Lessons from supersymmetric lattice quantum mechanics

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Joel Giedt, E.P., hep-th/0407135 [*JHEP09(2004)029*]
Joel Giedt, Roman Koniuk, Tsahi Yavin, E.P., hep-lat/0410041 [*JHEP12(2004)033*]
Joel Giedt, E.P., work in progress

related recent work:

Catterall (2001-)
(Cohen), Kaplan, Katz, Unsal (2002-);
Sugino (2004-)

(some) related older work:

Dondi, Nicolai (1977)
Elitzur, Rabinovici, Schwimmer (1982/3)
Sakai, Sakamoto (1983)

supersymmetric theories play a role in:

particle theory models

supersymmetric extensions of the standard model (MSSM, GUTs)

typically weakly coupled at TeV, but not always

hidden sectors used for dynamically breaking supersymmetry

models for studying strong-coupling phenomena

confinement, chiral symmetry breaking, and duality...

2d and 3d supersymmetric critical systems (tricritical Ising model)...

string theory

including fermions requires world-sheet supersymmetry, even if space time theory nonsupersymmetric - 2d supersymmetry important in many of these examples the supersymmetric theories are at strong coupling

generally, in supersymmetry,

nonperturbative effects are well understood via holomorphy and symmetries,
but not all desired aspects are under theoretical control

examples:

control over D-terms important for finding low-energy spectrum of strongly coupled supersymmetric theories, with or without dynamical supersymmetry breaking

numerous conjectures are based on symmetries and nonrenormalization,
but few explicit checks? proofs?

to address these issues, one requires a tool to study strong coupling:

- a nonperturbative definition...

- ways to extract nonperturbative information...

the lattice is the only* known nonperturbative definition of a general field theory and it would be of interest, and perphaps even useful, to have one for supersymmetric theories

*apart from
constructive field theory...
...?... string theory

Outline:

1. Problems and approaches

2. The essence of the recent ideas in a nutshell:

supersymmetric quantum mechanics on a "naive"
vs. "supersymmetric" lattice

3. General criteria and lessons:

what higher-dimensional theories do we expect to be able to study similarly?

with how much effort?

4. Outlook

main problem:

supersymmetry is a space-time symmetry
a lattice generally breaks space-time symmetries

restoration of Euclidean rotation symmetry in the continuum does not guarantee supersymmetry restoration,

e.g., in supersymmetric theories with scalars, no symmetry, other than supersymmetry itself, can forbid relevant supersymmetry breaking operators (``soft" scalar masses)

only 4d theory without scalars: pure SYM, where chiral symmetry of overlap fermions forbids the **single** relevant operator (gaugino mass)

Kaplan, Schmaltz/
Kogut, Vranas, 1990s

more specifically - important if we're to make progress:

variation of a supersymmetric action under supersymmetry is a total derivative iff the Leibnitz rule for spacetime derivative holds (in general, in interacting theories only)

however, **there exist no (ultra) local lattice derivatives that obey the Leibnitz rule!**

$$\Delta_- A_i = \frac{A_i - A_{i-1}}{a} \quad \rightarrow \quad \Delta_-(AB)_i = \Delta_- A_i \, B_i + A_i \, \Delta_- B_i + a \, \Delta_- A_i \, \Delta_- B_i$$

[Bartels and Bronzan, 1983, susy algebra with infinite range ``derivatives'' on an infinite lattice...]

Are we stuck, then? ...yes and no...

1. Problems and approaches to lattice supersymmetry

$$S = \int dx d\theta ... d\theta' \; F(\Phi)$$

generic SUSY action:
integral over superspace of a function of superfields

$$Q = \frac{\partial}{\partial\theta} + \theta\Gamma\frac{\partial}{\partial x}$$

SUSY generators:
differential operators acting on superfields

$$\delta_\epsilon \Phi = \epsilon Q \Phi$$

SUSY variation of the action:

$$\begin{aligned} \delta_\epsilon S &= \int dx d\theta ... d\theta' \left[F(\Phi + \epsilon Q\Phi) - F(\Phi)\right] \\ &= \int dx d\theta ... d\theta' \; \epsilon Q F(\Phi) \\ &= \int dx d\theta ... d\theta' \; \epsilon \left(\frac{\partial}{\partial\theta} + \theta\Gamma\frac{\partial}{\partial x}\right) F(\Phi) \; = 0 \end{aligned}$$

use the Leibnitz
rule for spacetime
derivatives

Dondi, Nicolai, 1977

$$\delta_\epsilon S \;=\; \int dx d\theta ... d\theta' \, \epsilon \left(\frac{\partial}{\partial \theta} + \cancel{\theta \Gamma \frac{\partial}{\partial x}} \right) F(\Phi) \;= 0$$

Moral: if the supersymmetry generator was simply $Q = \dfrac{\partial}{\partial \theta}$

the Leibnitz rule would not be needed for supersymmetry of the *interaction lagrangian* (it is really easy to have a *free* supersymmetric lattice theory!)

Hence, we could simply replace continuum coordinates by a set of discrete points, without destroying the nilpotent supersymmetry of the action.

This realization is not exactly new - early 1980s, but never pushed much, until recently.

What's new?

The essence of the modern developments in lattice supersymmetry is the

progress in our ability to write "supersymmetric" lattice actions

in the sense that a set of nilpotent anticommuting supercharges are exact at finite lattice spacing (not the entire algebra, however!)

-the ability to write "supersymmetric" lattice actions for many theories using a variety of new techniques

-the understanding of how the full supersymmetry algebra is restored in the continuum limit (*rather than the "one supersymmetry is better than none" attitude...*)

In this talk, I will illustrate these points.

will consider the example of supersymmetric quantum mechanics on an imaginary time lattice - details are the same as in all theories where the supersymmetry restoration is understood in detail

...one moral will be that a similar - no matter how tedious - analysis has to be performed in each case with "supersymmetric" lattice actions, preferably *before* starting simulations...

...will also formulate general criteria to tell when an exact "lattice supersymmetry" is possible...

...will briefly go over results for higher-dimensional theories...

“Euclidean” action - “1d QFT” of one boson and fermion (both periodic):

$$S = \int_0^\beta dt \left[\frac{1}{2}(\dot{x}^2 + h'^2(x)) + \bar{\psi}(\partial_t + h''(x))\psi \right]$$

$$\delta x = \epsilon_1 \psi + \epsilon_2 \bar{\psi} \ , \qquad \delta \bar{\psi} = -\epsilon_1(\dot{x} + h') \ , \qquad \delta \psi = -\epsilon_2(\dot{x} - h')$$

supersymmetry generated by two anticommuting parameters

ψ and $\bar{\psi}$ are independent fields in “Euclidean” space (as in any dimension)

$$h = \frac{1}{2} m x^2 + \tilde{h} \ , \qquad \tilde{h} = \sum_{n>2} \frac{g_n}{n} x^n$$

here,“superpotential” fixes all interactions - “1d Wess-Zumino model” useful to write supersymmetry transforms as:

$$Q_1 x = \psi \ , \qquad Q_1 \bar{\psi} = -(\dot{x} + h') \ , \qquad Q_1 \psi = 0 \ ,$$
$$Q_2 x = \bar{\psi} \ , \qquad Q_2 \bar{\psi} = 0 \ , \qquad Q_2 \psi = -(\dot{x} - h') \ , \quad Q_A S = \ 0 \ , \quad A = 1, 2$$

"NAIVE" *lattice action* = discretized continuum action, avoid doublers by Wilson term:

$$a^{-1} S = \frac{1}{2} \Delta^- x_i \Delta^- x_i + \frac{1}{2} h'_i h'_i + \bar{\psi}_i (\Delta^W(r)_{ij} + h''_i \delta_{ij}) \psi_j$$

$$\Delta_W(r) = \Delta^S - \frac{ra}{2} \Delta^- \Delta^+ \ , \quad \Delta^S A_i = \frac{A_{i+1} - A_{i-1}}{a}$$

can also discretize supersymmetry transforms [both procedures not unique!]:

$$Q_1 x_i = \psi_i, \qquad Q_1 \bar{\psi}_i = -(\Delta^+ x_i + h'_i), \qquad Q_1 \psi_i = 0$$
$$Q_2 x_i = \bar{\psi}_i, \qquad Q_2 \bar{\psi}_i = 0, \qquad Q_2 \psi_i = -(\Delta^+ x_i - h'_i)$$

Now, of course, these “lattice” Qs are not symmetries (or any Qs, as per general argument) the Q-variation of the lattice action is nonzero:

$$Q_A S = a\mathcal{Y}_A \equiv a\sum_i aY_{A,i} \to \int_0^\beta dt\, aY_A(t) \to 0\ , \quad A = 1,2$$

but vanishes in the classical continuum limit

What happens quantum mechanically?

one finds that $\langle Q_A S\rangle = a\langle \mathcal{Y}_A\rangle \sim a\cdot a^{-1} \to \text{finite}$

(despite the fact that this 1d theory has no UV divergences)

The violation of continuum supersymmetry Ward identity found numerically by Catterall in 2001 and attributed to *“large nonperturbative supersymmetry-breaking renormalization”*...

...will address shortly, but before that, how about finding a “supersymmetric” lattice action?

2. The essence... in a nutshell: "naive" vs."supersymmetric" lattice

two **nilpotent** real supercharges given as operators in superspace:

$$\{Q_1, Q_2\} = 2i\frac{\partial}{\partial t}$$

$$Q_1 = \frac{\partial}{\partial\theta^1} + i\,\theta^2\,\frac{\partial}{\partial t} = e^{-i\theta^1\theta^2\frac{\partial}{\partial t}}\,\frac{\partial}{\partial\theta^1}\,e^{i\theta^1\theta^2\frac{\partial}{\partial t}}$$

$$Q_2 = \frac{\partial}{\partial\theta^2} + i\,\theta^1\,\frac{\partial}{\partial t} = e^{i\theta^1\theta^2\frac{\partial}{\partial t}}\,\frac{\partial}{\partial\theta^2}\,e^{-i\theta^1\theta^2\frac{\partial}{\partial t}}$$

acting on real superfields: $\Phi(t,\theta^1,\theta^2) = x(t) + \theta^1\psi(t) + \theta^2\chi(t) + \theta^1\theta^2 F(t)$

[action is of the form discussed above, $S = \int dx d\theta ... d\theta'\ F(\Phi)$]

goal is to discretize time in a manner that preserves one of the two Q's, for example, Q1. Generally, a set of nilpotent anticommuting supercharges can always be simulateneously conjugated to pure θ-derivatives:

$$\frac{\partial}{\partial\theta^1} = e^{i\theta^1\theta^2\frac{\partial}{\partial t}}\,Q_1\,e^{-i\theta^1\theta^2\frac{\partial}{\partial t}}$$

in the conjugated basis, $Q = \frac{\partial}{\partial\theta}$, while the real superfield is:

2. The essence... in a nutshell: "naive" vs. "supersymmetric" lattice

$$\begin{aligned}\Phi'(t,\theta^1,\theta^2) &= e^{i\theta^1\theta^2\frac{\partial}{\partial t}}\,\Phi(t,\theta^1,\theta^2)\,e^{-i\theta^1\theta^2\frac{\partial}{\partial t}} = \Phi(t+i\theta^1\theta^2,\theta^1,\theta^2)\\ &= (x+\theta^1\psi)+\theta^2(\chi-\theta^1(i\dot{x}+F))\;.\end{aligned}$$

$$U(t) = x(t)+\theta\psi(t) \qquad\qquad \xi(t) = \chi(t)-\theta\left(F(t)+\dot{x}(t)\right)$$

and thus has two irreducible components w.r.t. Q [Euclidean, beginning from line above].
Q acts as a shift of theta, as a purely "internal" supersymmetry, so replacing continuum time with a lattice does not affect the action of Q.

$t^i,\ i=1,\ldots N$

$$\xi^i = \chi^i-\theta\left(F^i+\frac{x^i-x^{i-1}}{a}\right)$$

$$U^i = x^i+\theta\psi^i$$

$$\xi^i = \chi^i-\theta f^i$$

denoted $x(t^i)=x^i$,etc.

can now use these lattice superfields to write supersymmetric actions:

- bosonic
- Q-invariant
- local
- lattice translation invariant

then, the most general bilinear action consistent with above (+some discrete symmetry) is:

$$S = -\sum_i a \int d\theta \left(\frac{1}{2}\, \xi^i\, \frac{\partial}{\partial\theta}\xi^i + \xi^i\, \frac{1}{a}\Delta U^i + m\, \xi^i\, U^i \right)$$

$$\Delta U^i \equiv U^i - U^{i-1}$$

to include superpotential interactions $mU^i \rightarrow h'(U^i)$

- resulting lattice action has one exact nilpotent supersymmetry on the lattice
- useful also in more 1d examples on Riemannian manifolds....
- works almost like this for 2d WZ (LG) and other (2,2) 2d models...

[Giedt, E.P., hep-th/0407135]

Now, back to the restoration of supersymmetry in the quantum continuum limit...

2. The essence... in a nutshell: “naive” vs.“supersymmetric” lattice

$$a^{-1}S = \frac{1}{2}\Delta^- x_i \Delta^- x_i + \frac{1}{2}h'_i h'_i + \bar{\psi}_i(\Delta^W(r)_{ij} + h''_i \delta_{ij})\psi_j$$ “NAIVE”

$$a^{-1}S_{ca} = \frac{1}{2}\ (\Delta^W x_i + h'_i)^2 + \ \bar{\psi}_i(\Delta^W_{ij} + h''_i \delta_{ij})\psi_j$$ “SUPERSYMMETRIC”

- both actions have the same classical continuum limit

- continuum, as well as $a \to 0$ lattice, theories are finite, so all loop graphs are finite

- only two diagrams with nonnegative, D=0, lattice degree of divergence:

(all remaining diagrams have lattice-D<0 and thus by ”Reisz’s theorem,” 1988, approach their continuum values)

2. The essence... in a nutshell: "naive" vs. "supersymmetric" lattice

For "NAIVE" action fermion loop contribution on the lattice is twice that of the continuum: doubler do not decouple from D=0 graphs

- doubler contribution needs to be subtracted off via a finite counterterm
- with the counterterm added, the perturbative series on the lattice agrees with the continuum perturbation theory in the $a \to 0$ limit

a nonperturbative proof that the finite counterterm suffices to obtain the quantum continuum limit is given (our paper) via the transfer matrix

moreover, transfer matrix also suggests ways to improve the naive lattice action to $\mathcal{O}(a)$

For "SUPERSYMMETRIC" action D=0 parts of the graphs cancel between the boson and the fermion [i.e., ``lattice superfgraph" has D<0]

so **no need for counterterms**; a nonperturbative proof that the quantum continuum limit is as desired (our paper) via the transfer matrix

instead of presenting formalism, look at "experiment"...

2. The essence... in a nutshell: "naive" vs."supersymmetric" lattice

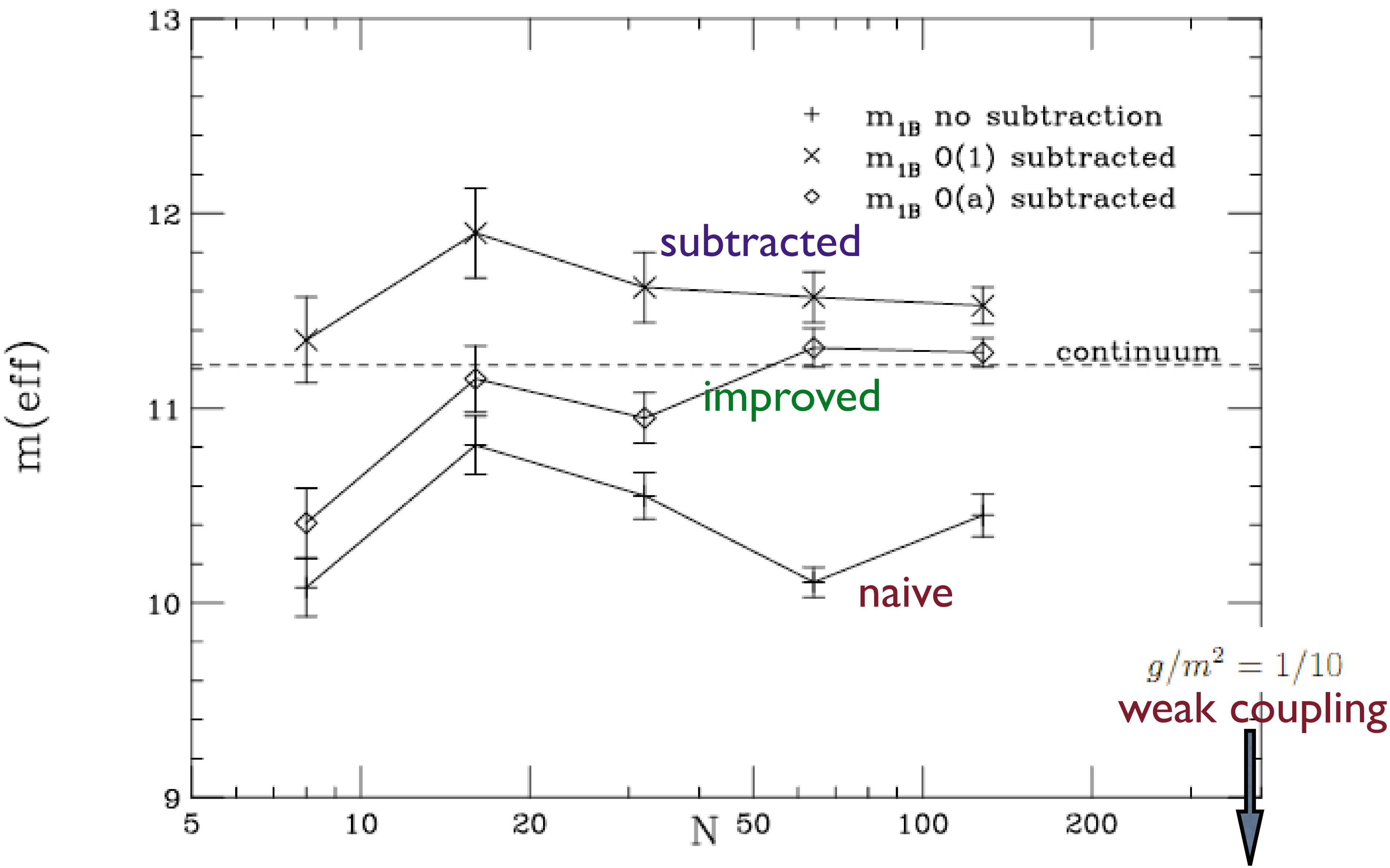


Figure 3: Leading boson mass for various forms of the naive action, with bare parameters $m = g = 10$. Large N corresponds to the continuum limit with $\beta = Na$ held fixed at $\beta = 1$. Lines are drawn to guide the eye.

2. The essence... in a nutshell: "naive" vs. "supersymmetric" lattice

IMPROVED "NAIVE" LATTICE ACTION (R=1)

$$a^{-1}S_{ca} = \frac{1}{2}\left(\Delta^{-}x_i\Delta^{-}x_i + h_i'h_i' + h_i''\right) + \bar{\psi}_i(\Delta^{-}_{ij} + h_i''\delta_{ij} + \frac{1}{2}ah_i''^2\delta_{ij})\psi_j$$

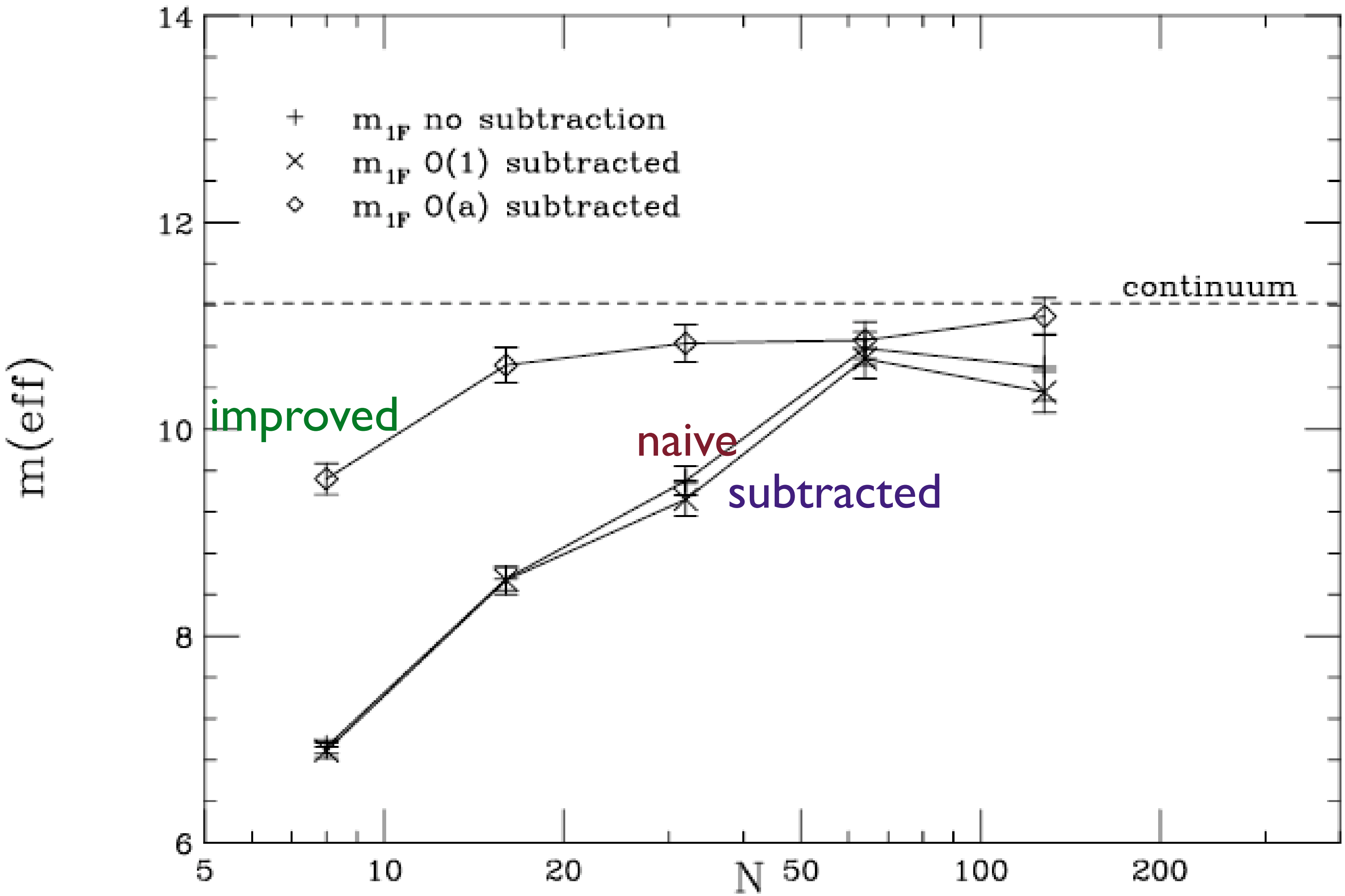
"NAIVE"

$$a^{-1}S = \frac{1}{2}\Delta^{-}x_i\Delta^{-}x_i + \frac{1}{2}h_i'h_i' + \bar{\psi}_i(\Delta^{W}(r)_{ij} + h_i''\delta_{ij})\psi_j$$

"SUPERSYMMETRIC"

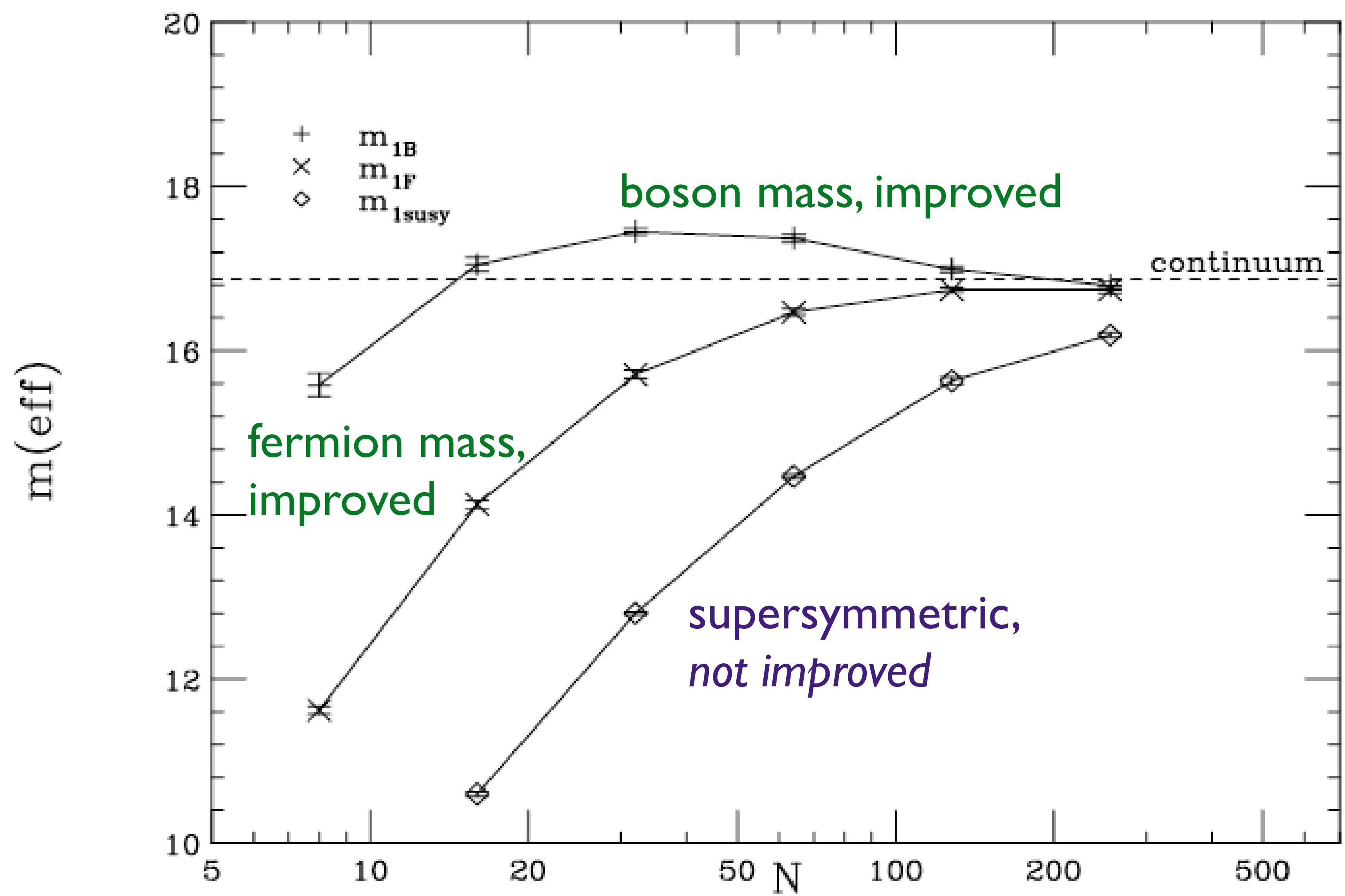
$$a^{-1}S_{ca} = \frac{1}{2}\left(\Delta^{W}x_i + h_i'\right)^2 + \bar{\psi}_i(\Delta^{W}_{ij} + h_i''\delta_{ij})\psi_j$$

2. The essence... in a nutshell: “naive” vs.“supersymmetric” lattice



weak coupling $g/m^2 = 1/10$

2. The essence... in a nutshell: “naive” vs.“supersymmetric” lattice



strong coupling: $g/m^2 = 1$

$g = 100, m = 10$

to summarize this part:

Supersymmetric lattice action works!

- lattice supersymmetry assures that D=0 diagrams cancel, counterterms not needed

- hence perturbative series approaches continuum one; in the continuum limit all continuum Ward identities should be reproduced, at least in perturbation theory

- nonperturbative proof in supersymmetric quantum mechanics via transfer matrix (”theory”) and simulations (”experiment”)

-“experiment” in 2d WZ models pending...
(no nonperturbative “theory” proof here!)

For what theories do we expect to be able to write supersymmetric lattice actions?

Formulated general criteria [Giedt, EP, hep-th/0407135]

1. Clearly, it is necessary that such nilpotent anticommuting* charges exist.
not enough!

2. If some interactions are given by integrals over restricted superspace (e.g., chiral), *there must exist a linear combination of nilpotent anticommuting Q's* **such that** *the Q-variation of these interactions is not a total derivative.*

*anticommuting is not a must; central charges are allowed on the r.h.s. of the anticommutator, but not derivatives [in gauge theories in WZ gauge: all up to gauge transforms] important example of the limitations imposed by criterion 2:

4-supercharge 3d and 4d theories; nilpotent anticommuting charges exist:

$$Q_\alpha \ (\alpha = 1, 2 \text{ is the } SL(2, C) \text{ index})$$

However, "1/3" of the lattice action (W*) will not be supersymmetric - as criterion II. is violated:
nilpotent-Q variation of antiholomorphic superpotential is a total derivative,
so requires Leibnitz rule (this only happens for integrals over restricted superspace)

In 3d WZ models, one can, perhaps, combine super-renormalizability with supersymmetry of K and W, but not of W*, to argue that fine tuning of counterterms can either be avoided or is one-loop only... future work...

as opposed to 4d WZ, the 3d models have interesting infrared dynamics: 3d supersymmetric "Wilson-Fischer" fixed points, where some anomalous dimensions are predicted by the 3d R-symmetry/anomalous dimension correspondence

Theories with nilpotent supercharges where superpotential interactions preserve some supersymmetry on the lattice exist --- most studied, (2,2) supersymmetry in 2d (include the already mentioned WZ models)

A supersymmetric lattice action can be written (old and new) and its renormalization studied (new).

In (2,2) nonlinear sigma models most likely will need fine tuning (generally not finite or even superrenormalizable... work in progress...)

On the other hand, restoration of continuum supersymmetry in WZ models works like in supersymmetric quantum mechanics:

- lattice perturbation theory reduces to continuum in small-a limit

- nonperturbative "experiment" in progress

(2,2) WZ models are interesting:

- depending on superpotential, conjectured to flow to N=2 CFT "minimal models"

- some are mirror duals to particular U(1) gauge theories with matter

the goal of the ongoing "experiment" is to verify the predicted values of critical exponents via a lattice simulation: **first direct "proof" of flow to minimal models**!

An important point that follows from our detailed study and has to be made is that the
nilpotent lattice supersymmetry is not enough to guarantee that desired continuum limit is achieved in all cases.

This is because we found that supersymmetric lattice actions with the desired supersymmetric Euclidean invariant continuum limit are not generic (same true in Kaplan et al. "deconstruction" orbifold models).

This *non-genericity is likely to be generic* and to persist in higher (d>2) dimensions and in gauge theories, with two major implications:

I.

non-finite theories will likely require fine tuning... manageable, perhaps, if superrenormalizable

II.

if the lattice theory is finite [in a precise technical sense, if lattice-D<0 for all integrals] will not need tuning: thus, in 3d and 4d, this approach is more likely to succeed in finite theories...in any case, they are the ones with most nilpotent Qs

3d 4-supercharge (N=2) and higher...
4d 16-supercharge (N=4), perhaps N=2 as well...

4. Outlook

...no implications (yet?) for the most interesting case of
4d N=1, chiral, supersymmetric theories....

The "other" theories, though, are also of interest at least from a formal point of view (string theory, AdS/CFT...), and so there have been

a number of recent proposals for lattice versions of theories with extended supersymmetry, preserving some exact nilpotent supersymmetry, similar in spirit to the (2,2) scalar-fermion theories discussed here:

Recent proposals:

Do they work?

(Cohen), Kaplan, Katz, Unsal (2002-):
4,8,16 supercharge SYM in 2-4 dimensions via deconstruction

difficult to simulate: complex fermion determinant (Giedt, 2003-4)

Sugino (2003-4):
4,8,16 supercharge SYM in 2-4 dimensions via "untwisted TFT"

unknown: renormalization, Euclidean invariance, chiral anomalies, positivity of determinant...?

Catterall (2004-):
4[16] supercharge SYM in 2[4] dimensions via Kahler-Dirac fermions

unknown: renormalization, chiral anomalies, integration contour, positivity of determinant..?

Giedt, E.P. (2004-):
2d (2,2) sigma model actions via Wilson or twisted mass term fermions

unknown: renormalization, Euclidean invariance, sigma model anomalies...?

More new proposals are certainly welcome.
(Certainly yet unwritten supersymmetric lattice actions exist!)

However, careful analysis of the renormalization, fermion det, etc., in each particular case needs to be performed.

From a theoretical point of view it would be interesting if some analytical results could be established using the theories formulated on the lattice [U(1) - WZ mirror symmetry???].

On the "experimental" side:
we will know the results of simulations of 2d WZ models soon (...)
first d>1 field theory example where the "supersymmetric" lattice methods are tried and (hopefully!) shown to work.