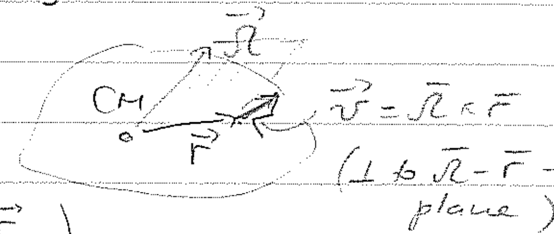
remember now our def. of $\vec{\Omega}$ in general:

an arbitrary point in the body has

$$\vec{v} = \vec{\Omega} \times \vec{r}$$



(we had $\vec{v} = \underbrace{\vec{V}_{CM}}_{\text{here this} \equiv 0} + \vec{\Omega} \times \vec{r}$) $\longrightarrow$ components of $\vec{r}$ are constant in (x_1, x_2, x_3) but NOT in (X, Y, Z), of course

consider an arbitrary point in the body w/ coordinates
$\vec{r} = (r_1, r_2, r_3)$ in $(x_1 x_2 x_3)$ - system

<u>Qu:</u> How will $\vec{r}$ change (in (XYZ)-system, of course) when $(x_1 x_2 x_3)$ rotates in $(\Delta\varphi, \Delta\psi, \Delta\theta)$?

<u>Ans:</u> Need to find $\vec{r} = (\tilde{X}, \tilde{Y}, \tilde{Z}) \leftarrow$ components of $\vec{r}$ in (X, Y, Z) and express them through (r_1, r_2, r_3) & (φ, ψ, θ) ; then

consider $\Delta \vec{r} = \dfrac{\partial \vec{r}}{\partial \theta} \Delta\theta + \dfrac{\partial \vec{r}}{\partial \varphi} \Delta\varphi + \dfrac{\partial \vec{r}}{\partial \psi} \Delta\psi$ and <u>show</u>

<u>that</u> $\Delta \vec{r} = \Delta\vec{\theta} \times \vec{r} + \Delta\vec{\varphi} \times \vec{r} + \Delta\vec{\psi} \times \vec{r}$, divide by Δt

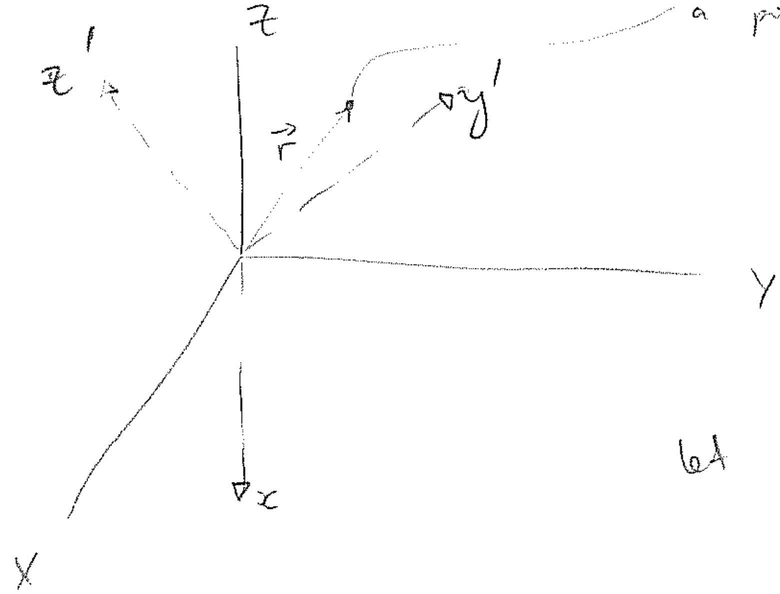
$$\Longrightarrow \vec{v} = \dot{\vec{\theta}} \times \vec{r} + \dot{\vec{\varphi}} \times \vec{r} + \dot{\vec{\psi}} \times \vec{r} \;, = \underbrace{(\dot{\vec{\theta}} + \dot{\vec{\varphi}} + \dot{\vec{\psi}})}_{} \times \vec{r}$$

(here $\Delta\vec{\theta}$ is a vector // to $\dot{\vec{\theta}}$ w/ magnitude $\Delta\theta$ etc.... for $\Delta\vec{\varphi}$, $\Delta\vec{\psi}$)

$$\Rightarrow \underline{\vec{\Omega} = \dot{\vec{\theta}} + \dot{\vec{\varphi}} + \dot{\vec{\psi}}}$$

Euler angles, SO(3), & angular velocity

as per (131.1) we have



a point of the body

$\vec{r} = (x_1' \; x_2' \; x_3')$ in body fixed

these #s do NOT change

let $\vec{r} = (X_1 \; X_2 \; X_3)$ in $(X \, Y \, Z)$ fixed

these change as body rotates

<u>Problem</u> : How does $\vec{r}$ change as body rotates?

<u>or</u>

How does $\vec{r}$ change as θ, φ, ψ change?

---

to <u>ANSWER</u> : (1) Find how X_1, X_2, X_3 depend on $x_1' \, x_2' \, x_3'$ (fixed) & θ, φ, ψ (orientation of $(x' y' z')$ wrt $(X Y Z)$).

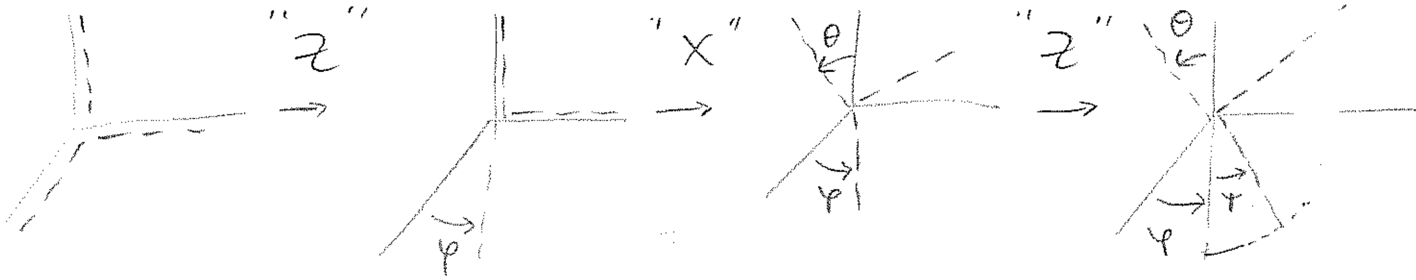
(2) Calculate derivatives wrt to θ, φ, ψ & show that

$$\Delta \vec{r} = \Delta t \vec{\Omega} \times \vec{r} \quad \rightarrow \quad \dot{\vec{r}} = \bar{\Omega} \times \bar{r}$$

w/ $\bar{\Omega} = \dot{\bar{\theta}} + \dot{\bar{\varphi}} + \dot{\bar{\psi}}$

hardest part is #(1).

We'll follow Euler's <u>Z - X - Z</u> construction

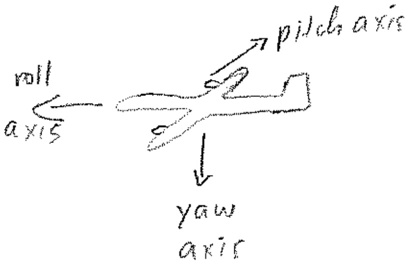


<u>a note on generality</u>: (θ, φ) describe most general orientation of z' axis (these are basically usual polar angles) - - allow z' to point anywhere

ψ, then describes general orientation of $x'y'$ around <u>general</u> z'

so $\theta \in (0, \pi)$, $\varphi, \psi \in (0, 2\pi)$ } give most general $x'y'z'$ orientation.

vs. airplane:

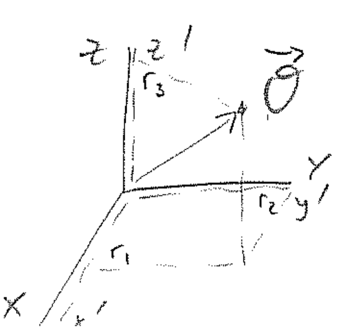


essentially, rotns. around 3 ⊥ axes.

To find coordinates of $\vec{r}$ (body-fixed) in (XYZ) for general (θ, φ, ψ)

---

follow "$Z^1 X Z^2$" <u>Auxiliary problem. $\vec{\theta}$: fixed in space</u>

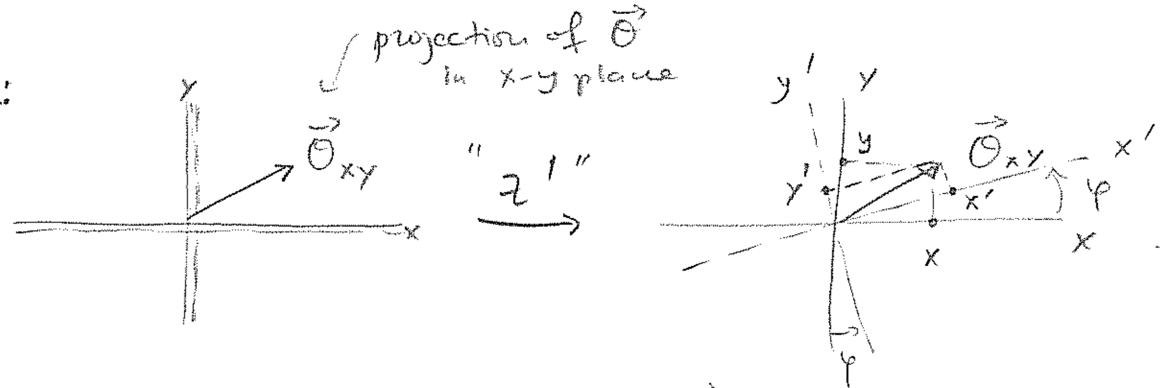


i.e. $\vec{\theta} = (x, y, z)$ in fixed @ every step.

initially $xyz = x'y'z'$ so

$\vec{\theta} = (x, y, z)$ also in (x', y', z')

Step z':



$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

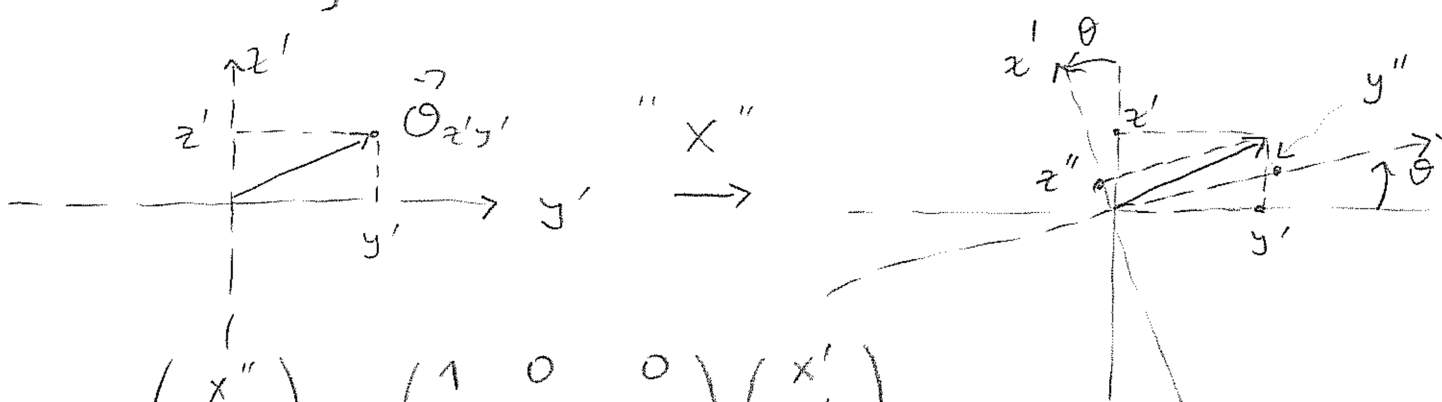
(from picture – – trigonometry!)

coordinates of $\vec{\theta}$ in $(x'y'z')$ after step z'

coordinates of fixed $\vec{\theta}$ in $(x'y'z')$ before step z'

(z' coordinate does not change!)

Step X: now x' coordinate will not change (since we're rotating on θ around x') – so plot $y'z'$ plane only



$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

– also trigonometry (picture.

$$y'' = \cos\theta\, y' + \sin\theta\, z'$$
$$z'' = -\sin\theta\, y' + \cos\theta\, z'$$

so
$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

fixed $\vec{\Theta}$ in $(x'y'z')$ after steps $Z^1 X$.

$$\boxed{\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\cos\theta\sin\varphi & \cos\theta\cos\varphi & \sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}}$$

fixed $\vec{\Theta}$ in (X, Y, Z)

<u>step Z^2</u> : rotate around z' on ψ – clearly the answer is

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\cos\theta\sin\varphi & \cos\theta\cos\varphi & \sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi & \cos\psi\sin\varphi + \sin\psi\cos\theta\cos\varphi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi & -\sin\psi\cos\varphi + \cos\psi\cos\theta\cos\varphi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix}}_{\text{matrix } 3\times3\text{, orthogonal, } \det = 1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

fixed $\vec{\Theta}$ in body fixed $(x'y'z')$ w/ (φ, θ, ψ)

$\| A_{ij} \|$

general (see argument p. 131.2)

fixed $\vec{\Theta}$ in (X, Y, Z)

Summary, so far:

$\vec{O}$ has coordinates $\{x_i, i=1,2,3\}$ ≈ (X,Y,Z)
$\vec{O}'$ —— " —— $\{x_i', i=1,2,3\}$ ≈ $(x'y'z')$

$$x_i' = A_{ij} \, x_j \qquad \Big(\sum_{j=1}^{3} \text{ understood} \Big)$$

A_{ij} depends on (θ, φ, ψ)

$$A(\theta,\varphi,\psi) \equiv B_z(\psi)\, B_x(\theta)\, B_z(\varphi) \quad \text{in matrix notation.}$$

$$B_z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

This is ~~for~~ a vector fixed in stationary frame.

We want a vector fixed in primed (Body-fixed frame)

→ well – very easy! – regard $\{x_i'\}$ as fixed and

express $\{x_i\}$ through $\{x_i'\}$:

$$x_i = (A^{-1})_{ij} \, x_j'$$

(**Equiv.** we can regard $(x'y'z')$ as fixed and have (X,Y,Z) perform an inverse rotation — replace $x \leftrightarrow x'$ & $\|A\| \to \|A^{-1}\|$)

So, using
$$\begin{cases} A^{-1} = B_z^{-1}(\varphi)\, B_x^{-1}(\theta)\, B_z^{-1}(\psi) \\ \& \quad B_{x,z}^{-1}(\alpha) = B_{x,z}(-\alpha) \end{cases}$$

we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{B_z(-\varphi)\, B_x(-\theta)\, B_z(-\psi)}_{\text{depend on time}} \underbrace{\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}}_{\text{fixed pt of body}}$$

depend on time

coordinates of body-fixed pt in stationary system w/origin placed @ c.m.

or

$$x^i = \left(B_z(-\varphi)\, B_x(-\theta)\, B_z(-\psi) \right)^{ij} x'^j$$

- let's now calculate $\{\dot{x}^i\}$
- $\{x'^j\}$ - fixed, (φ, θ, ψ) depend on t.

$(\dot{\varphi}, \dot{\theta}, \dot{\psi} \neq 0)$

thus:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi & -\sin\varphi\cos\theta & \sin\varphi\sin\theta \\ \sin\varphi & \cos\varphi\cos\theta & -\cos\varphi\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi\cos\psi - \sin\psi\sin\varphi\cos\theta & -\cos\varphi\sin\psi - \sin\varphi\cos\theta\cos\psi & \sin\varphi\sin\theta \\ \sin\varphi\cos\psi + \cos\varphi\cos\theta\sin\psi & -\sin\psi\sin\varphi + \cos\varphi\cos\theta\cos\psi & -\cos\varphi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}}_{\|A^{-1}\|} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\| A^{-1} \|$ time dependence is here (θ, φ, ψ change in time)

$$X_i = (A^{-1})^{(\theta,\varphi,\psi)}_{ij} x'^{j}$$

$\sum_{j=1}^{3}$ understood

as θ, φ, ψ change by $\Delta\varphi, \Delta\psi, \Delta\theta$ we have

$$\Delta x_i \equiv \left((A^{-1})_{ij}(\theta+\Delta\theta,\ \varphi+\Delta\varphi,\ \psi+\Delta\psi) - (A^{-1})_{ij}(\theta,\varphi,\psi) \right) x'_j$$

$$\approx \left(\frac{\partial A^{-1}}{\partial\theta}\right)_{ij} x'_j \Delta\theta + \left(\frac{\partial A^{-1}}{\partial\varphi}\right)_{ij} x'_j \Delta\varphi + \left(\frac{\partial A^{-1}}{\partial\psi}\right)_{ij} x'_j \Delta\psi$$

(+ quadratic terms - drop)

$$v_i = \frac{\Delta x_i}{\Delta t} \equiv \dot{x}_i = \left[\dot\theta \left(\frac{\partial A^{-1}}{\partial\theta}\right)_{ij} + \dot\varphi \left(\frac{\partial A^{-1}}{\partial\varphi}\right)_{ij} + \dot\psi \left(\frac{\partial A^{-1}}{\partial\psi}\right)_{ij} \right] x'_j .$$

This gives $\{\dot{x}_i\}$ expressed i.t.o. $\{x_i'\}$

expressing $x_j' = A_{jl}\, x_l$

we get

$$\dot{x}_i = \Big(\dot\theta \left(\frac{\partial A^{-1}}{\partial \theta}\right)_{il} A_{lm}$$

$$+ \dot\varphi \left(\frac{\partial A^{-1}}{\partial \varphi}\right)_{il} A_{lm}$$

$$+ \dot\psi \left(\frac{\partial A^{-1}}{\partial \psi}\right)_{il} A_{lm} \Big)\, x_m$$

or in m-x form:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left(\dot\theta\, \frac{\partial A^{-1}}{\partial \theta} \cdot A + \dot\varphi\, \frac{\partial A^{-1}}{\partial \varphi} \cdot A + \dot\psi\, \frac{\partial A^{-1}}{\partial \psi} \cdot A \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad ((*))$$

velocity of body-fixed point i

matrix multiplic

matrix multiplic.

coordinate of body-fixed point in body-fixed frame

((*)) will become, in vector notation

$$\vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \frac{d}{dt}\vec{r} = \vec{\Omega} \times \vec{r}$$

¢ we'll find explicitly $\vec{\Omega}$ as fn of θ, φ, ψ ¢ $\dot\theta\, \dot\varphi\, \dot\psi$

hence

$$\frac{d}{dt} x_i = \left(\dot{\varphi}\, \epsilon_{ijk}\, n_j^{\varphi} + \dot{\theta}\, \epsilon_{ijk}\, n_j^{\theta} + \dot{\psi}\, \epsilon_{ijk}\, n_j^{\psi} \right) x_k$$

where

$$\left.\begin{aligned} \vec{n}^{\varphi} &= (0,\ 0,\ 1) \\ \vec{n}^{\theta} &= (\cos\varphi,\ \sin\varphi,\ 0) \\ \vec{n}^{\psi} &= (\sin\varphi \sin\theta,\ -\cos\varphi \sin\theta,\ \cos\theta) \end{aligned}\right\}$$

components of $\vec{n}^{\varphi}, \vec{n}^{\theta}, \vec{n}^{\psi}$ in (X, Y, Z) system

in x', y', z' – see p. 131.1.

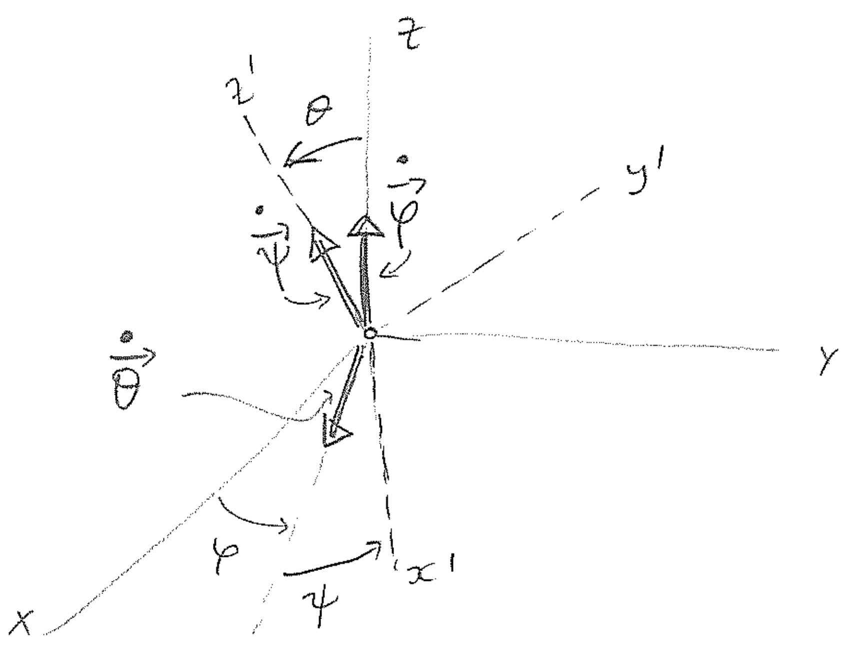
or

$$\frac{d}{dt}\vec{r} = \vec{\dot{\varphi}} \times \vec{r} + \vec{\dot{\theta}} \times \vec{r} + \vec{\dot{\psi}} \times \vec{r}$$

$$\vec{\dot{\varphi}} \equiv \dot{\varphi}\, \vec{n}^{\varphi}$$

$$\vec{\dot{\theta}} = \dot{\theta}\, \vec{n}^{\theta}$$

$$\vec{\dot{\psi}} = \dot{\psi}\, \vec{n}^{\psi}$$



$\vec{\dot{\theta}}, \vec{\dot{\psi}}, \vec{\dot{\varphi}}$ = the three components of angular velocity (NOT $\perp$!)