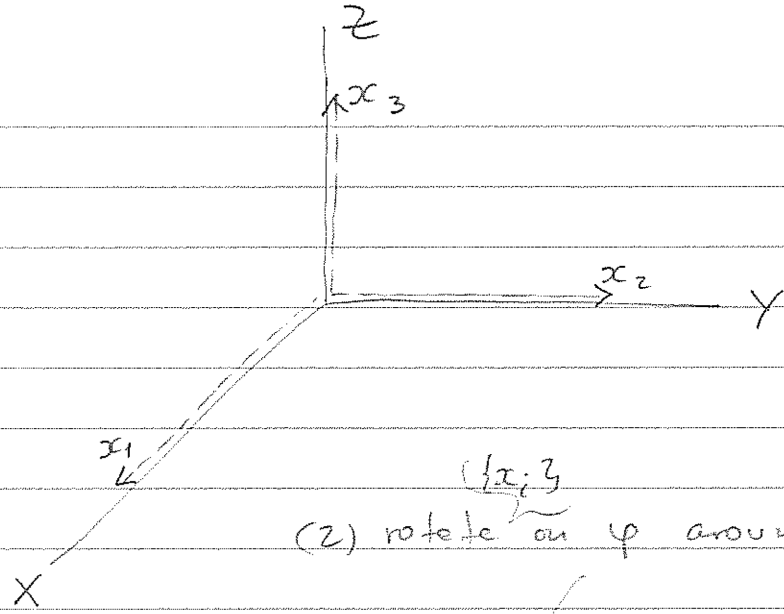
NOW, HOW DO WE PARAMETRIZE $\vec{\Omega}$??

Euler angles → in notation of book (XYZ) = fixed coordinate system w/ origin C.M.

$(x_1 x_2 x_3) = \{x_i\} =$

= body-fixed system, same origin CM

$(x_1 x_2 x_3)$ is rotated wrt (XYZ) in a general way, depending on three parameters; here's how to construct a general rotation →



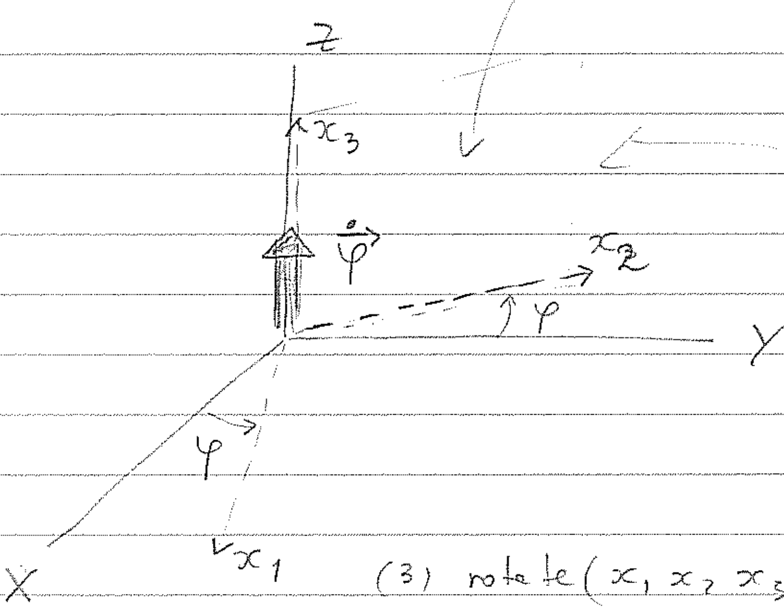
(1) 1st step

$(x_1 x_2 x_3) \equiv (XYZ)$

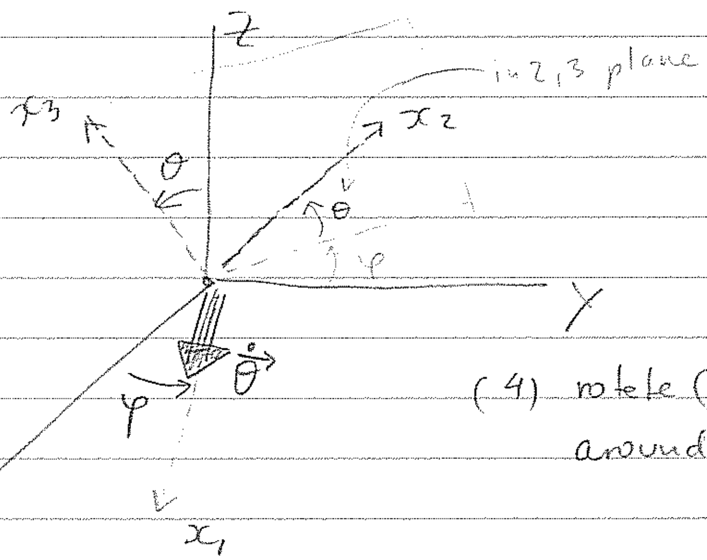
$\{x_i\}$

(2) rotate on φ around x_3

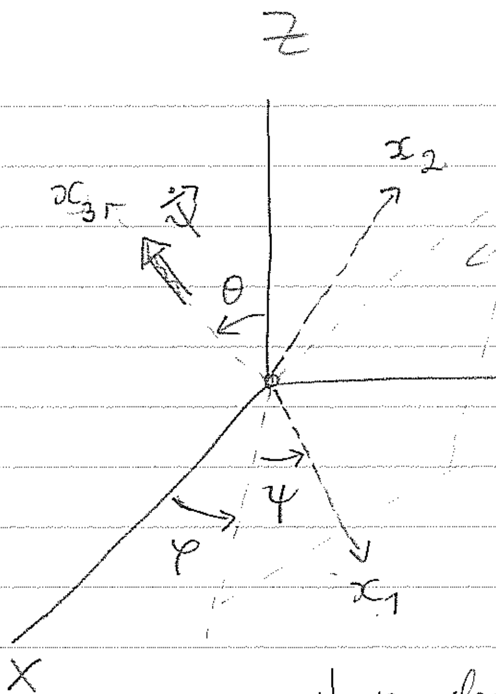
i.e. in $x_2 - x_3$ plane



(3) rotate $(x_1, x_2 x_3)$ on θ around x_1



(4) rotate $(x_1 x_2 x_3)$ on ψ around x_3, ie in $x_1 - x_2$ plane



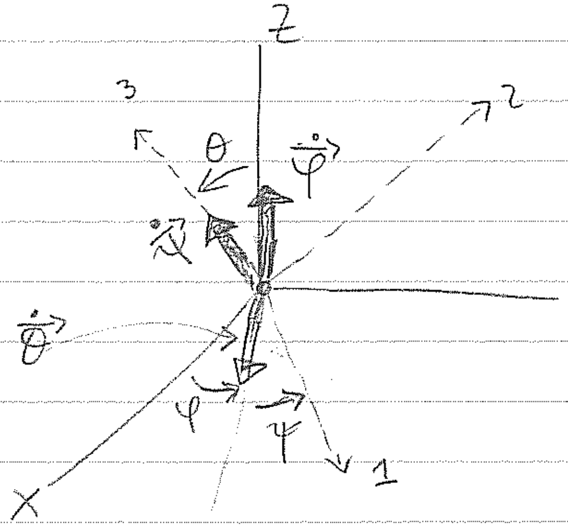
plane of $x_1 x_2$ of step 3

It is clear that this way a general orientation of (x_1, x_2, x_3) wrt (XYZ) is achieved $\left\{ \begin{array}{l} \varphi, \psi \in (0, 2\pi) \\ \theta \in (0, \pi) \end{array} \right\}$

"parametrize $\forall SO(3)$ group element 1↔1 (rotation)⟺$(\theta \varphi \psi)$

⟹ all rotns of $\mathbb{R}^3$ ≡ 3×3 matrices — special orthogonal — ones that preserve $(\vec{x} \cdot \vec{y})$ & have unit det.

Now:



let's express $\vec{\dot\theta}, \vec{\dot\varphi}, \vec{\dot\psi}$ (vectors!) into their components in $x_1\ x_2\ x_3$ frame:

$$\begin{aligned} \vec{\dot\theta} &= (\dot\theta \cos\psi, -\dot\theta \sin\psi, 0) \\ \vec{\dot\varphi} &= (\dot\varphi \sin\theta \sin\psi, \dot\varphi \sin\theta \cos\psi, \dot\varphi \cos\theta) \\ \vec{\dot\psi} &= (0, 0, \dot\psi) \end{aligned}$$

$\vec{\dot\varphi}$ in 1-2 plane is $\dot\varphi \sin\theta$, but then we rotated on ψ, so 1 & 2 components $\sim(\sin\psi, \cos\psi)$, respectively