From Lagrangian to Hamiltonian mechanics

Lagrangian mechanics: $\{q(t), \dot{q}(t)\}$

for every particle $+ \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$

Legendre transform

Hamiltonian mechanics: $\{q(t), p(t)\}$

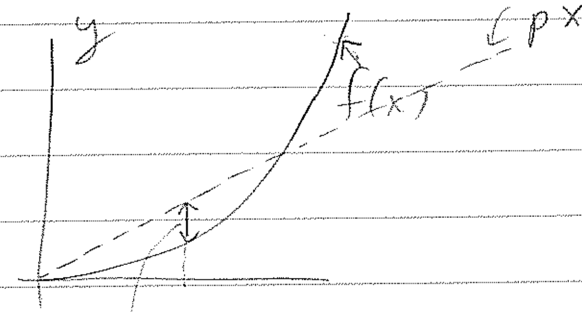
+ Hamilton's E.o.M

→ what's this?

Let $y = f(x)$ be s.t. $f''(x) > 0$

the Legendre transform of $f(x)$ is another

f-n, $g(p)$, defined as follows:



here $F(x,p) = xp - f(x)$

→ has a max w.r.t. x @ fixed p

i.e. $x(p)$: $F'_x(x,p) = 0$

i.e. $p = f'(x) \Rightarrow x = x(p)$

NB max is unique 'cause $f''(x) > 0$ – see pic.

Def: Legendre transform of $f(x)$ w/ $f''(x) > 0$ ⟶ $\underline{g(p) = F(x(p),p) = x(p)\,p - f(x(p))}$

So, given

$$f(x), \quad f''(x) > 0$$

$$g(p) = px - f(x), \text{ where}$$

$$x(p): \quad p = f'(x)$$

Claim: (a) $g(p)$ also obeys $g''(p) > 0$

(b) Legendre transform of $g(p) \longrightarrow f(x)$

i.e. $\underbrace{\text{"(Legendre)}^2 = \mathbb{1}\text{"}}$

it is a transform that's an "involution"

(applied twice gives back trivial transform, like a rot'n on π, reflection etc.)

Proofs: (a) $$g'(p) = \frac{d}{dp}\Big(x\,p - f(x)\Big) =$$

$$= \cancel{x'(p)\,p} + x(p) - \cancel{\underbrace{f'(x(p))}_{\equiv p;\ p = f'(x)}\,x'(p)}$$

$$\Rightarrow \quad g'(p) = x(p)$$

$$g''(p) = x'(p), \quad \text{but:} \quad p = f'(x(p))$$

$$\frac{d}{dp}\,p = \frac{d}{dp}\,f'(x(p))$$

$$1 = f''(x)\,x'(p)$$

$$\text{so} \quad x'(p) = \frac{1}{f''(x(p))}$$

$$g''(p) = \frac{1}{\underbrace{f''(x(p))}_{>0}} > 0. \quad \text{(a)} \ \blacksquare$$

(b) so Legendre of $g(p)$ is obtained by considering

$$\varphi(x,p) = px - g(p)$$

& solving for $p = p(x)$ $\left(\frac{d}{dp}\varphi(x,p)\Big|_{x=\text{const}} = 0\right)$

from $x = g'(p) \Rightarrow p(x)$

So, $\varphi(x) = p(x)x - g(p(x))$

(L. transf of $g(p)$)

where $x = g'(p(x))$

but recall $g(p) = px - f(x)$
where $p = f'(x)$
is used to find $x(p)$

so $\varphi(x) = \left(px - (px - f(x))\Big|_{p=f'(x)}\right)$ $\boxed{x = g'(p)}$

i.e. $x = \frac{d}{dp}\left(p\,x(p) - f(x(p))\right)$

need to solve for $p(x)$ & plug back in $\varphi(x)$

$$\cancel{x} = \cancel{x} + px' - f'(x)x'$$

$f'(x) = p$ gives p as f-n of x

so then $\varphi(x) = f(x)$ ■

— " —

Similar def for many variables.

$$L \overset{?}{\longleftrightarrow} H$$

$$L(q,\dot q) \underset{?}{\longleftrightarrow} H(q,p)$$

Consider 1d example 1st:

$L(q, \dot{q}, t)$

don't assume system is "conservative", i.e. allow L can explicitly depend on time, so that E NOT conserved (generally)

then $H(p,q) \equiv$ Legendre transf of $L(q\dot{q}t)$ wrt $\dot{q}$, i.e. q & t "go along for the ride" $\equiv$ some extra parameters L & H depend on

in formulae, $p \quad p \; x \quad f(x)$

$$H(p,\underbrace{q,t}_{\text{go along}}) = p\dot{q} - L(q,\dot{q},t)$$

where $p = \dfrac{\partial L(q,\dot{q},t)}{\partial \dot{q}} \Rightarrow \dot{q} = \dot{q}(q,p,t)$

So $\longrightarrow$ we have a fn of (p,q,t), called H.

if p,q,t change on dp, dq, dt we have

$$dH(p,q,t) = \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial t}dt$$

but $H(p,q,t) = p\dot{q} - L(q,\dot{q},t)$ w/ $\dot{q} = \dot{q}(p,q,t)$ via $p = \dfrac{\partial L(q\dot{q}t)}{\partial \dot{q}}$

What's $d\left(p\dot{q} - L(q,\dot{q},t)\right)\Big|_{p=\frac{\partial L}{\partial \dot{q}}} = ?$

well – just recall $\dot q = \dot q(q, p, t)$

$$d\left(p\dot q - L(q,\dot q, t)\right) =$$

$$= \left(\dot q + p\frac{\partial \dot q}{\partial p} - \frac{\partial L}{\partial \dot q}\frac{\partial \dot q}{\partial p}\right)dp$$
$$+ \left(p\frac{\partial \dot q}{\partial q} - \frac{\partial L}{\partial \dot q}\frac{\partial \dot q}{\partial q} - \frac{\partial L}{\partial q}\right)dq$$
$$+ \left(p\frac{\partial \dot q}{\partial t} - \frac{\partial L}{\partial \dot q}\frac{\partial \dot q}{\partial t} - \frac{\partial L}{\partial t}\right)dt$$

I recalled that $\dot q(p,q,t)$ in each case

BUT $p = \frac{\partial L}{\partial \dot q}$

so many terms cancel. (as shown)

$$d(p\dot q - L) = \dot q\, dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt$$

while we had

$$dH = \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial t}dt$$

So it must be that

$$\frac{\partial H}{\partial p} = \dot q$$

$$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} \quad = -\dot p$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

but now if $q, \dot q$ obey E.-L., we have $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot q}$

$$= \frac{d}{dt}p$$

hence, if $(q, \dot{q})$ obey

E-L. w/ $L(q, \dot{q}, t)$, i.e. $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$,

then (p, q) obey

$$\left\{ \begin{aligned} \dot{q} &= \frac{\partial H(p,q,t)}{\partial p} \\ \dot{p} &= -\frac{\partial H(p,q,t)}{\partial q} \end{aligned} \right\} \equiv \text{"Hamilton E.O.M."}$$

$$\left(\text{\& } \frac{\partial H(p,q,t)}{\partial t} = -\frac{\partial L(q, \dot{q}(p,q,t), t)}{\partial t} \right.$$
if there's explicit t - dependence $\Big)$

i.e. E-L. equations $\equiv$ Hamilton's eqns

(2nd order) in t $\qquad\qquad$ (1st order in t)

$$\left\{ \ddot{q} = f(q, \dot{q}, t) \right\} \qquad \left\{ \begin{aligned} \dot{q} &= f(p,q,t) \\ \dot{p} &= \tilde{f}(p,q,t) \end{aligned} \right\}$$

Ex. $\quad L = \frac{m\dot{q}^2}{2} - U(q)$

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \Rightarrow \dot{q} = \frac{p}{m}$$

$$H = p\dot{q} - L(\dot{q}, q) = p\frac{p}{m} - \frac{m}{2}\left(\frac{p}{m}\right)^2 + U(q)$$

$$H(p, q) = \frac{1}{2}\frac{p^2}{m} + U(q)$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \& \quad \dot{p} = -\frac{\partial H}{\partial q} = -U'(q)$$

so since $p = m\dot{q}$ (by $\frac{\partial H}{\partial p} = \dot{q} = \frac{p}{m}$)

we have $\dot{p} = m\ddot{q}$
& $\dot{p} = -U'(q)$ $\Big\} \Rightarrow m\ddot{q} = -U'(q)$

$\equiv$ (Newton's eqs.)

(of course) ☒

For many variables = SAME STORY =

$L = L(q_i, \dot{q}_i, t)$

$H(p_i, q_i, t) = \left(\sum_i p_i \dot{q}_i\right) - L(q_i, \dot{q}_i, t)$

where $p_i = \dfrac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i}$

solve for $\dot{q}_i = \dot{q}_i(p, q, t)$

(generally, can depend on ALL (p, q, t)'s)

& E-L are equivalent to

$$\begin{cases} \dot{p}_i = -\dfrac{\partial H(p, q, t)}{\partial q_i} \\[2ex] \dot{q}_i = \dfrac{\partial H(p, q, t)}{\partial p_i} \end{cases}$$

$\forall i$

in Hamiltonian form —

- conserved quantities occur where $H(p,q,t)$ is independent of some q_i —

- then

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0 \Rightarrow \underbrace{p_i = \text{const.}}_{\Downarrow \text{ conserved}}$$

<u>N.B.</u> ⇒ no "conserved" q_i's. → would mean H was indep. of some p_i's — but goes against $g''(p) > 0$ property needed to have a meaningful Legendre transform!

- as far as energy conservation, it is simply the statement that

$$\frac{\partial H(p,q,t)}{\partial t} = 0.$$

since $H = \underbrace{\left(\sum_i p_i \dot{q}_i\right) - L}$

& this is the quantity conserved when $\frac{\partial L}{\partial t} = 0$, i.e. <u>ENERGY</u>.

By involution property of Legendre transform, we also have that given $H(p,q) \longrightarrow$ get $L(q,\dot{q})$

by Legendre transf. H:

$$L(q,\dot{q}) = \dot{q}p - H(p,q)$$

where $\dot{q} = \dfrac{\partial H(p,q)}{\partial p}$

solve for $p = p(q,\dot{q})$

& plug to get $L(q,\dot{q})$

$= \dot{q}p - H(p,q)$

i.e. $L(q,\dot{q}) = \dot{q}\,p(q,\dot{q}) - H(p(q,\dot{q}),q)$

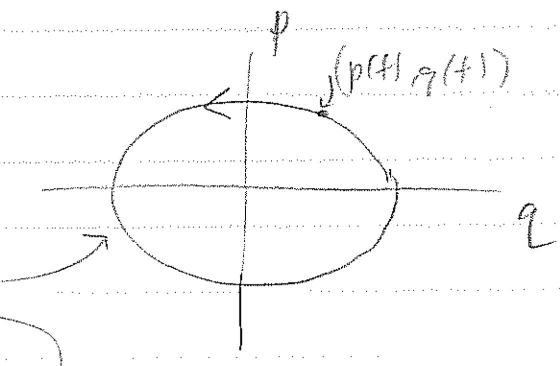
where $\dot{q} = \dfrac{\partial H(p,q)}{\partial p} \Rightarrow p(q,\dot{q})$

///

$\{p(t), q(t)\}$ — for N-particle in 3d — a $6N$ dim space called "phase space"

e.g.

$$H_{osc} = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$$



"phase trajectories"

$$\left.\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} \\ \dot{q} &= \frac{\partial H}{\partial p} \end{aligned}\right\}$$

given $p(0), q(0)$

$\underbrace{\text{find } p(t), q(t)}_{\text{unique solution}}$

$\underbrace{\text{maps} \begin{pmatrix}\text{points}\\ \text{phase}\\ \text{space}\end{pmatrix} \xrightarrow{\text{time}} \begin{pmatrix}\text{points in}\\ \text{phase}\\ \text{space}\end{pmatrix}}_{\text{"phase flow"}}$

"phase flow"

For most practical purposes

$$L = T(\dot{q}) - U(q) \quad (L = T - U)$$

&

$$H = T(p) + U(q) \quad (H = T + U)$$

Many applications and uses of $H(p,q,t)$ & phase flow exist (except of path ∫ — also needed H, though)

— we'll only notice that "normal" transition from classical to qu mechanics uses Hamiltonian, not Lagrangian

we always look @ $\underbrace{H(p,q)}_{\text{classical } H}$

then make $p, q \to \hat{p}, \hat{q} \to$ operators

& say $\underbrace{i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H}(\hat{p}, \hat{q}) |\psi\rangle}$

(Schr. equ $\equiv$ QM)

A structure which allows to easily "axiomatize" quantiz'n, i.e. transition from class. system ⇒ qu. system, appears if we consider how an arbitrary f-n of (p,q,t) changes along the Hamiltonian phase flow —

— e.g. $f(p,q,t)$, where p,q are sol'ns of Hamilton's eqns:

$$\frac{d}{dt} f(p,q,t) = \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t}$$

$$= -\frac{\partial f}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial f}{\partial t}$$

i.e

$$\frac{d}{dt} f(p,q,t) = \frac{\partial f}{\partial t} + \underbrace{\sum_i\left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}\right)}$$

if many p's, q's

$$\underbrace{[H,f]} = \sum_i\left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}\right)$$

Poisson bracket of $\underbrace{H(p,q,t) \;\&\; f(p,q,t)}_{\text{f'ns on phase space}}$

$$\frac{d}{dt}\left(f(p,q,t)\right) = \frac{\partial f}{\partial t} + [H,f]$$

so, using this language,

if $f(p,q)$ is an integral of motion

it's got $\frac{\partial f}{\partial t} = 0$

& $\frac{df}{dt} = 0$, i.e. must have $\underbrace{[H, f] = 0}$.

→ integrals of motion have zero Poisson brackets w/ H

Q.M.: conserved quantities commute w/ H ...!

so $\underbrace{[\ ,\ ]_{P.B.}}_{\text{yes!}} \longleftrightarrow$ commutator??

For arbitrary $f(p,q)$ & $g(p,q)$ define

$$[f, g] = \sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right)$$

$$[f, g] = -[g, f]$$

$$[f_1 + f_2, g] = [f_1, g] + [f_2, g]$$

in particular:

$$[q_i, q_j] = 0 \quad [p_i, p_j] = 0$$

$$[q_i, p_j] = -\delta_{ij}$$

"quantization" : P.B $\longleftrightarrow$ commutators.

$$[q_i, p_j] = -\delta_{ij}$$

$$\Updownarrow$$

$$[\hat{q}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$$

$$\left.\begin{array}{l} \hat{q}_i : q_i \times \\ \hat{p}_j : -i\hbar \dfrac{\partial}{\partial q_i} \times \end{array}\right\} \text{ when acting on } \psi(\{q\})$$

$$\left([M_x, M_y) = -M_z \text{ etc} \ldots \longleftrightarrow [\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z \text{ etc} \ldots \right)$$

We can also write H. eqns as

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \sum_j \left(\frac{\partial H}{\partial p_j} \overbrace{\frac{\partial q_i}{\partial p_j}}^{\delta_{ij}} - \frac{\partial H}{\partial q_j} \cancel{\frac{\partial q_i}{\partial p_j}}^{0} \right) \equiv [H, q_i]$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = \sum_j \left(\frac{\partial H}{\partial p_j} \underbrace{\cancel{\frac{\partial p_i}{\partial q_j}}}_{0} - \frac{\partial H}{\partial q_j} \underbrace{\frac{\partial p_i}{\partial p_j}}_{\delta_{ij}} \right) \equiv [H, p_i]$$

$$\left\{ \begin{array}{l} \dot{q}_i = [H, q_i] \\ \dot{p}_i = [H, p_i] \end{array} \right. \longleftrightarrow \text{(Heisenberg rep in Q.M.)}$$