Resurgence in QFT ?

summary :

+ some new math insight (1980's) into series "asymptotic" $\Longrightarrow$ "resurgent"

+ clearly relevant for finite-dim f.I.s ???

(±) evidence in QM, encoding of (all ?) nonpert phenomena in pert. series ... ?

(±), some evidence in QFT & hopes ... 2012 →

perturbation theory $f(g^2) = c_0 + c_1 g^2 + \ldots + c_n (g^2)^n + \ldots$

usually divergent (zero radius of convergence in g^2)

- exceptions: * lattice strong coupling exp.
  * small $(g^2 N)$ exp. (@ leading $\frac{1}{N}$)
  * OPE in CFT (not QCD!)
  * extended N≥2 4d susy

Dyson: QED, suppose $f(\alpha)$ had finite radius

then $f(-\alpha)$ - world where $e^+ e^+$ attract -

- but unstable no matter how small α

$\Rightarrow$ 0-radius

typical behavior of c_n @ $n \to \infty$ $\quad c_n \sim a^n n!$

(a - any sign - !)

Warning! : poses more issues than it solves ... (healthy!?)

so, we have

(1) $$f(g^2) = c_0 + c_1 g^2 + \cdots + \underbrace{a^n n! (g^2)^n}$$ (1)

- as $g^2 \to 0$ each consecutive term is smaller
- at fixed g^2 ($\ll 1$) there exists an order, n_*, s.t. terms w/ $n > n_*$ increase

use $n! \approx n^n e^{-n}$

$$|a|^n n! (g^2)^n \approx \left(\frac{|a|g^2}{e}\right)^n n^n = e^{n\left(\ln n + \log\left(\frac{|a|g^2}{e}\right)\right)}$$

$$\frac{d}{dn}\left(\cdots \right) = 0 \Rightarrow \quad \log n_* + 1 + \log\frac{|a|g^2}{e} = 0$$

$$\log n_* e = \log\frac{e}{|a|g^2}$$

truncating series at $n = n_*$ = "optimal truncation" $\longrightarrow$ $n_* = \dfrac{1}{|a|g^2}$:

$n = n_*$ term is of order $e^{n_*\left(\ln(n_* e) - \log e + \log\frac{|a|g^2}{e}\right)} = e^{-n_*}$

so accuracy is $\underline{\mathcal{O}}\left(e^{-\frac{1}{|a|g^2}}\right)$

<u>Textbook attitude on (1)</u> : (1) gives an asymptotic estimate of "true quantity" $f(g^2)$

(asymptotics in the sense of Poincare)

good for all practical purposes (?!)

But nonetheless, can ask : <u>is there a sense that one can recover true function from (1)?</u>

"Borel summation" is an attempt to associate an analytic f-n to the asymptotic expansion (1) [some mes. ex. of f]

definitions:

rewrite (1) as $\frac{f(t)}{c_0} = 1 + c_1 t + \ldots + c_n t^n$
(relabel coeffts)

($t \in$ Borel 'plane')

with $c_n \sim a_n^n n!$ in mind ("Gevrey-1")

define: $Bf(t) = 1 + \frac{c_1}{1!} t + \ldots + \frac{c_n}{n!} t^n + \ldots$ "Borel transform"

& its Laplace transform

"Borel sum"
$$F(g^2) = \frac{1}{g^2} \int e^{-t/g^2} Bf(t)\, dt$$
$(C: 0 \to \infty)$

important example: let $c_n = (-a)^n n! \Rightarrow Bf(t) = \frac{1}{1+at}$

assume for now $a>0 \Rightarrow f$ is sign alternating

then $F(g^2) = \frac{1}{g^2} \int_0^\infty e^{-t/g^2} \frac{dt}{1+at} = \int_0^\infty \frac{e^{-x}}{1+ag^2 x} dx$

for $a>0, g^2>0$ $F(g^2)$ exists (~some error f-n)

but numerically expanding denominator ⟶

we find

$$F(g^2) = \int_0^\infty \frac{e^{-x}}{1+ag^2x}\,dx \sim \sum_{n=0}^\infty (-ag^2)^n \underbrace{\int_0^\infty dx\, e^{-x} x^n}_{n!} \equiv f(g^2)$$

so, for now, $\int \mathcal{B}f(t)\, e^{-t/g^2}$ gives a 'nice' (?) way to generate asymptotic expansion $f(g^2)$ from an analytic f-n of g^2.

* if $\mathcal{B}f(t)$ has no singularities on $t \in (0,\infty)$ axis

$\equiv$ " $f(t)$ is Borel summable "

<u>Ex's</u> : • $V(x) = \frac{\omega^2}{2}x^2 + \lambda x^4$ (anharmonic oscillator)

is Borel summable for $\omega^2 \geq 0$

= for $\omega^2 > 0$ Borel sum $\longrightarrow$ exact answer

(numerical for $\omega^2 > 0$

= for $\omega^2 = 0$ Borel sum $\neq$ exact answer

(Voros, exactly solved homogeneous cases!)

(Mariño et al 1405...)

[these corresp. to C instantons ...]

exist @ $g^2 < 0$

include, continue..

Borel Σ in this case differs from exact solution by exponentially small terms

analytic

(add e^{-c/g^2} to $F(g^2)$ $\rightarrow$ ge f-n w/ same asymptotic expansion)

• $V(x) = -\frac{\omega^2}{2}x^2 + \lambda x^4$ "double-well" $\longrightarrow$

pert. gives a sign-non-alternating asymptotics →

→ let's dwell on this:

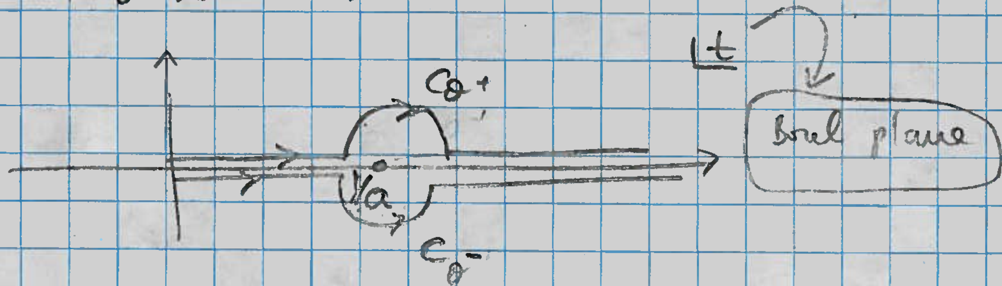
now we have

$$f(g^2) = \sum_{n=0}^{\infty} c_n (g^2)^n , \qquad c_n \sim a^n n! \quad , a>0$$

$$Bf(t) = 1 + \frac{c_1}{1!} t + \dots + a^n t^n + \dots$$

$$= \frac{1}{1-at} \qquad \text{a pole @ } t = a :$$

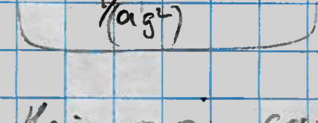
$$F(g^2) = \frac{1}{g^2} \int_{C:\,0\to\infty} e^{-t/g^2} \frac{1}{1-at} \, dt = \int_C \frac{e^{-x}}{1-ag^2 x} dx$$



we have now two

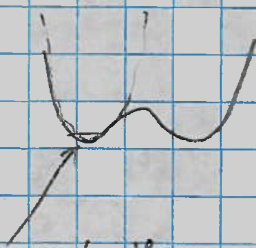
"lateral Borel sums", each of which has an imaginary part

$$F_+(g^2) - F_-(g^2) = \oint dx \frac{e^{-x}}{1-ag^2 x} = \frac{2\pi i}{ag^2} e^{-\frac{1}{ag^2}}$$



So $F_\pm(g^2)$ on their own can not represent physical

⇒ have an $\pm \frac{i\pi}{ag^2} e^{-1/ag^2}$ ambiguity quantity, which (must be real (energy.)



- pert. theory computes this, doesn't know about tunneling
- notice though, that imaginary ambiguity in the 'lateral' Borel sum has an exp. suppression factor which is exactly the $\overset{\text{pre due to}}{I - \bar{I}}$ tunneling; in this sense $F_{pert}(g^2)$ "knows" about instantons!!

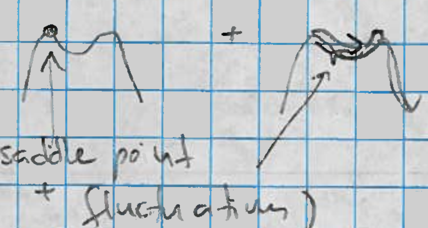
  ((not obvious; must study...))

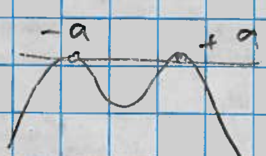
- Attempt to add nonperturbative correction to $f(g^2)$: (I-Ī action)

$$(2)\quad \left\{ f_{pert}(g^2) \Longrightarrow \sum_{n=0}^{\infty} c_n (g^2)^n + \ldots e^{-\frac{1}{ag^2}}\left(g^2 + \ldots + (g^2)^k c_k + \ldots\right) + \ldots e^{-\frac{2}{ag^2}}\left(g^2 + \ldots + (g^2)^p c_p + \ldots\right) + \ldots \right.$$

perturbation theory $+ (I\bar{I}) + (I\bar{I})^2 + \ldots$ - view as a semiclassical expansion of Euclidean path integral

$$\langle -a | e^{-HT} | (\pm) a \rangle = \sum_{\text{paths}} \quad + \quad + \ldots$$





This is, of course, known as the semiclassical expansion, which has traditionally been viewed as asymptotic.

The modern point of view is that they should be treated as exact representations of the quantity being calculated.

The formal expression (2) [all series are divergent!] is defined thru Borel resummation + analytic continuation in g^2 & converges to an unambiguous res. + equal to the exact quantity.

I think (fair to say) that, even in QM ([Alvarez, 2004; Dunne Unsal 2014] some indirect evidence!) this is still a conjecture. I'll show you a plausible way this may work. (It may, eventually, go over to QFT as well.)

to this end, begin in words & go back to our series (2)

(finite dim ∫s (functions) vie?)

$$f(g^2) \text{ "=" } \underbrace{1 + c_1 g^2 + \cdots + c_n (g^2)^n + \cdots}_{\text{Borel sum ambig.: } \pm i e^{-1/ag^2}} \quad \Leftarrow \text{perturbation series around trivial saddle}$$

(3)

$$+ \; e^{-\frac{1}{ag^2}} \underbrace{\cdots}_{\text{I-}\bar{\text{I}} \text{ ambiguity: } \mp i e^{-1/ag^2}} \left(\underbrace{g^2 + \cdots + \tilde{c}_k g^{2k} + \cdots}_{\text{Borel } \Sigma: \; \pm i e^{-2/ag^2}} \right) \quad \text{cancel}$$

$$+ \; e^{-\frac{2}{ag^2}} \underbrace{\cdots}_{(\text{I-}\bar{\text{I}})^2: \; \mp i e^{-1/ag^2}} \left(\underbrace{g^2 + \cdots + \tilde{\tilde{c}}_n g^{2n}}_{} + \cdots \right) \quad \text{cancel}$$

¡ cancellations of ambiguities ad infinitum !

* cancellation of Borel Σ ambiguities can only be established after considering complex g^2
* supposed to work for all orders in $(g^2)^k$ & $(e^{-1/(ag^2)})^k$ expansions
* the resulting sum of f (defined by Borel & Laplace w/ $\mathcal{C}$ contour) is called the <u>Borel - Ecalle sum</u>

  & is supposed to give the <u>exact</u> answer. (incl. strong coupling)

* series of the type (3) [including also $\ln \frac{1}{g^2}$'s] called "<u>resurgent transseries</u>" ← generalize (vs. old) asymptotics (Ecalle, 1980's)

[most 'reasonable' $f(z)$ have such representations]

To get a flavor $\Rightarrow$ a calculation (generalizable to QFT!)

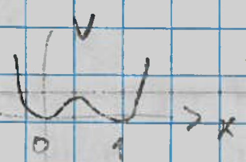
$$H = -\frac{g}{2}\left(\frac{d}{dx}\right)^2 + \frac{1}{g}V(x)$$

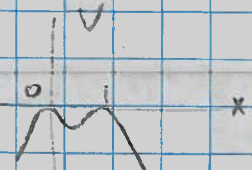
$$P\,\psi(x) = \psi(1-x)$$

$$L_{Eucl.} = \frac{1}{2g}\left[\dot{x}^2 + x^2(1-x)^2\right].$$

all known in QFT (non-localizable SUSY) involves this!!

$$V(x) = x^2(1-x)^2/2$$





Unsal, ED Schaefer...

including SYM — where independent check by SUSY !!

$$x_I(t) = \frac{1}{1+e^{-(t-t_0)}} \qquad t \to -\infty \quad x_I \to 0$$
$$\qquad\qquad\qquad\qquad\qquad\qquad +\infty \quad x_I \to 1$$

Bender Wu 1970's

Zinn-Justin 1980's

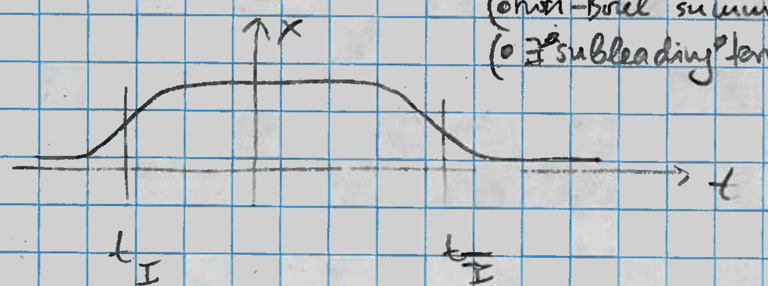
$$S_I = \frac{1}{6g}$$

$$E^{(0)}_{perturb.} \underset{k\to\infty}{\overset{k\text{-th order}}{\sim}} -\frac{3}{\pi}(3g)^k\, k!$$

(non-Borel summable)

(0 $\mathcal{I}$ subleading terms @ large k)

can be checked num. ~ 300 terms!



$$x_{I\bar{I}} = \frac{1}{1+e^{-(t-t_I)}} + \frac{1}{1+e^{t-t_{\bar{I}}}} \quad \text{exact as } |t_I - t_{\bar{I}}| \to \infty$$

$$S_{I\bar{I}} = \frac{1}{g}\Big(\underbrace{\frac{1}{3} - 2e^{-|t_I - t_{\bar{I}}|}}_{\text{attraction, sensible}} + \dots\Big)$$

exchange of Eucl. particle of unit mass ($V = x^2$)

holds at $|t_I - t_{\bar{I}}| \gg 1$

Consider thus the contribution of I & $\bar{I}$

to Z (appropriately defined - I will be sketchy ---

either $\quad Z = \mathrm{Tr}\, e^{-\beta H} = \sum_{N=0}^{\infty} e^{-\beta E_N}$

or $\quad \langle x=0 | e^{-TH} | x = 0 \text{ or } 1 \rangle$ ...)

- leading semiclassical contribution only -

  - action suppression
  - ✓ - zero-mode integrals
  - "quasi zero-mode" integrals

[ignore determinants of nonzero modes!]

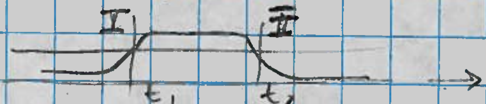
$$\left(\frac{1}{\sqrt{g\pi}}\, e^{-S_I} \int dt_0\right) \longleftarrow \text{ integration } \forall \text{ instanton}$$

t_0 - center (zero mode - collective coord.)

$S_I = \dfrac{1}{6g}$

$\dfrac{1}{\sqrt{g\pi}}$ - Jacobian [see Coleman]

so for $I \bar{I}$ we have



$$\underbrace{\left(\frac{1}{\sqrt{g\pi}}\, e^{-\frac{1}{6g}}\right)^2}_{\text{"fugacity"}} \int dt_1 \int dt_2 \; e^{-U_{int}(t_1 - t_2)}$$

"fugacity" ~ like a 1d classical grand canonical gas
(small ≡ dilute)

Usually ignore $U_{int} = -\dfrac{2}{g}\, e^{-|t_1 - t_2|}$ --- write as <u>virial exp.</u>:

$$\underbrace{\left(\frac{1}{\sqrt{g\pi}}\, e^{-\frac{1}{6g}}\right)^2 \int dt_1\, dt_2}_{} + \left(\frac{1}{\sqrt{g\pi}}\, e^{-\frac{1}{6g}}\right)^2 \int dt_1\, dt_2 \left(e^{-U_{int}} - 1\right)$$

$T^2 \left(\dfrac{1}{\sqrt{g\pi}}\, e^{-\frac{1}{6g}}\right)^2 \Rightarrow$ exponentiates & gives "dilute gas" E_{vac}
<u>(single - I contribution)</u>

focus on 2nd, usually ignored term.

$$\left(\frac{1}{\sqrt{g\pi}}\, e^{-\frac{1}{6g}}\right)^2 T \;\underbrace{\int_0^\infty dt \left(e^{-\left(-\frac{2}{g}e^{-t}\right)} - 1\right)}$$

← quasizero mode $\int$ -l

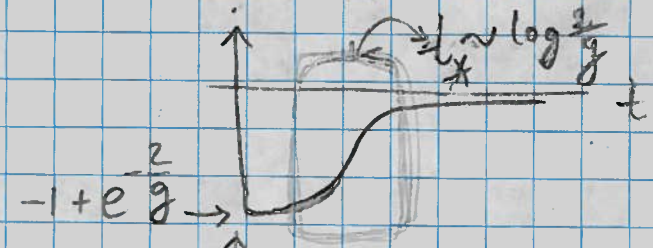
"C.M." anywhere

$$\int_0^\infty dt \left(e^{+\frac{2}{g}e^{-t}} - 1\right)$$

- attraction
- $t=0$ dominates $\int dt$ but there U_{int} makes no sense!

BZJ 1980-83 : turn into repulsion by analytic continuation : $g \to -g$

$$I(g) = \int_0^\infty dt \,\underbrace{\left(e^{-\frac{2}{g}e^{-t}} - 1\right)}$$



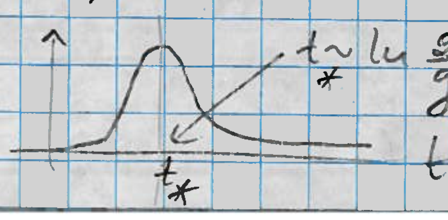
- dominated by -1 in regime where repulsion should forbid them to be (-1 was added to subtract noninteracting $I + I$)

- $\int$ by parts

$$I(g) = \int_0^\infty dt \left(e^{-\frac{2}{g}e^{-t}} - 1\right)$$

$$- \int_0^\infty dt \; t \frac{d}{dt}\left(e^{-\frac{2}{g}e^{-t}} - 1\right) \quad + \text{ ignore boundary (vanishes anyway)}$$

$$= - \underbrace{\int_0^\infty dt \; t \, \frac{2}{g} e^{-t} e^{-\frac{2}{g}e^{-t}}} = -\gamma_E + \log\frac{g}{2} + (\text{exp. small})$$



$t_* \sim \ln\frac{2}{g}$: size of molecule $\gg 1$

About $\int$: $I = -\frac{2}{g}\int_0^\infty dt\; t\, e^{-t}\, e^{-\frac{2}{g}e^{-t}} = (*)$

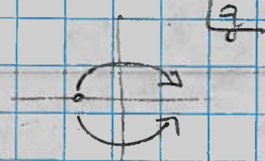
let $\zeta = e^{-t}$

$$(*) = \frac{2}{g}\int_0^1 d\zeta\, \ln\zeta\; e^{-\frac{2}{g}\zeta} = \log\frac{g}{2} - \gamma_E + Ei\left(-\frac{2}{g}\right)$$

[[* other ways to define by adding n_f ψ's, $n_f \to 0$]]

exp. small ignored

what we want is $I(-g)$:



$$I(-g) = -\gamma_E + \log\frac{g}{2} \pm i\pi$$

this is > 0

$I\bar{I}$ contribution generates ambiguous Im part.

so we have "$I\bar{I}$ molecule" $\sim T\,\frac{e^{-\frac{1}{3g}}}{g\pi}\left(\pm i\pi + \log g/2\right)$.

means contributes to E after exp.

$$E^0_{I\bar{I}} = \left(\pm i\pi + \log\frac{g}{2}\right)\frac{e^{-\frac{1}{3g}}}{g\pi}$$

are you surprised?

$E^0_{pert} = -\frac{3}{\pi}\sum_k \underbrace{(3g)^k}_{a} k! \Rightarrow$ (leading!)

$\Delta E^0[\text{Borel sum}] = -\frac{3}{\pi}\left(\pm\frac{i\pi}{3g}e^{-\frac{1}{3g}}\right)$.

from p. 9.

["uniform WKB" + subleading terms !!]

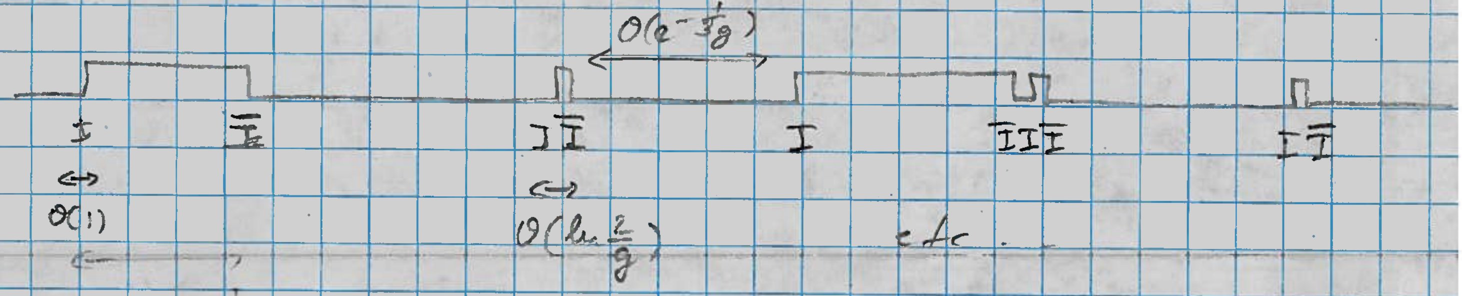
$\Delta E_{pert}[\text{Borel}] = \mp\frac{1}{g}e^{-\frac{1}{3g}}$.

This is largely early '80s et ff.

Let's recall the picture.

"vacuum" = Σ I's, Ī's, [IĪ]'s, [IĪI], etc...

typical configuration in path ∫:



$O(e^{-\frac{1}{5g}})$ – scale hierarchy – all kinds of "molecules" obtained from "bound states" of widely separated I's Ī's on p. (8).

$$f(g^2) = 1 + c_1 g^2 + \dots + c_n g^{2n} + \dots$$

$$+\, e^{-\frac{1}{ag^2}} (\ln g, g^2) \text{ expansion} + \dots$$

$$+\, e^{-\frac{2}{ag^2}} (\text{etc} \dots), \text{ e.g.}$$

Z-J conjecture:

$$E^{(0)}(g^2) = E^{(0)}_{pert}(g^2) + \sum_{\pm} \sum_{k=1}^{\infty} \sum_{l=1}^{k-1} \sum_{p=0}^{\infty} \left(e^{-\frac{c}{g^2}}\right)^k \frac{1}{g} \left(\ln \pm \frac{1}{g^2}\right)^l c^{\pm}_{k,l,p}\, g^{2p}.$$

(0-mode ∫.)

"resurgent series"

(k-instanton)

(possible molecules of k instantons)

(pert. series around I/molecules)

How to find? → • conjecture (1st): exact WKB"

• "uniform WKB"

Notice: • we checked $k=1,2$ leading-order cancellations

• Z-J expression

+ has been derived by "uniform WKB"

Alvarez 2004

(Dunne Unsal 2014)

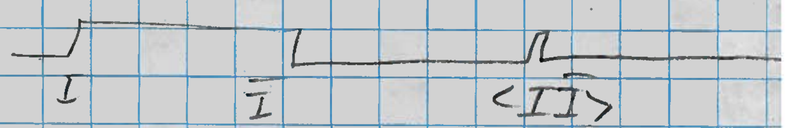
((+ pert. theory ≡ everything ??)) incl. use of analyt. cont, incl!)

+ checked to O(3-inst) numerically

+ details (beyond BZJ) not understood from path $\int$ perspective!

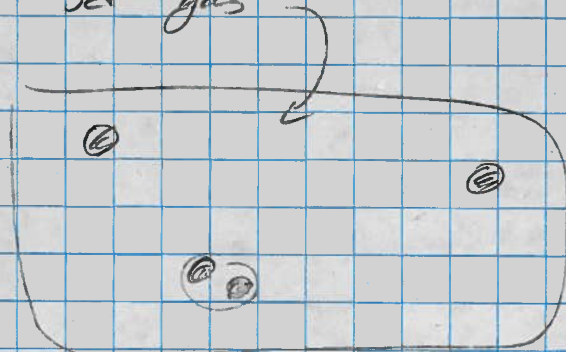
QFT ??

QM: 1d "gas"



etc.

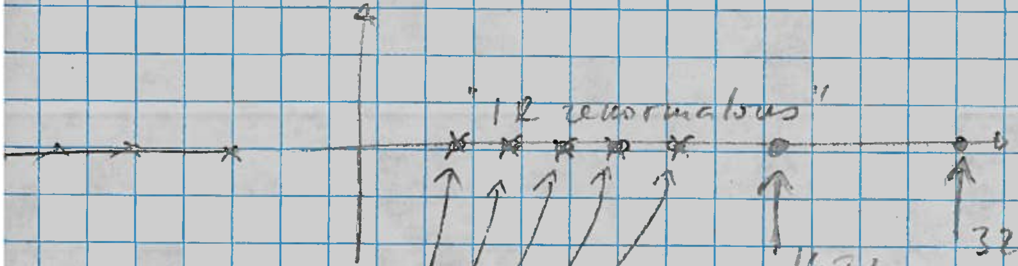
QFT (in best case) 3d "gas"



→ which? where?

35 yrs ago, 't Hooft @ Erice

("Can we make sense out of QCD?")



(for the Adler f-n)

$16\pi^2$ $\quad$ $32\pi^2$

$\left(S_{I\bar{I}} = \frac{8\pi^2}{g^2} \times 2 \right)$

$\frac{16\pi^2}{\beta_0} k$

$\left(\beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f \right)$

conjectured "IR renormalons related to confinement" ...

dream $\approx$ realized in theories on $\mathbb{R}^3 \times S^1$

* as opposed to $\mathbb{R}^4$ semiclassics OK (no IR problem!)

* BPST $\longrightarrow$ $\underbrace{M_1 \times M_2 \times \cdots \times M_{N_c}}_{\text{constituents}}$

(1) corresponding poles $\longrightarrow$ action $\frac{8\pi^2}{N_c g^2}$
$N_c \times$ closer to origin than BPST

(2) indeed related to confinement!

(3) various "Molecules" $\exists$

"magnetic" (Unsal 2007) (confinement)

"neutral" (Unsal et al 2011, ...) (center stab'n, deconf transition)

incl. 1m parts !: "semiclassical renormalons"? (Argyres Unsal 2012)

(4) take $R^1 \times S^1$ instead, where → QM

→ can verify cancel'n of ambiguities ("define QFT"???) (Dunne Unsal 2012)

dream: poles on $R^3 \times S^1$ ⟶ IR renormalons in R^4 (?)

(no phase transition suspected in some cases)

Current work: ◆ finite-d flds (eg Cherman, Koroteev, Unsal)

→ m-x models / strings / large coupling

→ SUSY theories w/ localization ⇒ (eg ABJM, $\frac{1}{N}$ exp'n)

◆ formul'n of BZJ in path $\int$?

me ??! ⟶ * Lefschetz thimbles?
* not true saddles excep @ ∞?
* SUSY ?

to end optimistic ⇒ we are learning unappreciated properties of QFT, I think!