in a noninertial frame moving w/ $\vec{V}(t), \vec{\Omega}(t)$ (151)

we found

$$L = \frac{m}{2}(\vec{v} + \vec{\Omega} \times \vec{r})^2 - m\vec{r}\frac{d\vec{V}}{dt} - U(|\vec{r}|)$$

hence $\boxed{\vec{p} = \frac{\partial L}{\partial \vec{v}} = m(\vec{v} + \vec{\Omega} \times \vec{r})}$ note again $\frac{\partial L}{\partial \vec{v}} \neq m\vec{v}$

& $$H \equiv \vec{p} \cdot \vec{v} - L(\vec{r}, \vec{v})\Big|_{\vec{v} = \frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r}} = H(\vec{p}, \vec{r})$$

$$H(\vec{p}, \vec{r}) = \vec{p} \cdot \left(\frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r}\right) - \frac{m}{2}(\vec{v} + \vec{\Omega} \times \vec{r})^2$$

$$+ m\vec{r}\frac{d\vec{V}}{dt} + U(|\vec{r}|)\Big|_{\vec{v} = \frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r}}$$

or $\vec{v} + \vec{\Omega} \times \vec{r} = \frac{\vec{p}}{m}$

$$= \frac{\vec{p}^2}{m} - \vec{p}(\vec{\Omega} \times \vec{r}) - \frac{m}{2}\left(\frac{\vec{p}}{m}\right)^2$$

$$+ m\vec{r}\frac{d\vec{V}}{dt} + U(|\vec{r}|) =$$

$$\boxed{H(\vec{r}, \vec{p}) = \frac{\vec{p}^2}{2m} + U(|\vec{r}|) - \vec{p}(\vec{\Omega} \times \vec{r}) + m\vec{r}\frac{d\vec{V}}{dt}}$$

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} = \frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r} \; (\equiv \vec{v}); \quad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} \longrightarrow$$

$$\mathring{\bar{p}} = -\frac{\partial U}{\partial \bar{r}} - m\frac{d\vec{V}}{dt} - \frac{\partial}{\partial \vec{r}}\left(\underbrace{-\bar{p}(\bar{\Omega}\times\bar{r})}\right)$$

$$- \frac{\partial}{\partial \bar{r}}\left(-\bar{r}\cdot(\bar{p}\times\bar{\Omega})\right)$$

$$\mathring{\bar{p}} = -\frac{\partial U}{\partial \bar{r}} - m\frac{d\vec{V}}{dt} + \bar{p}\times\bar{\Omega}$$

to compare w/ E.L.: use $\mathring{\bar{r}} = \frac{\bar{p}}{m} - \bar{\Omega}\times\bar{r}$,

take $\frac{d}{dt}\mathring{\bar{r}} = \mathring{\mathring{\bar{r}}} = \frac{1}{m}\mathring{\bar{p}} - \frac{d}{dt}(\bar{\Omega}\times\bar{r}) \quad \Big| \times m$

$$m\mathring{\mathring{\bar{r}}} = \mathring{\bar{p}} - m\mathring{\bar{\Omega}}\times\bar{r} - m\bar{\Omega}\times\mathring{\bar{r}} =$$

$\downarrow$ sub $\mathring{p}$ & express $\vec{p}$ via $\bar{v}, \bar{r}$

$$m\mathring{\mathring{\bar{r}}} = \underbrace{-\frac{\partial U}{\partial \bar{r}} - m\frac{d\vec{V}}{dt} + m(\vec{v} + \bar{\Omega}\times\bar{r})\times\bar{\Omega}}_{\mathring{\bar{p}}}$$

$$- m\mathring{\bar{\Omega}}\times\bar{r} - m\bar{\Omega}\times\bar{v}$$

$$= -\frac{\partial U}{\partial \bar{r}} - m\frac{d\vec{V}}{dt} + m\bar{v}\times\bar{\Omega} + m(\bar{\Omega}\times\bar{r})\times\Omega$$

$$- m\mathring{\bar{\Omega}}\times\bar{r} - m\bar{\Omega}\times\bar{v}$$

same as E.L.

$$m\mathring{\mathring{\bar{r}}} = -\frac{\partial U}{\partial \bar{r}} - m\frac{\partial\vec{V}}{\partial t} + 2m\bar{v}\times\bar{\Omega} + m(\vec{\Omega}\times\vec{r})\times\Omega - m\mathring{\bar{\Omega}}\times\bar{r}$$

if $\vec{V}$ & $\vec{\Omega}$ are constant, equ simplify – only Coriolis & centrifugal remain

$$m\ddot{\vec{r}} = -\frac{\partial U}{\partial \vec{r}} + \underbrace{2m\,\vec{v}\times\vec{\Omega}}_{\text{Coriolis}} + \underbrace{m\,\bar{\Omega}\times(\bar{r}\times\bar{\Omega})}_{\text{centrifugal}}$$

Now, before we go to gyroscopic applications, let me make an important side remark:

H in noninertial frame of p. 151 can be also written like this (bottom, boxed equ.)

(153.1)
$$H(\bar{r},\bar{p}) = \frac{1}{2m}\left(\bar{p} - m\bar{\Omega}\times\bar{r}\right)^2 + U(|\vec{r}|) + \\ + m\,\vec{r}\cdot\frac{d\vec{V}}{dt} - \frac{1}{2}m(\bar{\Omega}\times\bar{r})^2$$

(expand square & get equ from p. 151)

Recall Hamiltonian in El.M. field

(153.2) $$H(\bar{r},\bar{p}) = \frac{1}{2m}\left(\bar{p} - \frac{e}{c}\bar{A}\right)^2 + U(\bar{r}) + e\phi$$

el.m. potentials: vector & scalar

Equ (153.1) & (153.2) are <u>very</u> similar:

$$e\phi \longleftrightarrow m\left(\vec{r}\cdot\frac{d\vec{V}}{dt} - \frac{1}{2}(\bar{\Omega}\times\bar{r})^2\right)$$

$$\frac{e}{c}\vec{A} \longleftrightarrow m(\bar{\Omega}\times\bar{r})$$

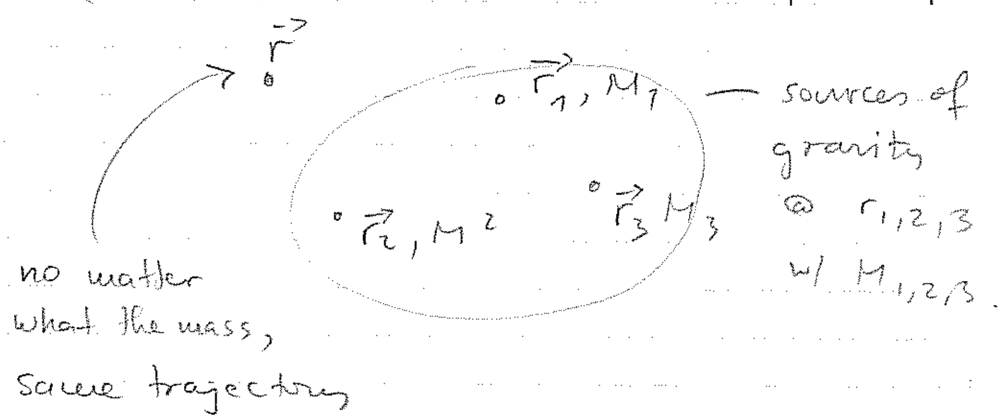
El.M. potentials "gauge fields"

potentials for inertial forces

This similarity is the formal expression of the fact that <u>inertial forces are proportional to the mass</u> and that all bodies feel the effect of inertial forces in the same manner – the acceleration due to inertial forces is independent of the mass of the body ≡ ≡ <u>just like for gravity</u>

in grav. field, also all bodies move identically, independent of their mass: $\cancel{m}\ddot{\vec{r}} = -\cancel{m}\vec{\nabla}\phi$

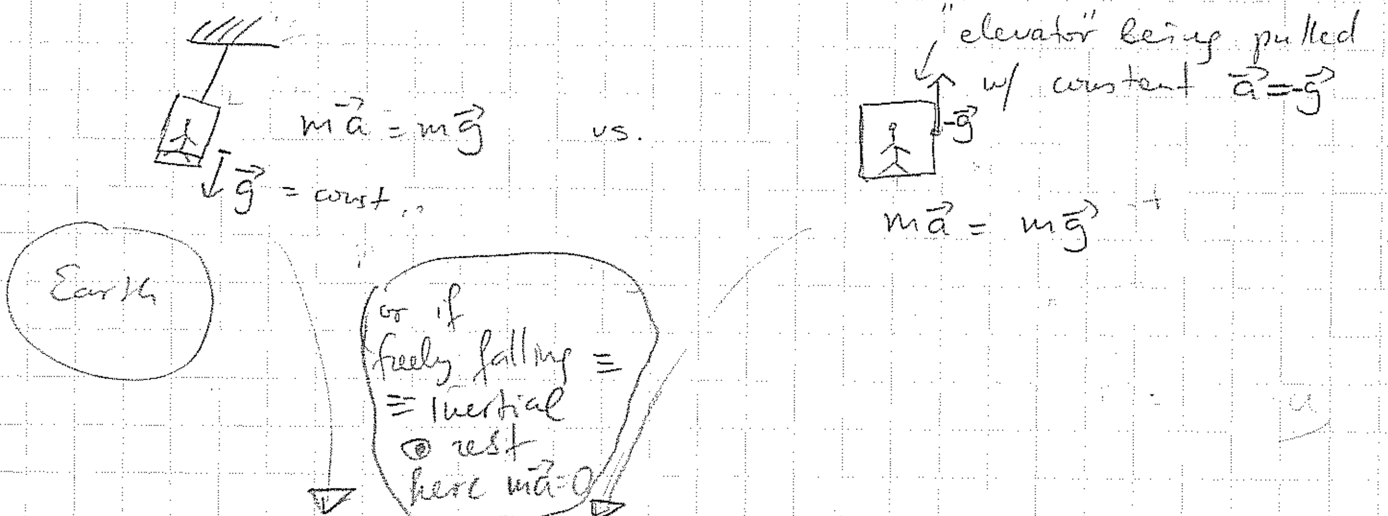
$$\phi(\vec{r}) = -\sum_a \frac{M_a G_N}{|\bar{r} - \bar{r}_a|} \quad \left(\begin{matrix}\text{gravitational}\\ \text{potential}\end{matrix}\right)$$



So, it appears that motion in a noninertial frame is similar to motion in an inertial frame, but with a gravitational field (at least as far as the $m\vec{r}\cdot\frac{d\vec{V}}{dt}$ term is concerned... – but identity of accel. under inertial forces for all bodies holds for $\vec{\Omega}$-dependent inertial forces as well)

and is reflected in the appearance of the $\vec{A} \sim \vec{\Omega} \times \bar{r}$ ("gravimagnetic") & $\phi \sim (\bar{\Omega} \times \bar{r})^2$ ("gravielectric")

The idea that inertial forces and gravitational fields are indistinguishable from the point of view of an observer able to see only a small region of space around his/er trajectory ("locally") is known as the "principle of equivalence"



* the observer in elevator can't tell whether s/he is in the left or right situation (so long as s/he can't peer outside & see e.g. the upcoming crash on the Earth → i.e. "locally" equivalent only)

Einstein idea was that every gravity field was locally equivalent to motion in an accelerating frame - which led to the notion that the gravitational field is different, it represents the curving of spacetime due to massive objects .... →

→ for people taking relativity / or GR

analogy w/ noninertial frame is in that

in such a frame we have

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$ (inertial frame Minkowski space interval)

frame rotating w/ $\vec{\Omega} \parallel \vec{z}$

$$x = x' \cos \Omega t - y' \sin \Omega t$$
$$y = x' \sin \Omega t + y' \cos \Omega t$$
$$z = z'$$

$$ds^2 = \left(c^2 - \Omega^2 (x'^2 + y'^2)\right) dt^2 - dx'^2 - dz'^2$$
$$+ 2\Omega y' dx' dt - 2\Omega x' dy' dt$$

now g_{ij} has

$$g_{00} \equiv 1 - \frac{\Omega^2 (x^2 + y^2)}{c^2}$$

like our ϕ ("gravielectric")

$$g_{x0} = 2(\vec{\Omega} \times \vec{r})_x \leftarrow \neq 0$$
$$g_{y0} = 2(\vec{\Omega} \times \vec{r})_y \leftarrow \neq 0$$

like our $\vec{A}$ ("gravimagnetic")

* so spacetime interval in noninertial frames has off-diagonal space-time dependent components

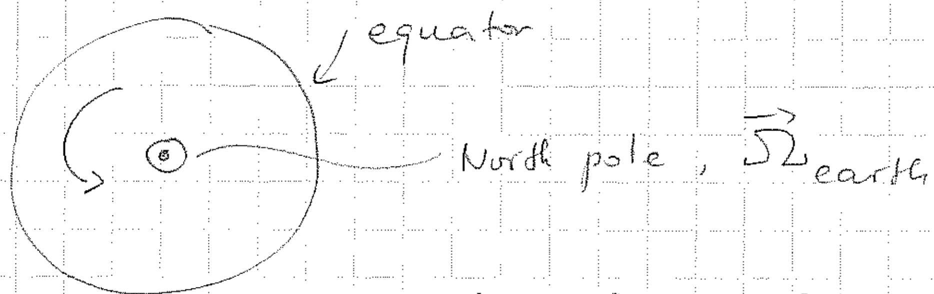
(Einstein's theory) ⇐

* accepting principle of equivalence, every gravitational field is then described by a general space-time dependent interval $ds^2 = g_{ij}(x)\, dx^i dx^j$
$\neq (+1, -1, 1, 1)$

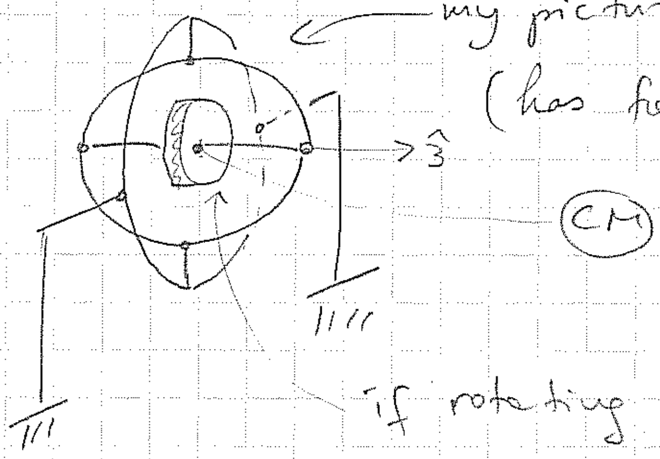
Wrapping up ---

"To those who study the progress of exact science, the common spinning top is a symbol of the labors & perplexities of men" – Maxwel

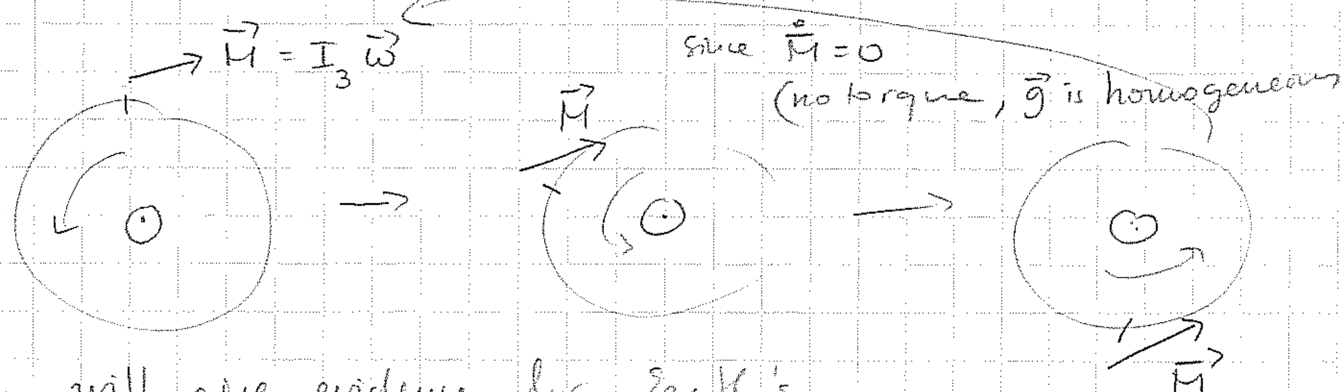
(1) a few words on gyrocompasses – instead of carefully solving equations (not enough time!)



my picture of a gimble
(has freedom to rotate in all directions)



if rotating & if friction small, on surface of Earth will keep its direction in space.

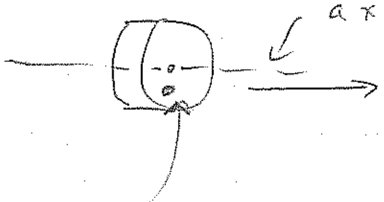


i.e. will give evidence for Earth's rotation (and less easily discernible for Earth's rot'n around Sun --), but not good for anything else

[in ~~an~~ the noninertial frame based on surface of Earth $\vec{M}$ changes, of course, due to torque from inertial forces (Coriolis/centrifugal)]

But can be made slightly more useful if modified a bit... imagine now CM is <u>NOT</u> in the center point around which the gyroscope can rotate, but instead

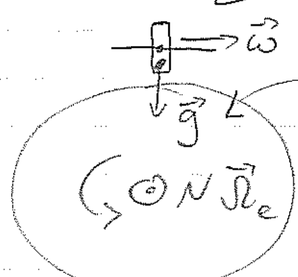
axis of symmetry/rotation



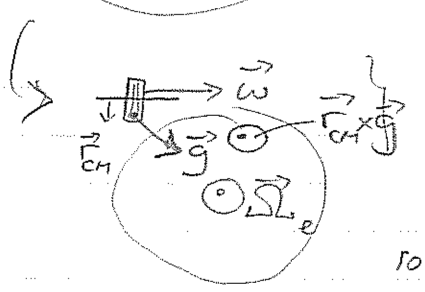
$\vec{\omega}$ - imagine rotating w/ ω <u>fast</u>

CM

starting position (for simplicity; on equator



no torque wrt center of rotation (some small one, // $\vec{\omega}$, but $\vec{\omega}$ is large, irrelevant)



after Earth's rotated a bit there's torque $\vec{r}_{CM} \times \vec{g}$

so $\vec{M} \approx I\vec{\omega}$ now $\dot{\vec{M}} = \mu \vec{r}_{CM} \times \vec{g}$

tries to push axis <u>north</u> (to meridian)

Always rotating towards meridian — points <u>N</u>

<u>Equations best studied in noninertial frame.</u>

--- (DIY__)

2 Most of this class we used the E.-L. equations (from the variational calculus combined w/ <u>extremum action principle</u>) to study "post-history" dynamics. I told you that the <u>modern understanding</u> of how this principle arises is only possible after Feynman's formulation of Quantum Mechanics via a path integral — where a quantum particle takes all paths — and only in the classical limit one finds that a single path dominates — the one that extremizes the action. But I didn't tell you the "pre-history" — Fermat's principle of geometric optics. Fermat argued that <u>light rays</u> (the stuff of interest of geometric optics, $\lambda \to 0$ limit of wave theory) <u>follow paths of minimal optical length</u>:

- let $n(\vec{r})$ be the (in general) position dependent refraction index

- let $\vec{r}(s)$ be the path a ray will follow, $\vec{r}(0) = \vec{r}_0$, $\vec{r}(s=1) = \vec{r}_1$

Fermat: $$\ell(\bar{r}_0, \bar{r}_1) = \int_0^1 ds \, \frac{d|\vec{r}|}{ds} \, n(\vec{r})$$

is minimal along the actual path.

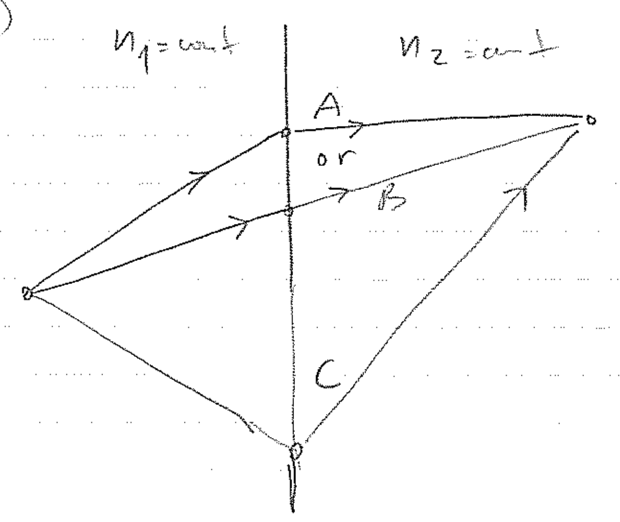
Ex: (1) $n(\vec{r}) = $ const.

$$\ell = \int_0^1 ds \, \frac{d|\vec{r}|}{ds} \equiv \int_{\vec{r}_0}^{\vec{r}_1} \sqrt{dx^2 + dy^2 + dz^2}$$

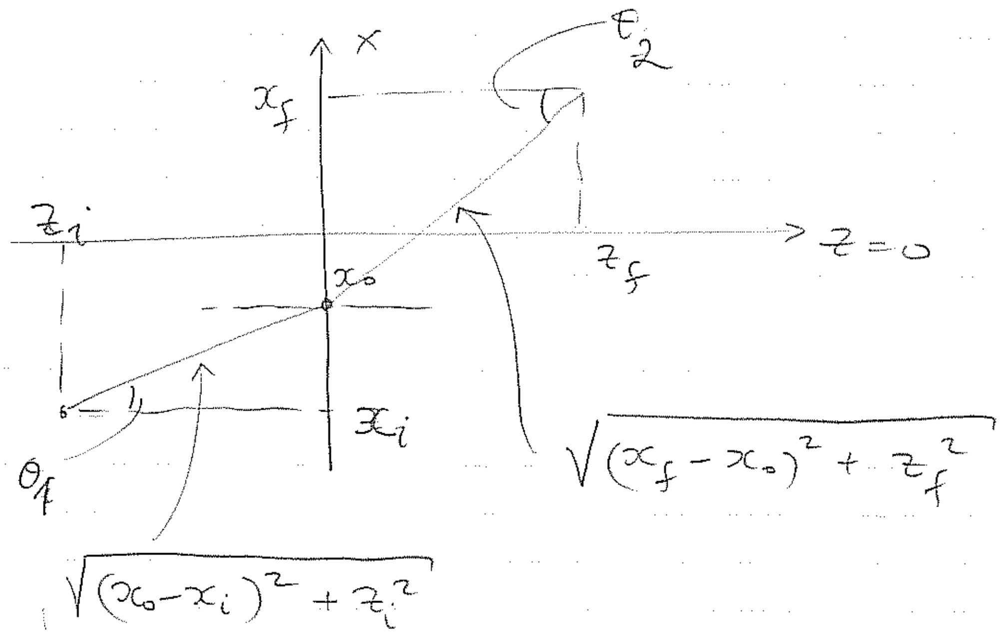
$$= \underbrace{\int_0^1 ds \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2}}_{\text{simply the geometric Euclidean length along the path from } \vec{r}_0 \text{ to } \vec{r}_1}$$

Tutorial 1: $\Rightarrow$ it is the straight line.

(2)



} which one is shortest?



$$\ell = n_1 \sqrt{(x_0 - x_i)^2 + z_i^2} + n_2 \sqrt{(x_f - x_0)^2 + z_f^2}$$

<u>$x_0 = ?$</u>

$$\frac{d\ell}{dx_0} = 0 \Rightarrow \frac{n_1 (x_0 - x_i)}{\sqrt{(x_0 - x_i)^2 + z_i^2}} - \frac{n_2 (x_f - x_0)}{\sqrt{(x_f - x_0)^2 + z_f^2}} = 0$$

$$n_1 \frac{x_0 - x_i}{\sqrt{(x_0 - x_i)^2 + z_i^2}} = n_2 \frac{x_f - x_0}{\sqrt{(x_f - x_0)^2 + z_f^2}}$$

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{n_2}{n_1} \longleftrightarrow \text{Snell's law}$$

If $n = n(\vec{r})$ one can find a differential equation ...

$$\ell = \int_0^1 ds \sqrt{|\dot{\vec{r}}|^2}\, n(\vec{r}(s)) \; ; \quad \dot{\vec{r}} = \frac{d\vec{r}}{ds}$$

this $\ell \equiv \ell[\vec{r}(s)]$ — a functional of the path

$$\ell[\vec{r}] = \int_0^1 ds \sqrt{\frac{d\vec{r}}{ds}\cdot\frac{d\vec{r}}{ds}}\; n(\vec{r}(s)) \;,\quad \text{w/. } \vec{r}(0) \;\&\; \vec{r}(1) \text{ fixed.}$$

$$\ell[\vec{r}+\delta\vec{r}] - \ell[\vec{r}] \equiv \delta\ell[\vec{r}] = 0 =$$

$$= \int_0^1 ds \left\{ \frac{\partial n}{\partial \vec{r}}\cdot\delta\vec{r}\sqrt{\frac{d\vec{r}}{ds}\cdot\frac{d\vec{r}}{ds}} + \right.$$

we $\int$ by parts, drop boundary terms

$$\left. + n(\vec{r})\frac{1}{\cancel{2}\sqrt{\frac{d\vec{r}}{ds}\cdot\frac{d\vec{r}}{ds}}}\,\cancel{2}\,\frac{d\vec{r}}{ds}\cdot\frac{d}{ds}\delta\vec{r} \right\}$$

$$= \int_0^1 ds\; \delta\vec{r}(s)\cdot\left\{ \frac{\partial n}{\partial\vec{r}}\sqrt{\dot{\vec{r}}^2} - \frac{d}{ds}\left(\frac{n\,\dot{\vec{r}}}{\sqrt{\dot{\vec{r}}^2}}\right)\right\}$$

As opposed to mechanics $s \in (0,1)$ is totally arbitrary — just a parameter along path, there is no sense that "time along ray is fixed", so free to choose s any way one wants.

If one thinks of s as "time" and $\dot{\vec{r}} = \frac{d\vec{r}}{ds}$ as "velocity", it is convenient to choose "velocity" along path at $\vec{r}$

to be equal to $n(\vec{r})$.

(thinking of it that way, clearly this is always possible!)

(explicitly, $ds = \dfrac{|d\vec{r}|}{n(\vec{r})}$, s found by $\int$)

then, $\dfrac{d}{ds}\left(\dfrac{n\,\dot{\vec{r}}}{\sqrt{\dot{\vec{r}}^2}}\right) = \sqrt{\dot{\vec{r}}^2}\,\dfrac{dn}{d\vec{r}}$

becomes

$$\begin{cases} \ddot{\vec{r}} = n\,\vec{\nabla} n \\ \dot{\vec{r}}^2 = n^2 \end{cases}$$

a particle of unit mass moves in potential $V(\vec{r}) = -\frac{1}{2}\,n^2(\vec{r})$

as a function of time s w/

total energy $E = \frac{1}{2}\,\dot{\vec{r}}^2 + V = \frac{1}{2}\,n^2 + \left(-\frac{1}{2}\,n^2\right) = $

$\boxed{\text{Snell's law "explanation"}}$ ← "particle" of light ("corpuscule") is "pulled" towards higher n (index of refraction) cause it "moves faster" & 'quickly' leaves the interface in $\perp$ direction ...

i.e. if n_2 - bigger, θ_2 - smaller

— as per Newton's
corpuscular theory of light
(wrong — but recovered in $\lambda \to 0$ limit
of geometric optics where Fermat's
principle applies) —

Moral: 1) principle of extremal action
has both pre-
and post- history ——
(Fermat & Feynman : optics & Q.M.

2) whether one formulates Class Mech
using L (or H) to a large degree an
aesthetic question, but:

(2.1) changes of variables easy w/ L
& symmetries
— polar, etc., generalized
— noninertial frames!
— all fundam. theories use it.

(2-2) Hamiltonian method allows for
many new results
- phase space useful concept
- transition to QM & QFT