The total vector $\vec{\Omega}$ is the $\overbrace{\text{sum}}^{\text{vector}}$ of the three angular velocities:

$$\vec{\Omega} = \vec{\dot{\theta}} + \vec{\dot{\varphi}} + \vec{\dot{\psi}} \quad \left(\text{see p. } 131.1\right)$$

and so its components are, in 1,2,3 - body fixed frame:

$$\vec{\Omega} = \big(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi,$$
$$-\dot{\theta}\sin\psi + \dot{\varphi}\sin\theta\cos\psi,$$
$$\dot{\psi} + \dot{\varphi}\cos\theta\big)$$

So for a symmetrical top w/ $I_1 = I_2 = I \neq I_3$,

$$T = \frac{1}{2}\left[I\left(\Omega_1^2 + \Omega_2^2\right) + I_3\,\Omega_3^2\right]$$
$$= \frac{1}{2} I\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + \frac{1}{2} I_3\left(\dot{\psi} + \dot{\varphi}\cos\theta\right)^2$$

Using these "Euler angle coordinates" we can study the motion of a free symmetrical top (see 133+ pages)

- no ext forces
- assume CM motion "taken out", e.g. assume CM is at rest, so XYZ is a CM-based fixed coordinate system
- since free, $\vec{M}$ does not change in time — a fixed vector (components of $\vec{M}$ wrt (XYZ) = const)

- let's take $\vec{M}$ along Z-axis of (X,Y,Z) system
- let's take $\hat{3}$ of $(x_1 x_2 x_3)$ along symmetry axis of top

then we have, on one hand

$(x_1 x_2 x_3)$ components, change

(XYZ) component, constant in time

$$M_3 = M\cos\theta$$

$$M_2 = M\sin\theta\cos\psi$$

$$M_1 = M\sin\theta\sin\psi$$

on the other hand $M_i = I_i\,\Omega_i$ for (X_1, X_2, X_3) system components

$$M_3 = I_3(\dot\psi + \dot\varphi\cos\theta) \qquad (\text{using } \vec\Omega \text{ from p.132})$$

$$M_2 = I(-\dot\theta\sin\psi + \dot\varphi\sin\theta\cos\psi)$$

$$M_1 = I(\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi)$$

plug in $M_{1,2,3}$ thru $(M, \theta, \varphi, \psi)$:

$$\begin{cases} M\cos\theta = I_3(\dot\psi + \dot\varphi\cos\theta) \\ M\sin\theta\cos\psi = I(-\dot\theta\sin\psi + \dot\varphi\sin\theta\cos\psi) \\ M\sin\theta\sin\psi = I(\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi) \end{cases}$$

$c_\theta \equiv \cos\theta$
$s_\theta \equiv \sin\theta$
etc...

$$(**)\quad \begin{cases} \dot\psi + \dot\varphi c_\theta = \dfrac{M}{I_3} c_\theta \\ -\dot\theta s_\psi + \dot\varphi s_\theta c_\psi = \dfrac{M}{I} s_\theta c_\psi \quad \Big|\ \times c_\psi \\ \dot\theta c_\psi + \dot\varphi s_\theta s_\psi = \dfrac{M}{I} s_\theta s_\psi \quad \Big|\ \times s_\psi \end{cases} \Rightarrow \begin{cases} \dot\psi + \dot\varphi c_\theta = \dfrac{M}{I_3} c_\theta \\ -\dot\theta s_\psi c_\psi + \dot\varphi s_\theta c_\psi^2 = \dfrac{M}{I} s_\theta c_\psi^2 \\ \dot\theta c_\psi s_\psi + \dot\varphi s_\theta s_\psi^2 = \dfrac{M}{I} s_\theta s_\psi^2 \end{cases}$$

(OK, except @ $\theta = 0$!)

add last two: $\dot\varphi = \frac{M}{I}$

plug in 1st: $\boxed{\dot\psi = -\dot\varphi c_\theta + \frac{M}{I_3} c_\theta = M\left(\frac{1}{I_3} - \frac{1}{I}\right)\cos\theta}$

plug $\boxed{\dot\varphi = \frac{M}{I}}$ into last two equs in $\overset{**}{\nabla}$

$$-\dot\theta s_\psi + \frac{M}{I} s_\theta c_\psi = \frac{M}{I} s_\theta c_\psi$$

$$\dot\theta c_\psi + \frac{M}{I} s_\theta s_\psi = \frac{M}{I} s_\theta c_\psi$$

which mean $\dot\theta s_\psi = 0$

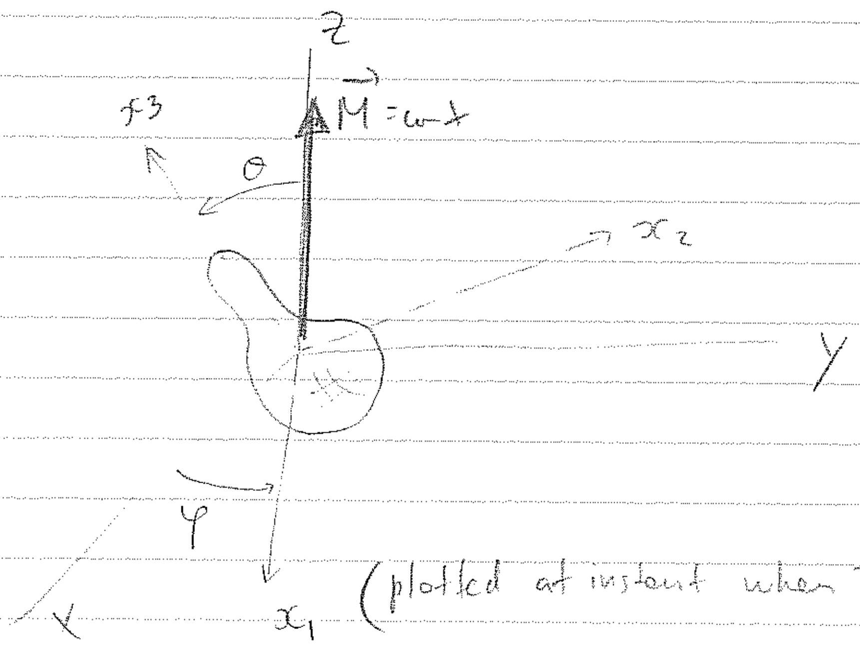
$\dot\theta c_\psi = 0$ hence $\dot\theta = 0$

$\longrightarrow$ $\dot\varphi = \text{const}$, $\dot\psi = \text{const}$, $\dot\theta = 0$, plug into Ω_i of p 132,

and hence $\Omega_3 = \frac{M}{I_3}\cos\theta$ $\longrightarrow$ constant since $\dot\theta = 0$

$$\left.\begin{aligned}\Omega_2 &= \frac{M}{I}\sin\theta\cos\psi \\ \Omega_1 &= \frac{M}{I}\sin\theta\sin\psi\end{aligned}\right\}$$

$\dot\theta = 0$ but $\dot\psi = \text{const}$, so $\Omega_{1,2}$ rotates w/ constant speed in 1-2 plane



(plotted at instant when $\psi = 0$)

at time

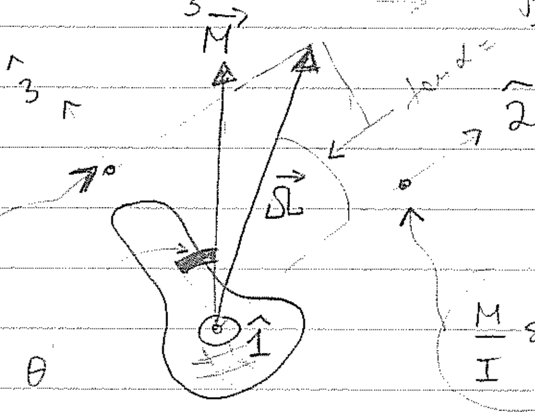
when $\psi = 0$ $\Omega_1 = 0$

$$\Omega_2 = \frac{M}{I}\sin\theta$$

$$\Omega_3 = \frac{M}{I_3}\cos\theta$$

$\rightarrow \tan\alpha = \frac{\Omega_3}{\Omega_2} = \frac{I}{I_3}$

so we have



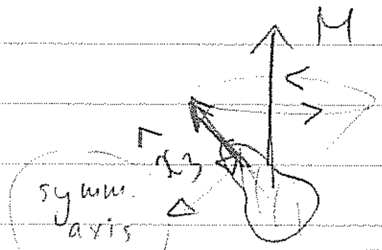
$\Omega_3 = \frac{M}{I_3}\cos\theta$

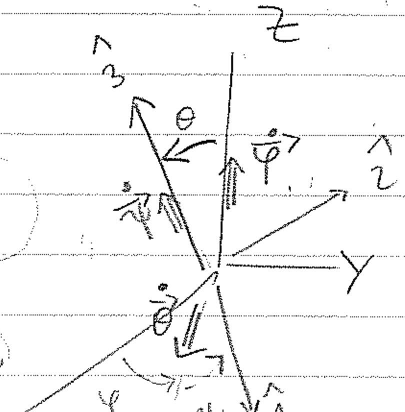
body rotates around symm. axis w/ Ω_3

$\frac{M}{I}\sin\theta = \Omega_2$

(I have chosen $I \ll I_3$ in this picture)

- so at every instant $\vec{M}$, $\vec{\Omega}$, $\underbrace{\hat{3}}_{\text{symmetry axis}}$ are in one plane
- velocity at any point on symmetry axis is $= \vec{\Omega} \times \vec{r}$ and is $\perp$ to the plane, causing plane to rotate
- since $\theta = $ const this means $\hat{3}$ symmetry axes precesses around $\vec{M}$, the velocity of precession being $\dot{\varphi} = \frac{M}{I}$;





$$\vec{\Omega} = \vec{\dot{\varphi}} + \vec{\dot{\psi}} + \underset{0}{\underset{\parallel}{\vec{\dot{\theta}}}}$$

$$\bar{v} = \bar{\Omega} \times \bar{r}$$

for $\bar{r}$ on $\hat{3}$, we have

$$\bar{v} = \underbrace{\vec{\dot{\varphi}} \times \hat{3}}_{\text{this pushes } \hat{3} \text{ axis to precess}} + \underbrace{\vec{\dot{\psi}} \times \hat{3}}_{\text{does nothing}}$$

the N pole

Claim: the Earth precesses like this, as $I_1 = I_2 \neq I_3$! $\frac{I_1 - I_3}{I_3} \approx -0.0033$

period ~ 430 days

Now, s'ppose we don't have a free top (symmetric or otherwise) ... we need eqns of motion – in general, $\vec{V}_{CM}$ is not constant and neither is $\vec{M}$, as there may be an external potential, leading to forces & torques.

To describe motion we need $\vec{R}_{CM}(t)$; $(\theta, \varphi, \psi)(t)$ – – so 6 equations, really needed $\Longrightarrow$ since $\dot{\vec{R}}_{CM} = \vec{V}_{CM}$ & $\mu \vec{V}_{CM} = \vec{P}_{CM}$ = momentum of body, & since $\dot{\theta}, \dot{\varphi}, \dot{\psi}$ & θ, φ, ψ determine $\vec{M}$ (in (x_1, x_2, x_3))

we'll need to know

$$\left.\begin{aligned} \frac{d}{dt}\vec{P}_{CM} &= ? \\ \frac{d}{dt}\vec{M} &= ? \end{aligned}\right\}$$

6 2nd order diff. eq'ns for X_{CM}, Y_{CM}, Z_{CM} & θ, φ, ψ.

$\Rightarrow$ the E.O.M. for a rigid body

Since

$\vec{P}_{CM} = \sum_a \vec{p}_a$, & we have $\dot{\vec{p}}_a = \vec{f}_a \leftarrow$ force on a-th particle due to all other parts of body + external forces so $\dot{\vec{P}}_{CM} = \sum_a \vec{f}_a = \vec{F}$ = total force on body

but now if there are no external forces acting on the body, r.h.s. should be zero since momentum as a whole is conserved: in other words, the forces between the parts of the body do not contribute to $\vec{F}$

– so it's only the external forces that do contribute !! assuming all forces are potential ones, we have

$U = U(\vec{\rho}_1, \ldots, \vec{\rho}_N)$ for the potential energy of the body (this can include int. term's)

If every position $\vec{\rho}_i$ gets shifted by $\delta\vec{R}$ we have

e.g. body shifts as a whole on $\delta\vec{R}$ – – so CM does, too

$$\delta U = U(\vec{\rho}_1 + \delta\vec{R} \ldots \vec{\rho}_N + \delta\vec{R}) =$$

$$= \delta\vec{R} \cdot \sum_a \frac{\partial U}{\partial \vec{\rho}_a} =$$

$$= \delta\vec{R} \cdot \left(-\sum_a \vec{f}_a\right) = -\vec{F} \cdot \delta\vec{R}$$

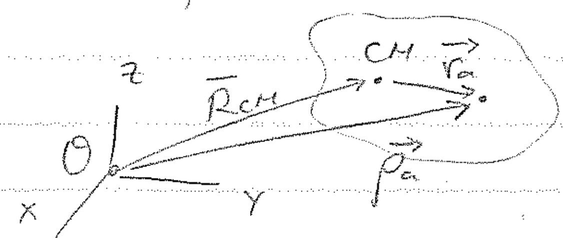
so $\vec{F} = -\frac{\delta U}{\delta \vec{R}}$ & one can think of this as due to a term in $L = T - U$ s.t. $U = U(\vec{R}_{CM})$

T: $\frac{1}{2}\mu \vec{V}_{CM}^2 + (\text{rotn})$; $\vec{R}_{CM}$: CM position.

so $\frac{d}{dt}\frac{\partial L}{\partial \vec{V}_{CM}} = \frac{\partial L}{\partial \vec{R}_{CM}} \iff \frac{d}{dt}\vec{P}_{CM} = -\frac{\partial U(\vec{R}_{CM})}{\partial \vec{R}_{CM}}$

Similar story holds for $\dot{\vec{M}}$ -->

from p. 127



general formula for $\dot{\vec{M}}$ wrt fixed O

$$\vec{M}_O = \sum_a \vec{\rho}_a \times \vec{p}_a \quad (\vec{p}_a = m\vec{v}_a)$$

$$\frac{d}{dt}\vec{M}_O = \sum_a \dot{\vec{\rho}}_a \times \vec{p}_a + \sum_a \vec{\rho}_a \times \dot{\vec{p}}_a =$$

$$= \underbrace{\cancel{\sum_a \vec{v}_a \times \vec{p}_a}}_{0 \text{ since } \vec{v}_a \parallel \vec{p}_a} + \sum_a \vec{\rho}_a \times \dot{\vec{p}}_a = \overbrace{\sum_a \vec{\rho}_a \times \vec{f}_a = \dot{\vec{M}}_O}$$

; since $\vec{\rho}_a = \vec{R}_{CM} + \vec{r}_a$

$$= \vec{R}_{CM} \times \left(\sum_a \dot{\vec{p}}_a\right) + \sum_a \vec{r}_a \times \vec{f}_a$$

$$\dot{\vec{M}}_O = \underbrace{\vec{R}_{CM} \times \vec{F}} + \sum_a \vec{r}_a \times \vec{f}_a$$

since $\vec{F}$ total force on body (external),
this is torque on the body as a point @ CM
wrt O

for $\vec{M}$ defined as angular mom-m wrt CM, we
have
only 2nd term: $\dot{\vec{M}} = \underbrace{\sum_a \vec{r}_a \times \vec{f}_a}_{\Sigma \text{ of torques of forces on } \forall \text{ particle}} = \sum_a \underbrace{\vec{k}_a}_{\text{torque on a-th one}} = \vec{K}$

only the torques due to external forces will matter (as if no ext. torques, $\dot{\vec{M}} = 0$ by momentum conservation)

so we have

$$\frac{d\vec{M}}{dt} = \underbrace{\vec{K}}_{\text{total torque on body due to } f_{\text{external}}}$$

$$\underbrace{\bar{K} = \sum_a \bar{r}_a \times \bar{f}_a}$$

here both $\vec{M}$ & $\vec{K}$ are wrt CM.

if $\vec{f}_a = m_a \vec{g}$, for example, we have

$$\vec{K} = \sum_a \vec{r}_a \times \vec{g} \cdot m_a = \underbrace{\left(\sum_a \vec{r}_a m_a\right)}_{0 \text{ in CM}} \times \vec{g}$$

$\Rightarrow$ homogeneous gravitational field does not apply torque wrt CM

$\rightarrow$ but it does wrt any other point —

e.g. $\vec{M}_O = \vec{R}_{CM} \times \sum_a \vec{f}_a = \vec{R}_{CM} \times \mu \vec{g}$

(clearly this is because $\vec{M}$, and $\vec{K}$ depend on the point wrt which they are defined)