kinematic effects in non inert. frames

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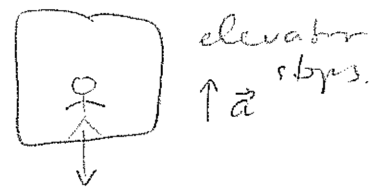
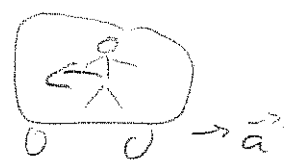
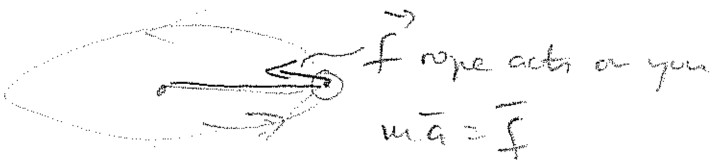


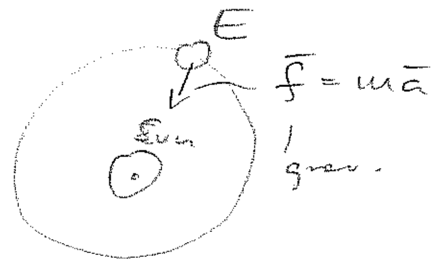
* bus starts



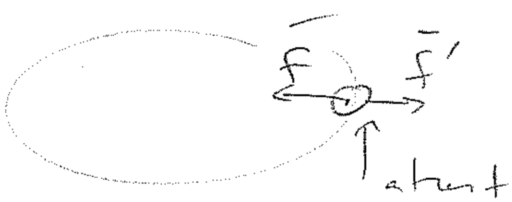
* rope//



or



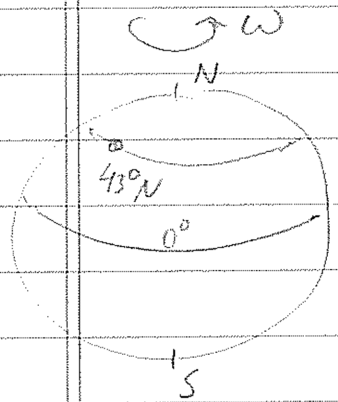
but in a rot. frame



not done w/ rigid body quite yet —

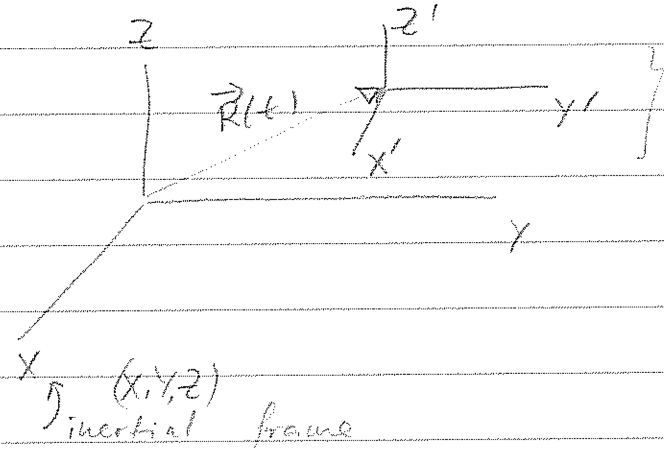
— but first — **noninertial frames**

e.g.:

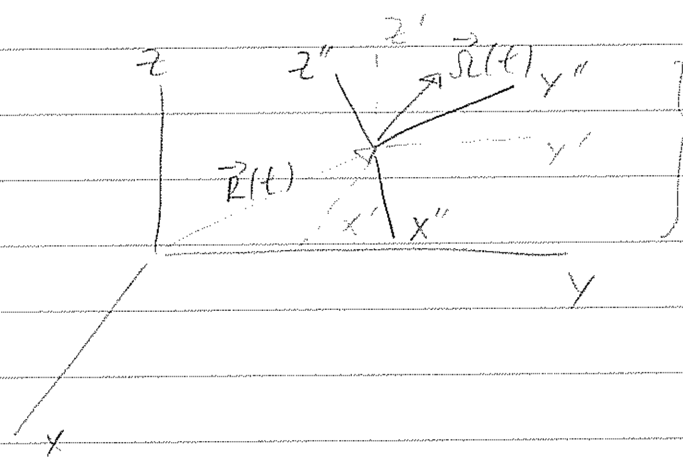


What is L in a noninertial frame?

most general case is shown in 2 steps:



$\dot{\vec{R}}(t) = \vec{V}(t)$
prescribed motion of (X',Y',Z') wrt (X,Y,Z) w/ parallel axes



$(X''Y''Z'')$ same origin as $(X'Y'Z')$ but rotates around it w/ $\vec{\Omega}(t)$ — also prescribed

* e.g. if Sun is taken inertial (XYZ), $(X'Y'Z')$ – Earth moving 'round Sun, $(X''Y''Z'')$ – a system rotating w/ the Earth

* going to an arbitrary frame is a change of coordinates, so action & E-L. equs are obtained by changing coordinates in L, by knowing relation of $X''Y''Z''$ to XYZ ones (dep. on $\vec{R}(t)$, $\vec{\Omega}(t)$,

original $L = \frac{1}{2} m \vec{v}^2 - U(\vec{r})$ in (XYZ)

then we have $\vec{r}' = \vec{r} - \vec{R}(t)$ ← prescribed position &
$\vec{v}' = \vec{v} - \vec{V}(t)$ ← velocity of $X'Y'Z'$

$L' = L(\vec{r}(\vec{r}'), \vec{v}(\vec{v}'))$

so
$$L' = \frac{1}{2} m (\vec{v}' + \vec{V}(t))^2 - U(\vec{r}' + \vec{R}(t))$$

$$= \frac{1}{2} m \vec{v}'^2 + \underbrace{m \vec{v}' \cdot \vec{V}(t)} + \underbrace{\frac{1}{2} m \vec{V}^2(t)}$$
$$- U(\vec{r}' + \vec{R}(t))$$

this is a total derivative – $\vec{V}^2(t) = f(t) = \frac{d}{dt} \int dt f(t)$ – drop

$$m \vec{v}' \cdot \vec{V}(t) = \underbrace{m \frac{d}{dt}(\vec{r}' \cdot \vec{V}(t))}_{\text{drop}} - \vec{r}' m \frac{d\vec{V}}{dt}$$

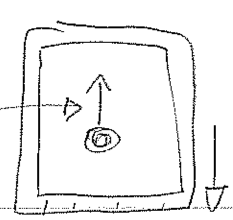
so $L' = \frac{1}{2} m \vec{v}'^2 - \underbrace{U(\vec{r}' + \vec{R}(t))} - m\vec{r}' \frac{d\vec{V}(t)}{dt}$

- for a single particle
- for many, w/ $U(\vec{r}_i - \vec{r}_j)$ would have $U(\vec{r}_i' - \vec{r}_j')$ i.e. not explicitly t-dependent

after step-1:
What are the EOM? $m \frac{d\vec{v}'}{dt} = - \frac{\partial U(\vec{r}' + \vec{R}(t))}{\partial \vec{r}'} - m \frac{d\vec{V}(t)}{dt}$

→ In an accelerated frame ⇒ like a uniform potential field force applied to mass and opposite to acceleration

$\vec{f}_{in} = -m\vec{g}$



i.e. in falling elevator frame

an example of an "inertial force"

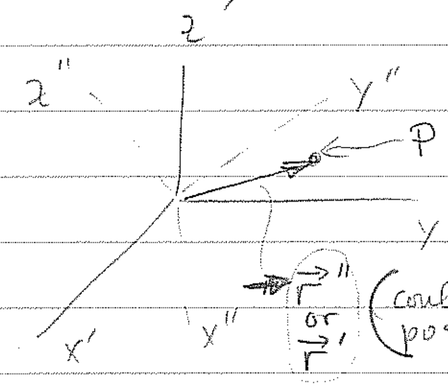
Now to step-2: for this step it is best to imagine that:

(a) there's more than one particle & (two-body) U depends on $\vec{r}_i - \vec{r}_j = \vec{r}_i{}' - \vec{r}_j{}'$

(b) U is independent of orientation of X''Y''Z'' wrt X'Y'Z' (e.g. it is central w/ center @ $\vec{R}(t)$)

(A more general case can, of course be handled, but leads to messy formulae; if/only when needed!)

Now, (to find L'') we need velocity of particle wrt X'Y'Z', $\vec{v}'$, i.t.o. velocity relative to X''Y''Z'' & the rotation of X''Y''Z'' wrt X'Y'Z'



(could've called it $\vec{r}$, doesn't matter, it's the position of P vector, can be studied in ' or '' component

Remember HW.5, #2 → we know for P, an arbitrary point w/ coordinates (x_1', x_2', x_3') in $(X'Y'Z')$ & (x_1'', x_2'', x_3'') in (X, Y, Z), that

$$x_i'' = \sum_{j=1}^{3} A_{ij}(\theta, \varphi, \psi)\, x_j' \quad \text{or} \quad x_j' = \sum_{l=1}^{3} A^{-1}_{jl}(\theta, \varphi, \psi)\, x_l''$$

As opposed to rigid body, here x_l'' can depend on t as well (here's not fixed, t-dep)

So now $\dot{x}_j'$ are the components of velocity wrt $(X'Y'Z')$ in $(X'Y'Z')$-frame.

So we have $$v_j' = \frac{d}{dt} x_j' = \sum_{l=1}^{3} \frac{d}{dt}\left(A^{-1}_{jl}(\theta,\varphi,\psi)\, x_l''\right)$$

$$= \sum_{l=1}^{3} \frac{d}{dt}\left(A^{-1}_{jl}(\theta,\varphi,\psi)\right)\cdot x_l''$$

this is the same as in #2 of HW.5, as x_l'' is not differentiated — you showed this term was $\vec{\Omega}\times\vec{r}''$

$$+ \underbrace{\sum_{l=1}^{3} A^{-1}_{jl}(\theta,\varphi,\psi)\, \dot{x}_l''}$$

this is a new term: $\dot{x}_l''$ is "components of velocity wrt $X''Y''Z''$ in $X''Y''Z''$"

$A^{-1}\cdot\dot{x}''$ is "components of same velocity in $X'Y'Z'$

So in vector notation, we have

$$\vec{v}' = \vec{\Omega}\times\vec{r}'' + \vec{v}''$$

velocity of particle in $X'Y'Z'$

velocity due to rotation of $X''Y''Z''$ wrt $X'Y'Z'$ (nonzero even if @ rest in $X''Y''Z''$)

velocity in $X''Y''Z''$

back to:

$$L' = \frac{1}{2} m \vec{v}'^2 - m \vec{r}' \cdot \frac{d\vec{V}(t)}{dt} - U(|\vec{r}'|)$$

$$L'' = L'(\vec{v}' = \vec{\Omega} \times \vec{r}'' + \vec{v}'';\ \underbrace{\vec{r}' = \vec{r}''}_{\text{tautology}})$$

$$L'' = \frac{1}{2} m (\vec{v}'' + \vec{\Omega} \times \vec{r}'')^2 - m \vec{r}'' \cdot \frac{d\vec{V}}{dt} - U(|\vec{r}''|)$$

$$\boxed{L'' = \frac{1}{2} m \vec{v}''^2 + m \vec{v}'' \cdot (\vec{\Omega} \times \vec{r}'') + \frac{1}{2} m (\vec{\Omega} \times \vec{r}'')^2 - m \vec{r}'' \cdot \frac{d\vec{V}}{dt} - U(|\vec{r}''|)}$$

so there are extra terms in L'', in the final frame which is moving w/ $\vec{V}(t)$ & rotating w/ $\vec{\Omega}(t)$

given functions of t

to find the E.-L. equs use:

$$\frac{\partial L''}{\partial \vec{v}''} = m \vec{v}'' + m \vec{\Omega} \times \vec{r}''$$

to do $\frac{\partial L}{\partial \vec{r}''}$ note that $(\vec{\Omega} \times \vec{r}'')^2 = \vec{\Omega}^2 (\vec{r}'')^2 - (\vec{\Omega} \cdot \vec{r}'')^2$

& that $\vec{v}'' \cdot (\vec{\Omega} \times \vec{r}'') = \vec{r}'' \cdot (\vec{v}'' \times \vec{\Omega})$

Quick proof of latter:

$$\bar{v}''\cdot(\bar{\Omega}\times\bar{r}'') = \sum_{ijk} \epsilon_{ijk} v_i'' \Omega_j r_k''$$

$$= \sum_{ijk} r_k'' \epsilon_{kij} v_i \Omega_j = \bar{r}''\cdot(\bar{v}\times\bar{\Omega})$$

So $$\frac{\partial L}{\partial \vec{r}''} = m\vec{v}''\times\vec{\Omega} + \underbrace{m\vec{r}''(\vec{\Omega}^2) - m\vec{\Omega}(\vec{\Omega}\cdot\vec{r}'')}$$

$$- m\frac{d\vec{V}}{dt} - \frac{\partial U(|\vec{r}''|)}{\partial \vec{r}''}$$

this is $\equiv$

$m(\vec{\Omega}\times\vec{r}'')\times\vec{\Omega}\Big]_{\ell^{th}\text{ component}}$

$m\,\epsilon_{ijk}\Omega_j r_k''\epsilon_{\ell im}\Omega_m =$

$= m\,\underbrace{\epsilon_{\ell im}\epsilon_{kij}}_{\delta_{\ell k}\delta_{mj}-\delta_{\ell j}\delta_{mk}}\Omega_j r_k''\Omega_m$

$= m\left(r_\ell''\Omega^2 - \Omega_\ell(\bar{\Omega}\cdot\bar{r})\right)$ ∎

$$= m\vec{v}''\times\vec{\Omega} + m(\vec{\Omega}\times\vec{r}'')\times\vec{\Omega}$$

$$- m\frac{d\vec{V}}{dt} - \frac{\partial U}{\partial \vec{r}''}$$

So $\underline{\frac{d}{dt}\frac{\partial L}{\partial \vec{v}''}} = \frac{\partial L}{\partial \vec{r}''}$ gives:

$$m\dot{\vec{v}}'' + m\dot{\vec{\Omega}}\times\vec{r}'' + m\vec{\Omega}\times\vec{v}'' =$$

$$= m\vec{v}''\times\bar{\Omega} + m(\bar{\Omega}\times\bar{r}'')\times\bar{\Omega} - m\frac{d\bar{V}}{dt} - \frac{\partial U}{\partial \bar{r}''}$$

$$m\dot{\vec{v}}'' = -\frac{\partial U}{\partial \vec{r}''} - m\frac{d\vec{V}}{dt} - m\dot{\vec{\Omega}}\times\vec{r}'' + 2m\vec{v}''\times\vec{\Omega}$$

$$+ m\vec{\Omega}\times(\vec{r}''\times\vec{\Omega})$$

acceleration inertial force

Coriolis, $\perp \vec{v}''$

non-uniform rotation inertial force

centrifugal

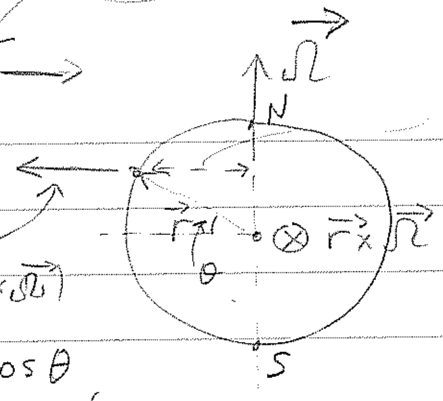
potential force

!! $\boxed{**}\rightarrow$!!

at this point it's good to trop the "sin $\bar{r}$ & $\bar{v}$

$\vec V = 0$, base at center || if base on surface $\dot{\vec V} \neq 0$, as $\vec V$ rotates, $\dot{\vec V}$ term gives centrifugal ... (150)

back to Earth



$\Omega \sim 10^{-4} \frac{1}{s}$

$\left(\frac{2\pi}{day} \approx \frac{2\pi}{10^5 s} \right)$

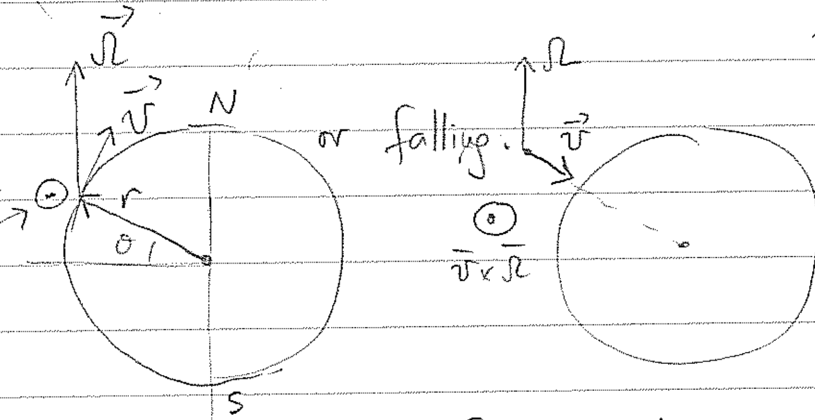
$m \vec\Omega \times (\vec r \times \vec\Omega)$

$m \Omega^2 r \cos\theta$

– centrifugal force ($\ll mg$, since $g \sim 10 \frac{m}{s^2}$ $\quad \Omega^2 r \sim 10^{-8} \times 10^6 \frac{m}{s^2} \sim 10^{-2} \frac{m}{s^2}$)

small 3 orders of magnitude

moving on a meridian, say:



or falling.

$2m \vec v \times \vec\Omega$

Coriolis : moving North ⇒ Eastward etc...

If there's no translational or rotational acceleration,

$\dot{\vec\Omega} = \dot{\vec V} = 0$

$$L = \frac{1}{2} m \vec v^2 + m \vec v \cdot (\vec\Omega \times \vec r) + \frac{1}{2} m (\vec\Omega \times \vec r)^2 - U$$

energy is actually conserved $\rightarrow$ $\vec p = \frac{\partial L}{\partial \vec v} = m(\vec v + \vec\Omega \times \vec r)$,

$$E = \vec p \cdot \vec v - L = m \vec v^2 + \cancel{m(\bar\Omega \times \bar r) \cdot \bar v} - \frac{1}{2} m \vec v^2 - \cancel{m \vec v \cdot (\bar\Omega \times \bar r)}$$

$$- \frac{1}{2} m (\bar\Omega \times \bar r)^2 + U$$

$$E = \frac{1}{2} m \vec v^2 + U - \underbrace{\frac{1}{2} m (\bar\Omega \times \bar r)^2}$$

(centrifugal potential energy $\sim \Omega^2 r^2$)