Midterm Exam, PHY 450S, Relativistic Electrodynamics
Monday, March 7, 2011
One double-sided $8'' \times 11''$ aid sheet allowed. No calculators.

1. Alice and Bob move in opposite directions around a circular ring of radius R. The ring is at rest in an inertial frame. They move with the same constant speed V as measured in the inertial frame. Each carries a clock which they synchronize to zero time at the moment they are at the same position at the ring. Bob predicts that when they meet, Alice's clock will read less than his because of the time dilation arising because she has been moving with respect to him. Alice predicts that Bob's clock will read less, using the same reasoning. Clearly, they can't both be right. Explain what's wrong with their arguments. Calculate the actual reading of their clocks at the spacetime point of their meeting.

30 points

Hint: the use of spacetime diagrams is helpful, encouraged, and rewarded.

2. A pion at rest decays into a muon and a neutrino. The pion mass is m_π and the muon mass is m_μ. Assume that the neutrino is massless. Find the energy of the outgoing muon in terms of m_π and m_μ.

25 points

3. We know from class that one pair of the Maxwell equations is, in four-vector notation and using Einstein's summation convention:

$$\frac{\partial}{\partial x^i} F^{ik} = \frac{4\pi}{c} j^k \ . \tag{1}$$

1. Show that the "continuity equation" $\frac{\partial}{\partial x^p} j^p = 0$ is actually a consequence of (1).

10 points

2. Recal the physical significance of j^0 and $\vec{j}$ and explain the physical meaning of the relation obtained in 1.

10 points

4. Imagine that there was a Lorentz scalar field in nature, which was sourced by particles, similar to the way the electromagnetic field is. Assume that the superposition principle was obeyed by that field to good accuracy, requiring that the action be quadratic in the scalar field, just like for the electromagnetic one.

Then, Lorentz invariance and the superposition principle require that the action for the system composed of the scalar field (whose action is given by the first term in (2) below, where M is called the "mass," or the inverse range, of the scalar field) and a particle of mass m (whose familiar action is given by the second term in (2)), which interacts with the scalar field (last term in (2), the strength of the coupling of the charged particle to the scalar field is called λ) is:

$$S = \int d^4x \left(\frac{1}{2}\partial_i\phi\partial^i\phi - \frac{1}{2}M^2\phi^2 \right) - mc \int ds - \lambda \int ds\phi(x(\tau)) , \qquad (2)$$

where $\partial^i \equiv g^{ij}\partial_j$. The integrals in the last two terms in (2) are, of course, taken over the worldline of the particle, $x^i(\tau)$. This is all similar to the Maxwell-charged particle action, except for the replacement of the 4-vector with a 4-scalar field.

Note that for a scalar field there is no gauge invariance, hence a term quadratic in the field ($M^2\phi^2$) is allowed in the action, along with the kinetic term ($\partial_i\phi\partial^i\phi$). Also, the sign of the kinetic term is different from that of the electromagnetic field, and is determined by the usual requirement that the coefficient of $\dot{\phi}^2$ in the Lagrangian (i.e., the kinetic energy of the field ϕ) be positive. The sign of the $M^2\phi^2$ term is determined by requiring that the field be non-tachyonic; this will not be of essence now, though.

1. Requiring that the action S be extremal with respect to arbitrary variations of the particle worldline, show that the equation of motion for the particle in an external scalar field $\phi(x)$ is:

$$(mc + \lambda\phi)\frac{du^i}{ds} = \lambda(\partial^i\phi - u^iu^j\partial_j\phi).$$

Comment on the similarities and differences with the Lorentz force.

13 points

2. The other degree of freedom is $\phi(x)$. Impose the usual conditions that at spatial infinity $\phi \to 0$ and that at the initial and final time slices the variation $\delta\phi = 0$. Show, using the variational principle for $\phi(x)$, that the equation of motion for the scalar field is:

$$(\partial_i\partial^i + M^2)\phi(x) = -\lambda\rho(x),$$

where $\rho(x) = \int ds\delta^{(4)}(x - x(\tau))$.

12 points

The equation you obtained in 2. is known as the Klein-Gordon equation. If you consider a static particle located, for simplicity, at $\vec{x} = 0$, the scalar field it creates is given by the Yukawa potential: $\phi(r) = -\frac{\lambda}{4\pi}\frac{e^{-Mr}}{r}$, where $r = |\vec{x}|$, which explains why M was called an "inverse range" of the field. If you want some bonus points (and much respect), derive this result.

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Total number of points:  30+25+20+25 = 100.
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