at a formal level, we've solved the problem →

→ we can calculate $r(t)$ & then $\varphi(t)$

by integrating (numerically - since in general $U(r)$ we can't take the $\int$-ls) & then plotting.

But using E.-L. equ's, let's 1st do a qualitative analysis for general $U(r)$ & then go & integrate equs on p. 64-65 for specific $U(r)$.

Now: $L(r,\dot{r},\varphi,\dot{\varphi}) = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\varphi}^2}{2} - U(r)$

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} \Rightarrow 0 = \frac{d}{dt}(mr^2\dot{\varphi}) = 2mr\dot{r}\dot{\varphi} + mr^2\ddot{\varphi}$$

we know this equation is solved by

(M = conserved angular mom-m)

$$\dot{\varphi} = \frac{M}{mr^2} \Rightarrow \text{since } \ddot{\varphi} = -\frac{2M}{mr^3}\dot{r}$$

Check:

$$\& \; mr^2\ddot{\varphi} = -\frac{2M}{r}\dot{r}$$

$$+$$

$$\& \; 2mr\dot{r}\dot{\varphi} = \frac{2\cancel{m}r\dot{r}M}{\cancel{m}r^2}$$

$$0 = 0 \;. \quad \checkmark$$

So in

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}}, \text{ which gives } mr\dot{\varphi}^2 - U'(r) = m\ddot{r}$$

we simply substitute $\dot{\varphi} = M/mr^2$ to get an equ for $r(t)$ →

thus, we get

$$m\ddot{r} = -U'(r) + \frac{m r M^2}{m^2 r^4} =$$

$$= -U'(r) + \frac{M^2}{m r^3}$$

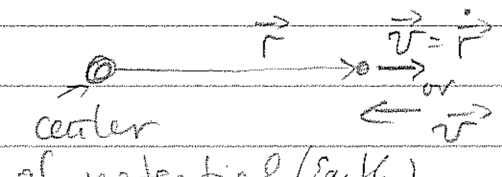
$$m\ddot{r} = -\frac{d}{dr}\left(U(r) + \frac{1}{2}\frac{M^2}{m r^2}\right) \leftarrow \left(\text{since } -\frac{d}{dr}\frac{1}{r^2} = \frac{2}{r^3}\right)$$

- but this is like 1d motion of a particle w/ coordinate r in a potential $U(r) + \frac{1}{2}\frac{M^2}{mr^2}$
- the only different thing is that $r > 0$ rather than $-\infty < r < \infty$

$$m\ddot{r} = -\frac{d}{dr}\Big(\underbrace{U(r)}_{\text{actual potential}} + \underbrace{\tfrac{1}{2}\frac{M^2}{mr^2}}_{\text{"repulsive centrifugal barrier"}}\Big)$$

"repulsive centrifugal barrier" } this one exists so long as $M \neq 0$

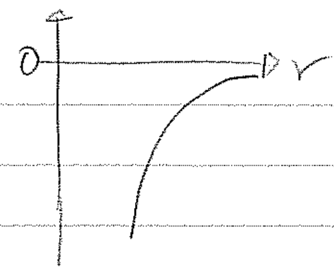
such initial conditions have $M=0$ }



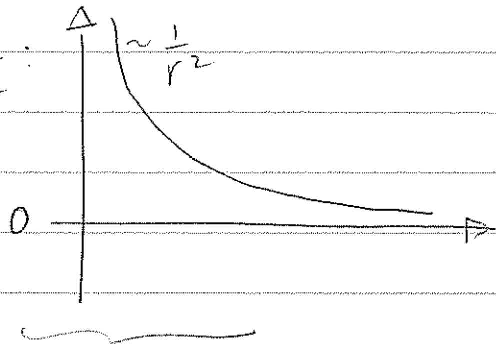
$\bar{r} \times \bar{v} = 0$ (must be collinear)

(or $\vec{v}$ must vanish $\Rightarrow$ dropping something to Earth)

Consider first attractive $U(r)$:

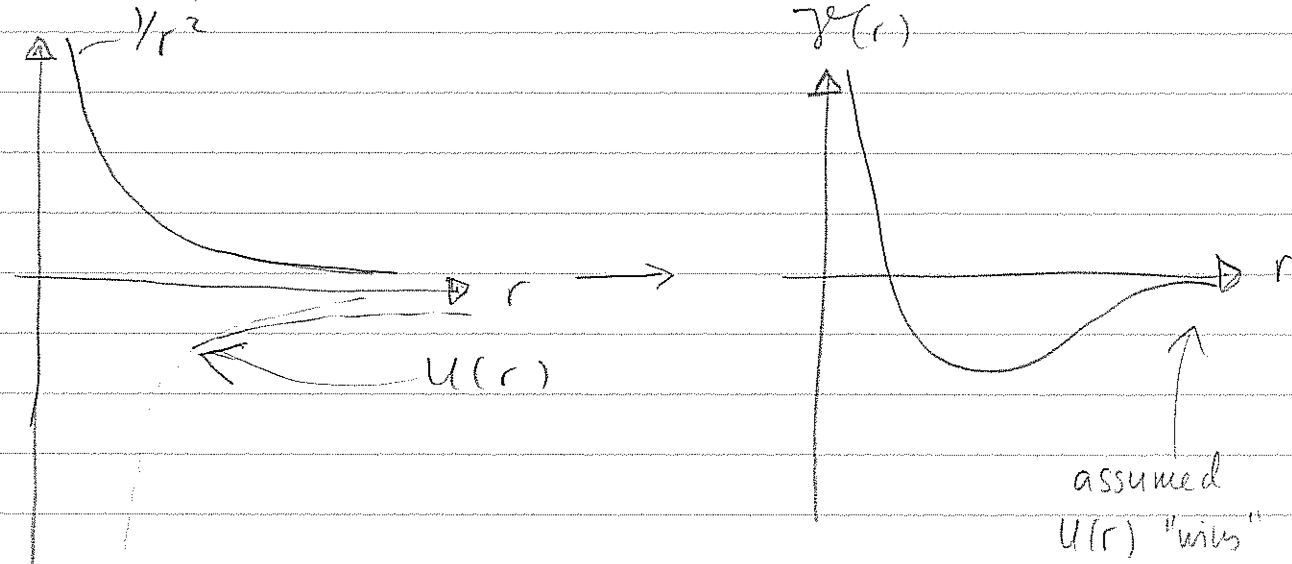


$\frac{M^2}{2mr^2}$:



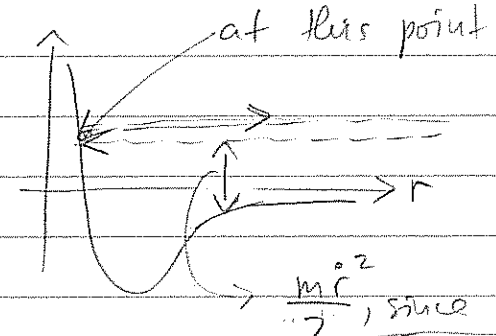
total effective $\mathcal{V}(r) = U(r) + \frac{M^2}{2mr^2}$ is the graphical Σ of the two.

- let 1st $U(r)$ (the attractive one) be such that as $r \to 0$ it is less singular than $1/r^2$ – in other words, centrifugal barrier always wins (for such cases) at small enough r:



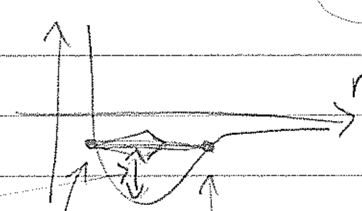
So there are a few possibilities, depending on the energy of the particle (which is conserved)

- $E > 0$



semi infinite motion: particle comes from ∞, bounces back & goes off to ∞

- $E < 0$



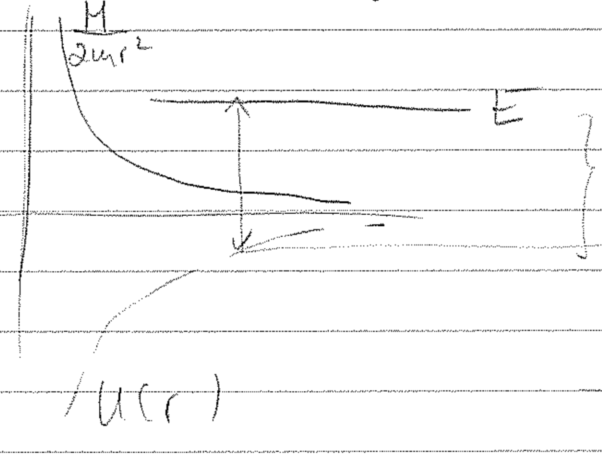
"bound" motion: particle bounces back & forth, r coordinate "oscillates"

$\frac{m\dot r^2}{2}$ "pericenter" (perihelium) "apocenter" (apohelium) (not a harmonic motion!)

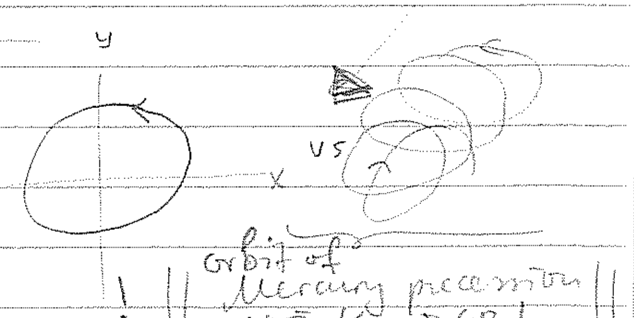
$\dot r = 0$ at "turning points" (NOT NECESSARILY CLOSED ORBIT!!)

must study $\dot\varphi$ → $\boxed{1/r \text{ \& } r^2 \text{ ONLY}}$

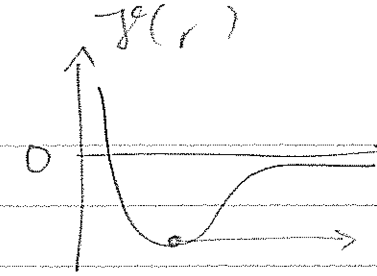
On the other hand, true kinetic energy, which also involves the "centrifugal energy" (i.e. energy in angular motion) is given by



true kinetic energy



! || orbit of Mercury precession — NOT 1/r → GR! ||



$\to E$ if E is negative & equal to the minimum of $\mathcal{V}(r) = \frac{M^2}{2mr^2} + U(r)$

then r is fixed, motion is on a circular orbit $r = \text{const}$

$$\underbrace{\dot{\varphi} = \frac{M}{mr^2}}_{\text{constant angular velocity}} = \text{const}$$

Value of r is min of $\mathcal{V}(r)$:

$$\mathcal{V}'(r) = 0 \Rightarrow -\frac{M^2}{mr^3} + U'(r) = 0$$

$$-\frac{M^2}{mr^3} = -U'(r) = f(r)$$

$$\underbrace{-mr\dot{\varphi}^2}_{\text{(balanced by inertial force)}} = \underbrace{f(r)}_{\text{(attractive force)}}$$

true for any M, changing M changes picture qualitatively only!

— this was for $U(r)$ which was $|U(r)| < \frac{1}{r^2}$ as $r \to 0$

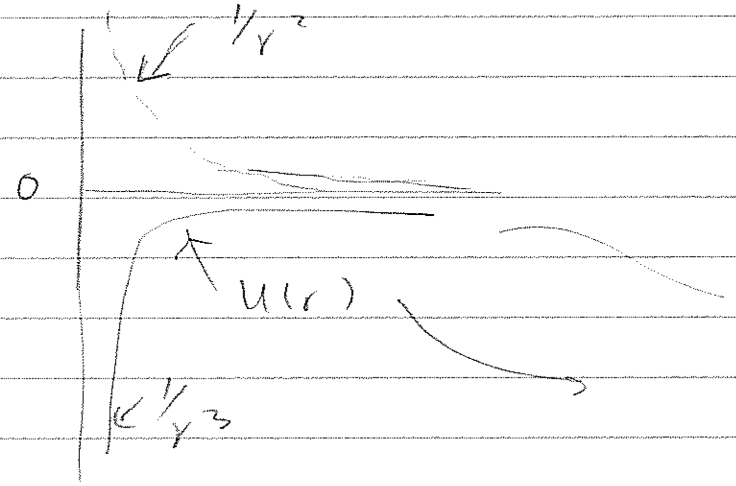
(less singular than $1/r^2$, so centrifugal barrier dominates as small r & falling off slower than $1/r^2$ @ ∞, so $U(r)$ dominates @ ∞.

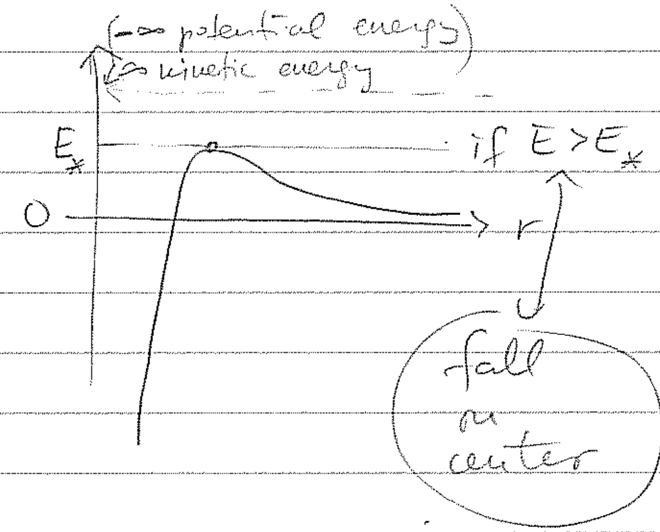
($\frac{1}{r}$ attractive obeys this)

If $\mathcal{U}$ = say = $\frac{1}{r^3}$ at small r

($p-n$ due to pion exchange in Nuclear physics
$\mathcal{U}(r) \sim \frac{1}{r^3} e^{-r/a}$
type)

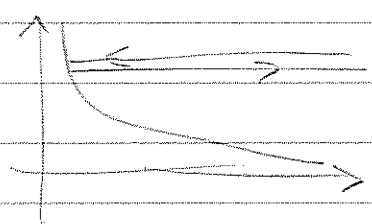
then picture is different:





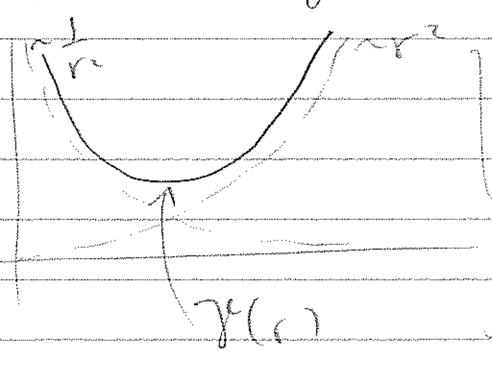
For a repulsive $V(r)$
((Coulomb, say)), qualitatively

$\mathcal{V}(r)$ looks like:



always ∞ motion

If $V(r) = \frac{1}{2} k r^2$ (a springlike potential $\equiv$ 3d harmonic oscillator!)



always finite motion — harmonic in each direction (x & y decouple, easier to solve in Cartesian!)

Before we leave @ Kepler's laws & virial theorem – one application of all this –

Galaxy $\xrightarrow{\quad r \quad} U(r) \sim \frac{G_N M}{r}$

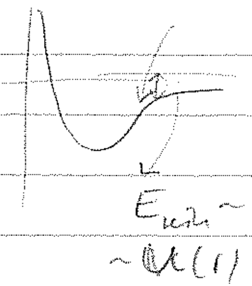
M(ass) "halo"

$\longleftrightarrow$ 10's kpc's

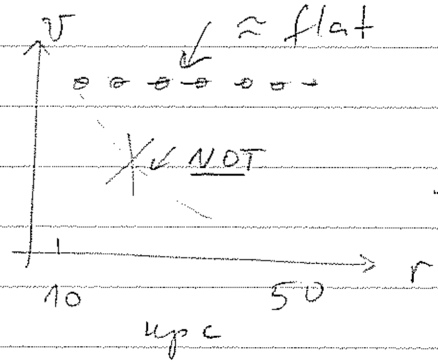
gas cloud -- expect

$v^2 \sim \frac{G_N M}{r}$

$v \sim \frac{1}{\sqrt{r}}$



but instead:



galaxy rotation curves

Suggests, if $v \approx$ const wrt r $\rightarrow$ then $M \sim r$

(mass within r)

$M \approx \int^r \rho(r) r^2 dr \Rightarrow \rho(r) \sim \frac{1}{r^2}$

historically 1st evidence for DM $\Longleftrightarrow$ "dark matter halo" (with density profile $\rho(r)$)

exact form of $\rho(r)$ subject of intensive current studies

(of which I know little $\longrightarrow$ MECHANICS VERY RELEVANT)

Mechanical similarity & virial theorem

* $L \to \text{const} \times L$ does not change E.O.M.

since $\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}$ same

$\underbrace{L \to \text{const} \times L}$ useful to deduce interesting things in case when potential is a homogeneous function.

homogeneous f-n: $f(x)$ - homogeneous of degree k

if $f(\alpha x) = \alpha^k f(x)$

$\left(f(x) = x^k \text{ - homogeneous of degree } k \right)$

$f(x_1, x_2, \ldots x_N)$ hom. of degree k if

$$f(\alpha x_1, \alpha x_2, \ldots, \alpha x_N) = \alpha^k f(x_1 \ldots x_N)$$

e.g.

$f(x_1 \ldots x_N) = x_1^{a_1} x_2^{a_2} \ldots x_N^{a_N}$ + similar terms w/ different a's

$(a_1 + a_2 + \ldots + a_N = k)$

(e.g. $x^3 + xyz + xy^2 + zx^2 \ \ldots$ etc)

If $f(\alpha x) = \alpha^k f(x)$, let's take $\frac{d}{d\alpha}$ of this relation —

— $\left(\frac{d}{d\alpha} \alpha^k = k \alpha^{k-1} \right) \Rightarrow$ $\frac{df(\alpha x)}{d(\alpha x)} \frac{d(\alpha x)}{d\alpha} = k \alpha^{k-1} f(x)$

$$\underbrace{\frac{df(\alpha x)}{d(\alpha x)}}_{''} x = k \alpha^{k-1} f(x) \longrightarrow$$

put $\alpha = 1 \Rightarrow \underbrace{x \frac{df}{dx} = k f}$

Euler's theorem on homog. fn.

if many variables:

$$f(\alpha x_1 \dots \alpha x_N) = \alpha^k f(x_1 \dots x_N)$$

same: $\left.\frac{d}{d\alpha}\right|_{\alpha=1}$ $\quad x_1 \frac{\partial f(x_1 \dots x_N)}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} f(x_1 \dots x_N) +$

$$\dots + x_N \frac{\partial}{\partial x_N} f(x_1 \dots x_N) = k f(x_1 \dots x_N)$$

$= U(\bar{x}_1 \dots \bar{x}_N)$

— " —

Assume that $L = \sum_a \frac{m_a \vec{v}_a^{\,2}}{2} - \sum_{a<b} \underbrace{U(\vec{x}_a - \vec{x}_b)}$

homogeneous fn

like $\frac{1}{|\bar{x}_a - \bar{x}_b|}$ or any other

we have

$$U(\alpha \vec{x}_1 \dots \alpha \vec{x}_N) = \alpha^k U(\vec{x}_1 \dots \vec{x}_N)$$

$k = -1$ (Coulomb, Newton, say)

rescale coordinates $\vec{x}_i$ by α $\quad$ } $\vec{v}_i$ rescaled by $\frac{\alpha}{\beta}$

rescale time t by β

kinetic energy rescaled by $\frac{\alpha^2}{\beta^2}$ & U – by α^k

→ but then, if we choose

$\frac{\alpha^2}{\beta^2} = \alpha^k$, the resulting rescaling

of $x \to \alpha x$, $t \to \beta t$ will be a rescaling of L by α^k

$\to \beta^2 = \alpha^{2-k} \to \beta = \alpha^{1-k/2}$

Hence for $U(\{\vec{x}\})$ homogeneous of degree k, we

have that under $\vec{x}_a \to \alpha \vec{x}_a$

$t \to \alpha^{1-k/2} t$, $\mathcal{L} \to \alpha^k \mathcal{L}$

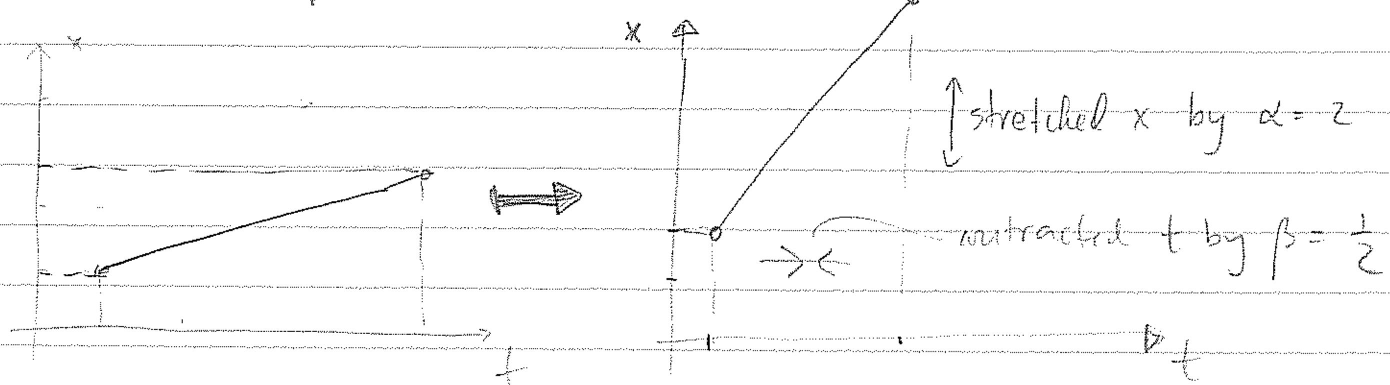
but this means — since EOM for $\mathcal{L}$ & $\alpha^k \mathcal{L}$ are

identical — that if $\vec{x}(t)$ is a solution of EOM,

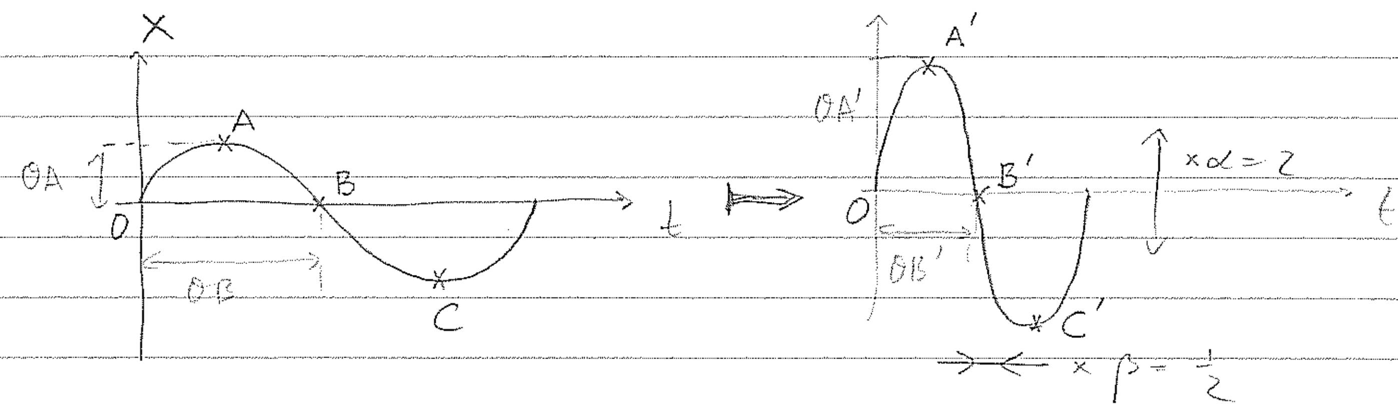
so $\underbrace{\vec{x}'(t) = \alpha \vec{x}(\beta t)}$, w/ $\beta = \alpha^{1-(k/2)}$

i.e. there is a whole family of geometrically similar trajectories

Ex: let $\alpha = 2$ $\beta = 1/2$ take a linear function 1st.



or take



this means that typical spatial size $\dfrac{OA'}{OA} = \alpha$

& typical times $\dfrac{OB'}{OB} = \beta = \alpha^{1-k/2}$

(as per $L \to L\alpha^k$)

$$\underbrace{\frac{OA'}{OA}}_{\alpha} = \underbrace{\left(\frac{OB'}{OB}\right)^{\frac{2}{2-k}}}_{\left(\alpha^{\frac{2-k}{2}}\right)^{\frac{2}{2-k}}}$$

& therefore

or, more generally $\dfrac{l'}{l} = \left(\dfrac{t'}{t}\right)^{\frac{-2}{k-2}}$

or, as in book: $\boxed{\dfrac{t'}{t} = \left(\dfrac{l'}{l}\right)^{1-\frac{k}{2}}}$

times of motion between corresponding points on two similar paths

ratio of linear dimensions of two paths

from this relation, we can look @ how

$\frac{v'}{v}$ scales $\rightarrow \left(\text{since } v \sim \frac{l}{t} \right) \rightarrow \left(\frac{v'}{v}\right) = \left(\frac{l'}{l}\right) \frac{1}{\left(\frac{t'}{t}\right)} =$

$$= \frac{\frac{l'}{l}}{\left(\frac{l'}{l}\right)^{1-\frac{k}{2}}} = \left(\frac{l'}{l}\right)^{k/2}$$

so

$$\boxed{\frac{v'}{v} = \left(\frac{l'}{l}\right)^{\frac{k}{2}}}$$

while energies' ratio is $\sim \frac{v'^2}{v^2} = \left(\frac{l'}{l}\right)^{k}$ $\left(E = \underbrace{\frac{mv^2}{2} + U}_{\text{same scaling}} \right)$

& angular momentum is $\sim \left(\frac{l'}{l}\right)\left(\frac{v'}{v}\right) =$ $(M = |\bar{r} \times \bar{p}|)$

$$= \left(\frac{l'}{l}\right)\left(\frac{l'}{l}\right)^{k/2} = \left(\frac{l'}{l}\right)^{1+\frac{k}{2}}$$

$$\frac{v'}{v} = \left(\frac{l'}{l}\right)^{k/2}$$

$$\frac{E'}{E} = \left(\frac{l'}{l}\right)^{k}$$

$$\frac{M'}{M} = \left(\frac{l'}{l}\right)^{1+k/2}$$

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-\frac{k}{2}}$$

$\rightarrow$ What does this imply?

(1) let $U(x) \sim x^2$ (harmonic oscillator, any dimension)

$$\Downarrow$$

$$k = 2$$

for two similar trajectories, we find that

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-\frac{2}{2}} = 1 \Rightarrow$$

$\Rightarrow$ period independent on amplitude

(time between corresponding points) (size of trajectory)

(2) let $\underbrace{U = -Fx}$ (homogeneous $\vec{E}$-field, or homogeneous gravity field)

$$k = 1$$

$$\frac{t'}{t} = \left(\frac{\ell'}{\ell}\right)^{1-\frac{1}{2}} = \sqrt{\frac{\ell'}{\ell}}$$

e.g. $\left(\text{ratio of two times to fall}\right) \sim \sqrt{\text{ratio of heights from which one falls}}$

(3) let $U \approx \frac{1}{r}$, $k = -1$

$$\frac{t'}{t} = \left(\frac{\ell'}{\ell}\right)^{1+\frac{1}{2}} = \left(\frac{\ell'}{\ell}\right)^{\frac{3}{2}}$$

$$\left(\frac{t'}{t}\right)^2 = \left(\frac{\ell'}{\ell}\right)^3$$

$\left(\text{ratio of periods of two orbits}\right)^2 \sim \left(\text{ratio of radius of orbit}\right)^3$

(Kepler's 3rd law)

(4) "Virial theorem"

kinetic energy T is a homogeneous f-n of $\vec{v}_a$ of order 2

$\left(T = \sum_a \frac{m_a \vec{v}_a^{\,2}}{2}\right.$, clearly so $\left.\right)$

so, by Euler's theorem on p. 74 we

have $\underbrace{\sum_a \vec{v}_a \cdot \frac{\partial T}{\partial \vec{v}_a}}_{\overset{\shortparallel}{\vec{p}_a}} = 2T$ (assuming $L = T - U$)

(indep. of $\vec{v}_a$)

we have $2T = \sum_a \vec{v}_a \cdot \vec{p}_a =$

$$= \sum_a \left(\frac{d}{dt} (\vec{p}_a \cdot \vec{r}_a) - \left(\frac{d}{dt} \vec{p}_a\right) \cdot \vec{r}_a \right)$$

$$= \frac{d}{dt}\left(\sum_a \vec{p}_a \cdot \vec{r}_a \right) - \sum_a \dot{\vec{p}}_a \cdot \vec{r}_a$$

"Lemma":

let's average w.r.t time, for any $f(t)$, $\bar{f} \equiv \lim_{\tau \to \infty} \int_0^\tau dt\, f(t) \Big/ \tau$

also if $f(t) = \frac{dF(t)}{dt}$, & $F(t)$ is bounded, we

have

$$\bar{f} = \lim_{\tau \to \infty} \int_0^\tau dt \frac{dF(t)}{dt} \Big/ \tau = \lim_{\tau \to \infty} \underbrace{\frac{F(\tau) - F(0)}{\tau}}_{\text{since } F(\tau) \text{ is bounded}} \to 0$$

Now, we assume that the system we're looking at is executing bounded motion (e.g. no particles $\to \infty$, ever $\Rightarrow$ e.g. only closed trajectories, no ∞ ones) then we look at

$$2T = \frac{d}{dt}\left(\sum_a \vec{p}_a \vec{r}_a\right) - \sum \dot{\vec{p}}_a \cdot \vec{r}_a$$

and let's average it wrt time (i.e. average both sides)

since finite motion of finite energy, both $\vec{p}_a$ & $\vec{r}_a$ are $< \infty$, $\forall a$, hence the $\frac{d}{dt}$ term averages to $\emptyset$.

so we have

$$\overline{2T} = -\overline{\sum_a \dot{\vec{p}}_a \vec{r}_a}, \quad \text{now } \dot{\vec{p}}_a = \frac{\partial U}{\partial \vec{r}_a}$$

$$\overline{2T} = \underbrace{\overline{\sum_a \frac{\partial U}{\partial \vec{r}_a} \cdot \vec{r}_a}}$$

This is always true — only used that $T \sim v^2$ —

if U is homogeneous of degree k, then by Euler theorem we know

$$\sum_a r_a \frac{\partial U}{\partial r_a} = k U$$

so we have

$$\underline{\overline{2T} = k \overline{U}}$$

this is called the "virial th-m"

it says that the time averages of T & U in a potential homogeneous of degree k are simply related; since $\overline{E} = E = \overline{T} + \overline{U}$ we can also say $E = \overline{T} + \overline{U} = \frac{k}{2}\overline{U} + \overline{U}$

$$\rightarrow \overline{U} = \frac{2}{k+2} E$$

$$\text{or} \quad E = \overline{T} + \frac{2}{k}\overline{T} \quad \text{so}$$

$$\overline{T} = \frac{k}{2+k} E$$

<u>Ex:</u> $k = 2$ (harmonic osc.) $\quad \overline{U} = \overline{T} = \frac{1}{2}E$

$k = -1$ (Newton, Coulomb) $\quad \overline{U} = 2E$
$\overline{T} = -E$ (total $E < 0$ for finite motion)

one application of this fact →

→ <u>equipartition in stat. mech</u> ⇒ we know $\overline{\frac{m\vec{v}^2}{2}} = \frac{3}{2} k T_{temper.}$

(statistical average ≡ ≡ time average in the equilibrium system)

$$\overline{T_x} = \overline{\frac{m v_x^2}{2}} = \frac{1}{2} k T_{emperature}$$

so if x is a vibrational direction

$$\overline{T_x} = \overline{U_x} = \overline{\frac{m\omega^2 x^2}{2}} = \frac{1}{2} k T_{emperature} \text{ as well}$$

Another ex. of scaling, not involving homogeneous potentials →

two particles, same U (same ext. force)
w/ different mass

$$\underbrace{m\ddot{x}}_{} = f$$

$$\left.\begin{aligned} m &= \alpha m \\ t &= \sqrt{\alpha}\, t \end{aligned}\right\} \text{ leaves l.h.s invt.}$$

so if $m_1 = \alpha\, m_2$

$$t_1 = \sqrt{\alpha}\, t_2$$

$$\left(\frac{t_2}{t_1}\right) = \frac{1}{\sqrt{\alpha}} = \left(\frac{m_2}{m_1}\right)^{1/2}$$

so if 1 is 4x heavier than 2
it will pass same orbit (in the same field)
2 x slower than 2

——— // ———

——— // ———

back to analysis of central-field motion.