The wonders of supersymmetry: from quantum mechanics, topology, and noise, to (maybe) the LHC

Erich Poppitz

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Nov. 21, 2005	Paper **1** to **25** of **13979**
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what is supersymmetry?

what is it good for?

why we have studied it for so long?

(will we ever stop?)

supersymmetry is a quantum mechanical space-time symmetry, which relates bosons and fermions

curiously, it was not discovered in quantum mechanics first, but in string theory and quantum field theory

Ramond 1971
Golfand, Likhtman 1972
Wess, Zumino 1974

nevertheless, some of its most important applications to date involve quantum mechanics

Witten 1981

moreover, supersymmetric quantum mechanics has all features that make it clear why supersymmetry is interesting to particle physicists, as well as to other fields

the harmonic oscillator

$$V = \frac{1}{2}\omega^2 x^2$$

in quantum mechanics, we know that

$$H_B = -\frac{1}{2}\hbar^2 \frac{d^2}{dx^2} + \frac{1}{2}\omega^2 x^2 = \hbar\omega\left(b^+ b + \frac{1}{2}\right)$$

bosonic

$$b = \sqrt{\frac{\hbar}{2\omega}}\frac{d}{dx} + \sqrt{\frac{\omega}{2\hbar}}\, x$$

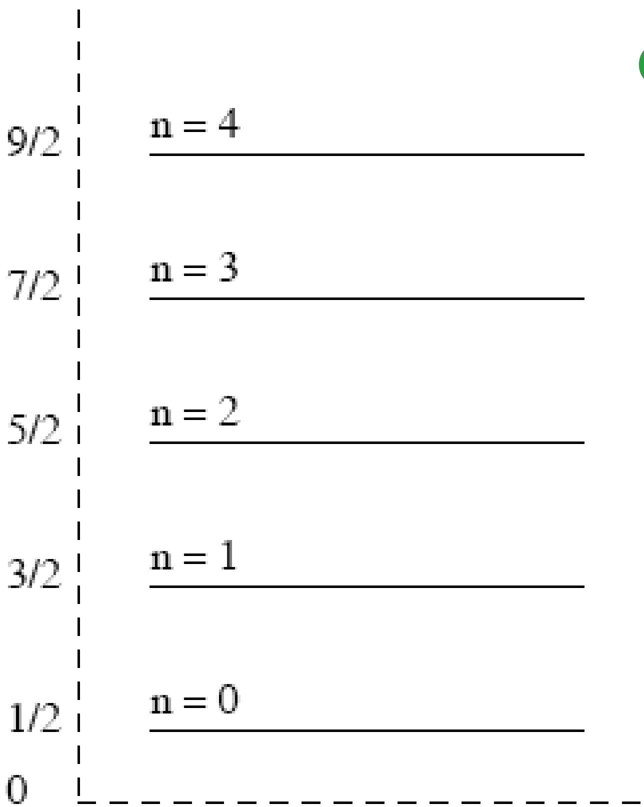
$$b^+ = -\sqrt{\frac{\hbar}{2\omega}}\frac{d}{dx} + \sqrt{\frac{\omega}{2\hbar}}\, x$$

$\left[\frac{d}{dx}, x\right] = 1$ implies $[b, b^+] = b\,b^+ - b^+ b = 1$

$$[b, b^+] = b b^+ - b^+ b = 1$$

bosons: commutation relations
between creation and annihilation operators

energy spectrum of bosonic oscillator



$$E_n^B = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$

$$|n\rangle \sim (b^+)^n |0\rangle$$

$$b|0\rangle = 0$$

what are fermions? they obey Pauli principle!

$$f^2 = 0 \qquad f^{+2} = 0$$

explicitly:

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$f^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

fermions: **anti**-commutation relations between creation and annihilation operators

$$\{f, f^+\} = f f^+ + f^+ f = 1$$

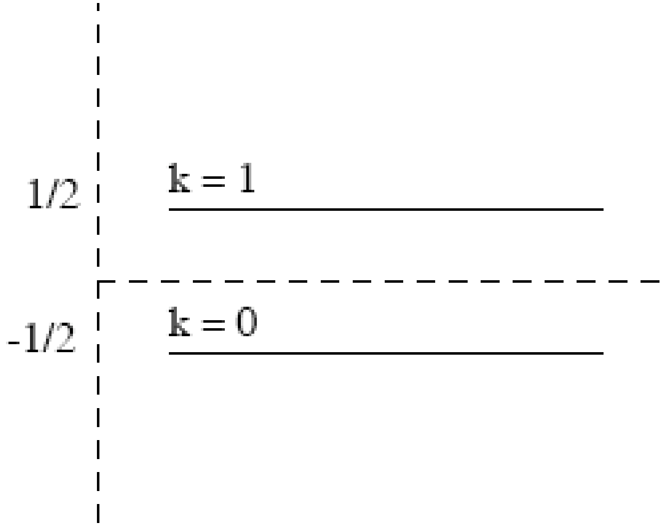
$$H_F = \hbar\omega\left(f^+ f - \frac{1}{2}\right)$$

fermionic

the -1/2 may look cooked-up, but it comes from quantizing the Dirac equation+C-symmetry

we'll simply define fermionic oscillator that way

energy spectrum of fermionic oscillator



$$E^F_k = \hbar\omega\left(k - \frac{1}{2}\right), \quad k = 0, 1$$

$$|1\rangle = f^+ |0\rangle$$

$$f|0\rangle = 0$$

$$H_B = \hbar\omega\left(b^+ b + \frac{1}{2}\right) \qquad H_F = \hbar\omega\left(f^+ f - \frac{1}{2}\right)$$

add them up: the supersymmetric oscillator

$$H_{susy} = \hbar\omega\left(b^+ b + f^+ f\right) = \hbar\omega\left(b^+ f + f^+ b\right)^2$$

Proof:

$$\left(b^+ f + f^+ b\right)^2 = \cancel{b^+ f b^+ f} + b^+ f f^+ b + f^+ b b^+ f + \cancel{f^+ b f^+ b}$$

$$= b^+ b\, f f^+ + b b^+ f^+ f =$$

$$= b^+ b\left(\cancel{-f^+ f} + 1\right) + \left(\cancel{b^+ b} + 1\right) f^+ f =$$

$$= b^+ b + f^+ f \quad \blacksquare$$

$$H_{susy} = \hbar\omega(b^+ b + f^+ f) = \hbar\omega(b^+ f + f^+ b)^2$$

introducing: the supersymmetry generators

$$Q = b^+ f \qquad Q^+ = f^+ b \qquad Q^2 = Q^{+2} = 0$$

$$H_{susy} = \hbar\omega(Q + Q^+)^2 = \hbar\omega(\cancel{Q^2} + QQ^+ + Q^+Q + \cancel{Q^{+2}})$$

$$= \hbar\omega\{Q, Q^+\}$$

this is not as silly as it looks!

we just found one of the simplest SUPERALGEBRAS!

$$\{Q, Q^+\} = H \qquad Q^2 = Q^{+2} = 0$$

$$[Q, H] = [Q^+, H] = 0$$

superalgebra has even (H) and odd (Q) elements, obeying commutation or anticommutation relations

we know that Hamiltonian generates time translations -

$H_{susy} = \hbar\omega(Q + Q^+)^2$ - took square root of time translations!

similar to the Hamiltonian H, the supersharges Q can be thought as of generating translations in a "fermionic dimension" of spacetime, making it thus into a superspace while abstract, superspace is quite a useful concept, when handled with care

Superspace [8] is the greatest invention since the wheel [9].

Physica 15D (1985) 289–293
North-Holland, Amsterdam

STUPERSPACE

V. GATES†, Empty KANGAROO‡, M. ROACHCOCK*, and W.C. GALL*
California Institute of Technology, Pasadena, CA 91125, USA

Pentember, 1999

the simplest superalgebra

$$\{Q, Q^+\} = H \qquad Q^2 = Q^{+2} = 0$$

$$[Q, H] = [Q^+, H] = 0$$

reveals a very general algebraic structure

from D=1 (QM) to D=11 (M-theory)!

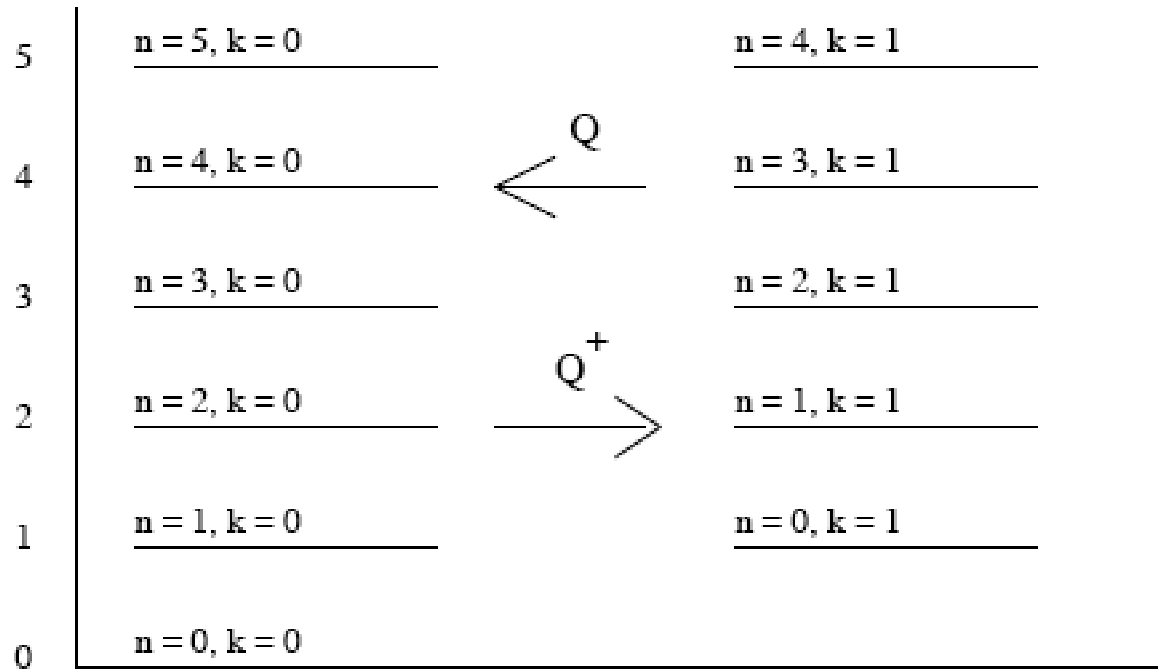
so far, we learned that supersymmetry is:

quantum mechanical, space-time, relates bosons and fermions...

but what are its implications? generic properties?

energy spectrum of supersymmetric oscillator

$$E^{SUSY}_{n,k} = \hbar\,\omega\,\,(n+k)$$



1 all energy levels positive or zero

2 zero point energy is gone! (true only if unbroken supersymmetry)

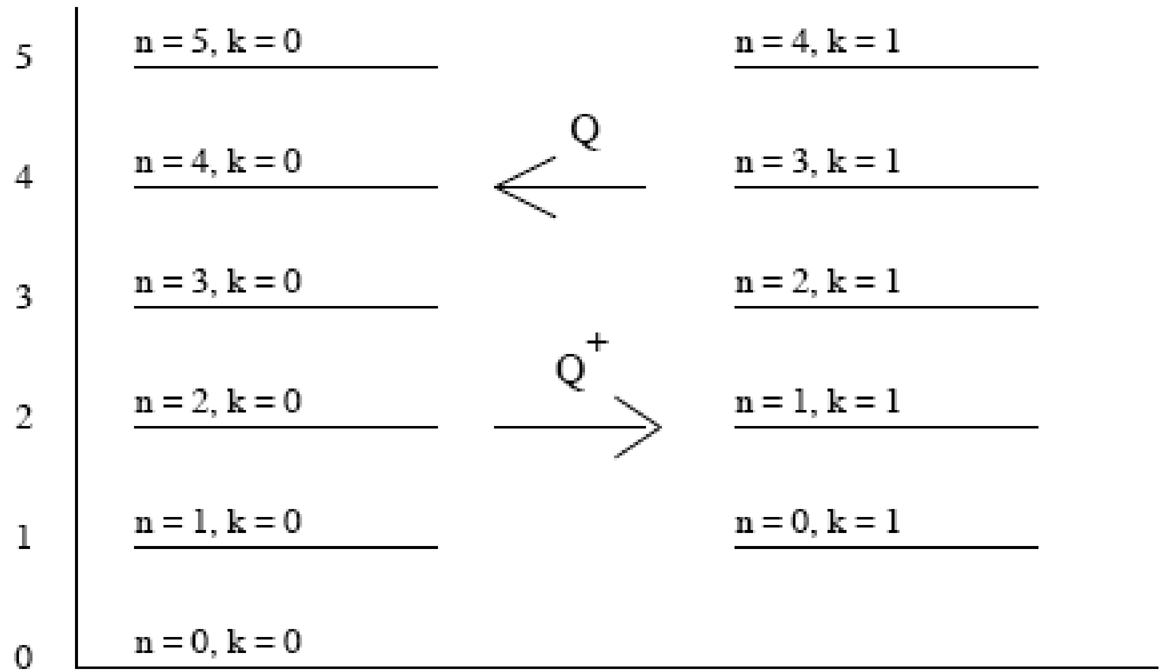
3 all **nonzero** energy levels degenerate, related by Q "supermultiplets"

1,2,3: general features, not just this example!

energy spectrum of supersymmetric quantum field theory

$$\sum$$

modes of field
in a box

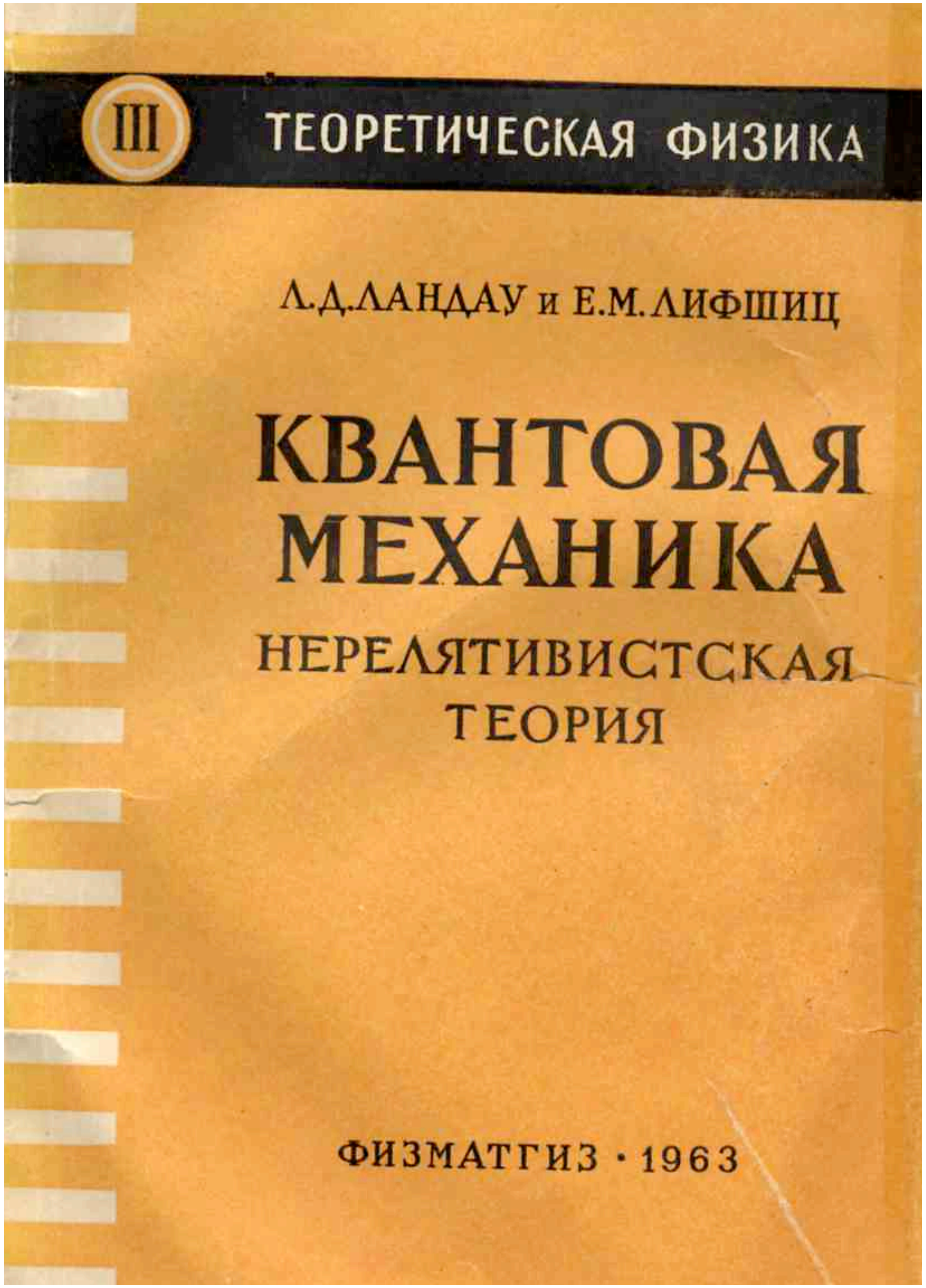


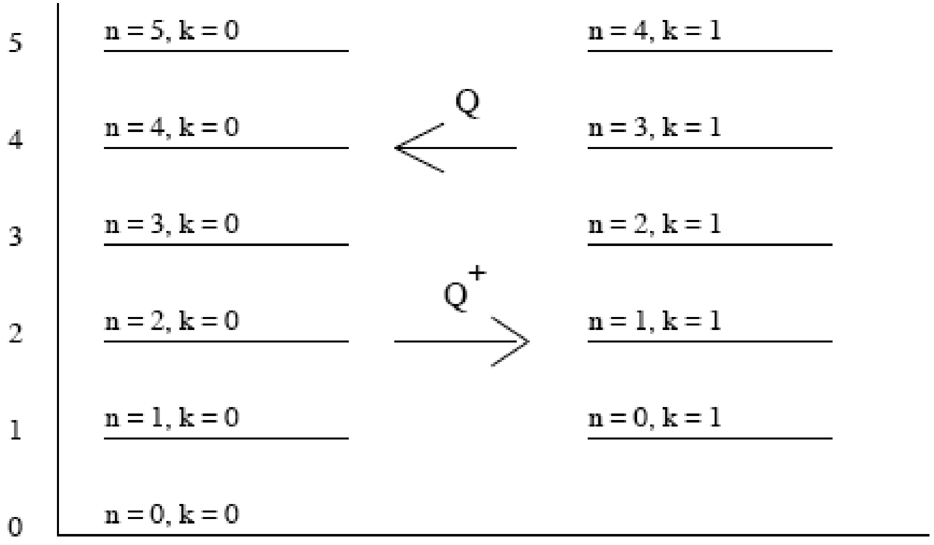
if nature was supersymmetric, degenerate supermultiplets would imply the existence of superparticles - *selectron, photino,...*

alas "sligthly nondegenerate"- the LHC may find them, if they exist...

given we are some time away from studying the LHC data,

is there supersymmetry in real physical systems?

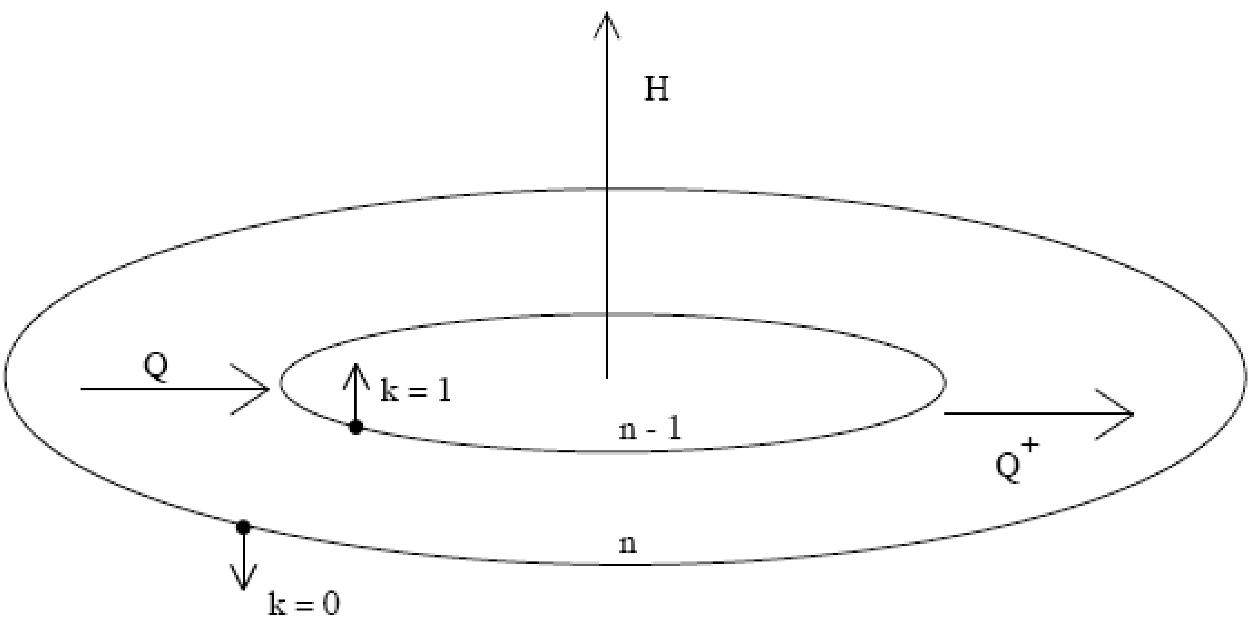




g=2 Landau levels
are supersymmetric

$$\omega = \frac{e H}{m_e c}$$

magnetic field, not Hamiltonian



spin and velocity precess at the same rate
tiny difference due to (g-2) from radiative corrections, use to measure it

a slightly more fancy (but somewhat less "real-world") example is that of **emergent supersymmetry**

some critical lattice systems in two space dimensions give rise to a supersymmetric "QFT" system, exploiting the equivalence:

D-dim classical equilibrium stat mech = (D-1)+1-dim quantum field theory in Euclidean space, i.e. QFT with t -> it

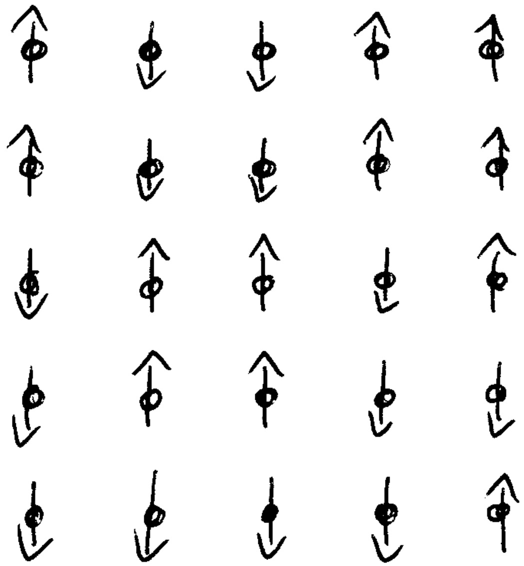
very general correspondence, only illustrate on 2-dim Ising model example

$$H = J \sum_{NN} \sigma_i \sigma_j \qquad \sigma_i = \pm 1$$

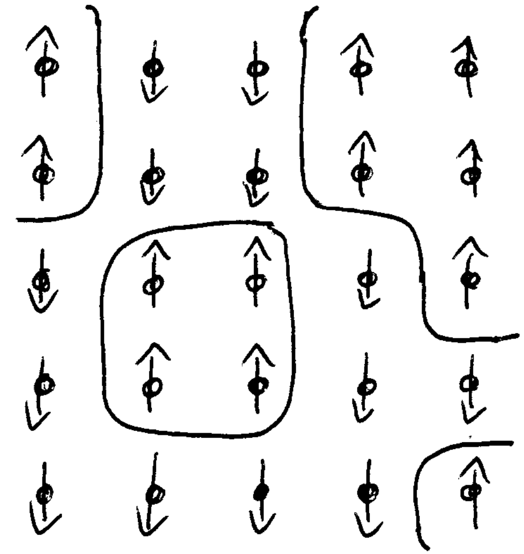
- magnetic moments
- lattice particles

Boltzmann partition function: $$Z = \sum_{\{\sigma\}} e^{-\frac{J}{kT}\sum_{NN}\sigma_i\sigma_j}$$

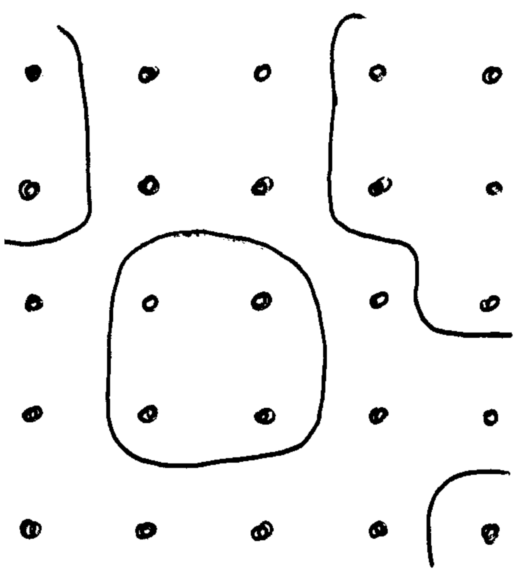
any spin configuration



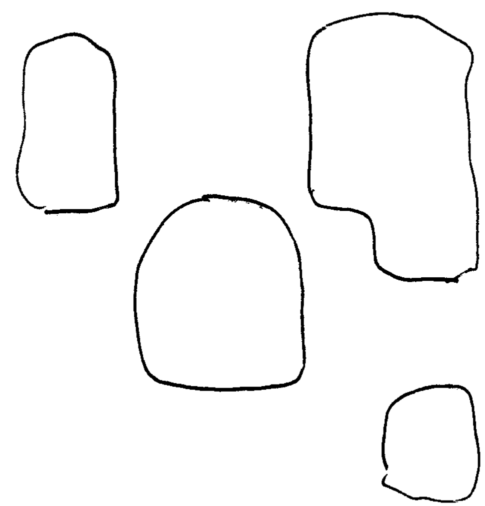
is completely specified by



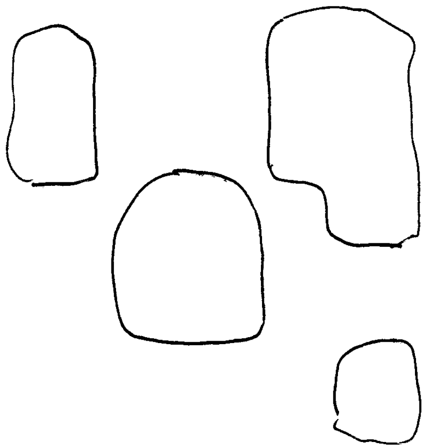
the loops around up-spin islands,



so the Ising model Z is really a Boltzmann-weighted sum over all possible closed-loop configurations



closed loops = world lines of virtual particle-antiparticle pairs -



- precisely the stuff the vacuum is made of!

thus:

2-dim Ising = (1+1)-dim Euclidean QFT

a closer look reveals that:

2-dim Ising = 2 (1+1)-dim massive free majorana **fermions** of spins 1/2, 3/2

now, we know that: low-T - magnetization, mostly large loops
high-T - random orientations, mostly small loops
in between - T_c - loops of all sizes

when loops of all sizes contribute to Z -> no scale in the problem at criticality

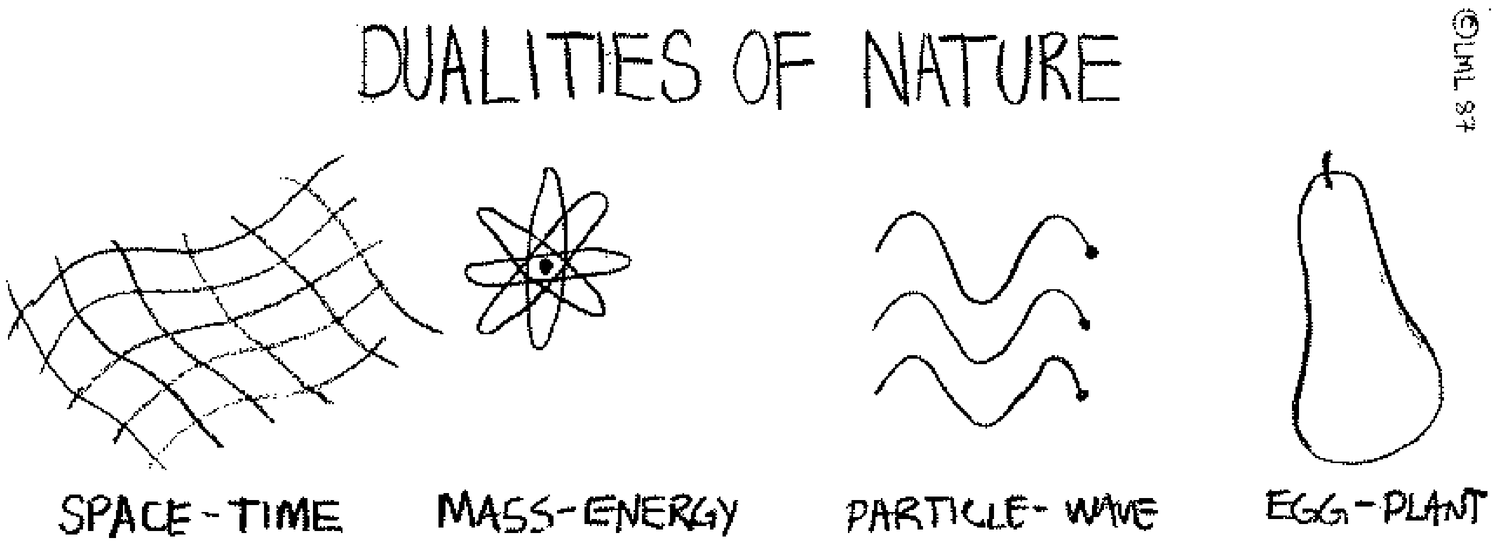
spin-1/2 fermion becomes massless -> scale and conformal invariance "emerge" at critical temperature

Onsager, 1944

use conformal QFT to compute critical exponents of classical spin systems, lattice gases, etc!

Polyakov, 1970
Belavin, Polyakov, Zamolodchikov, 1984

another example of the many



(courtesy of Prof. A. Steinberg)

the 2d Ising example shows how QFT with fermions can be relevant for describing classical critical phenomena

since we've already got the fermions and the emergent conformal invariance at criticality - could it happen that in a lattice model modified to also have bosons, a conformal invariance + supersymmetry *(= superconformal)* would emerge at criticality?

Friedan, Qiu, Shenker 1985 -yes: ***tricritical Ising model*** at tricritical point is equivalent to a (1+1)-dim "N=1 supersymmetric" conformal quantum field theory

$$H = J \sum_{NN} \sigma_i \sigma_j + K \sum_{NN} \sigma_i^2 \sigma_j^2 + \Delta \sum \sigma_i^2$$

$$\sigma_i = -1, 0, +1$$

tricritical Ising aka "Blume-Emery-Griffiths" model, e.g. phases of He^3 -He^4 mixtures

supersymmetry predicts relations between different critical exponents

has a 2-dim system with above H been realized experimentally at tricritical point?

answer appears to be NO, so far... perhaps possible via cold atoms?

in "real" QFT (Minkowskian)

e.g., so-called "N=1 supersymmetry in 4d"
we similarly take square root of space-time translations, introduce new "fermionic dimensions" of spacetime, etc.,
all this most easily done by simply "covariantizing" our QM result:

$$H = P_0 \longrightarrow P_\mu \qquad \{Q, Q^+\} = H$$

$$Q \longrightarrow Q_\alpha \qquad \downarrow$$

$$Q^+ \longrightarrow \bar{Q}_{\dot\alpha} \qquad \{Q_\alpha, \bar{Q}_{\dot\alpha}\} = -2i\,\sigma^\mu_{\alpha\dot\alpha} P_\mu$$

Haag, Lopushansky, Sohnius 1975: "super-Poincare" is the most general nontrivial extension of Poincare algebra consistent with relativistic S-matrix...

- find that all particles come in supermultiplets
- the already mentioned *selectron, photino,* and other *"superpartners"*
- but of course, can not be exactly degenerate
- may be they will be seen at the LHC, may be not,...

not my main topic today

instead, stick with QM and illustrate some of the uses of supersymmetry

one of the great simplifications that supersymmetry brings is in finding ground state wave functions - the first thing you want to know about any theory!

$$H_{susy} = \hbar\omega\,(b^\dagger b + f^\dagger f) =$$

$$-\ \ldots$$

$$= \hbar\omega\,(Q + Q^\dagger)^2$$

$$= \hbar\omega\,\{Q\,,\,Q^\dagger\}$$

this implies that the ground state energy is zero **if and only if** *the ground state is annihilated by both Q and Q*$^+$

then one can look for the ground state by solving not

$$H|0\rangle = 0$$ - a second order diff. equation

but by solving a first order equation instead! $$Q|0\rangle = 0$$

this (or rather, similar simplifications in more complex cases) has led to some great progress in studying the ground states of supersymmetric systems

from **strongly coupled QFT** - *the "supersymmetric cousins" of QCD*

to **string theory...**

also led to great progress in mathematics and mathematical physics related to computing topological properties, such as various topological indices...

a concept introduced by Witten plays a crucial role in these developments

to explain, go back to QM...

QM of a particle of spin-1/2 moving on a line, with potential and "spin-orbit" interaction, both determined by one function:

W(x) - the **SUPER**potential

$$\Psi = \begin{pmatrix} \psi(x_1) \\ \psi(x_2) \end{pmatrix}$$

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\left(\frac{dW(x)}{dx}\right)^2 + \frac{1}{2}\sigma_3 \frac{d^2W(x)}{dx^2}$$

consider one hermitean linear combination of supercharges
(turns out to be sufficient)

$$Q = -\frac{i}{\sqrt{2}}\sigma_1 \frac{d}{dx} + \frac{1}{\sqrt{2}}\sigma_2 \frac{dW(x)}{dx}$$

then, because $H=Q^2$, ground state (recall, it has E=0) wave function obeys:

$$Q|0\rangle = 0$$

$Q|0\rangle = 0$ -> $\left(-i\sigma_1 \frac{d}{dx} + \sigma_2 W'(x)\right)\Psi_0(x) = 0$

-> $-i\sigma_1 \boxed{\left(\frac{d}{dx} - \sigma_3 W'\right)\Psi_0(x) = 0}$ which is immediately solved:

$$\Psi_0(x) = e^{\sigma_3 W(x)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e^{W(x)} c_1 \\ e^{-W(x)} c_2 \end{pmatrix}$$

normalizable zero energy state exists iff **= *unbroken supersymmetry***

$c_1 = 0$, $W(x) \to \infty$ as $|x| \to \infty$

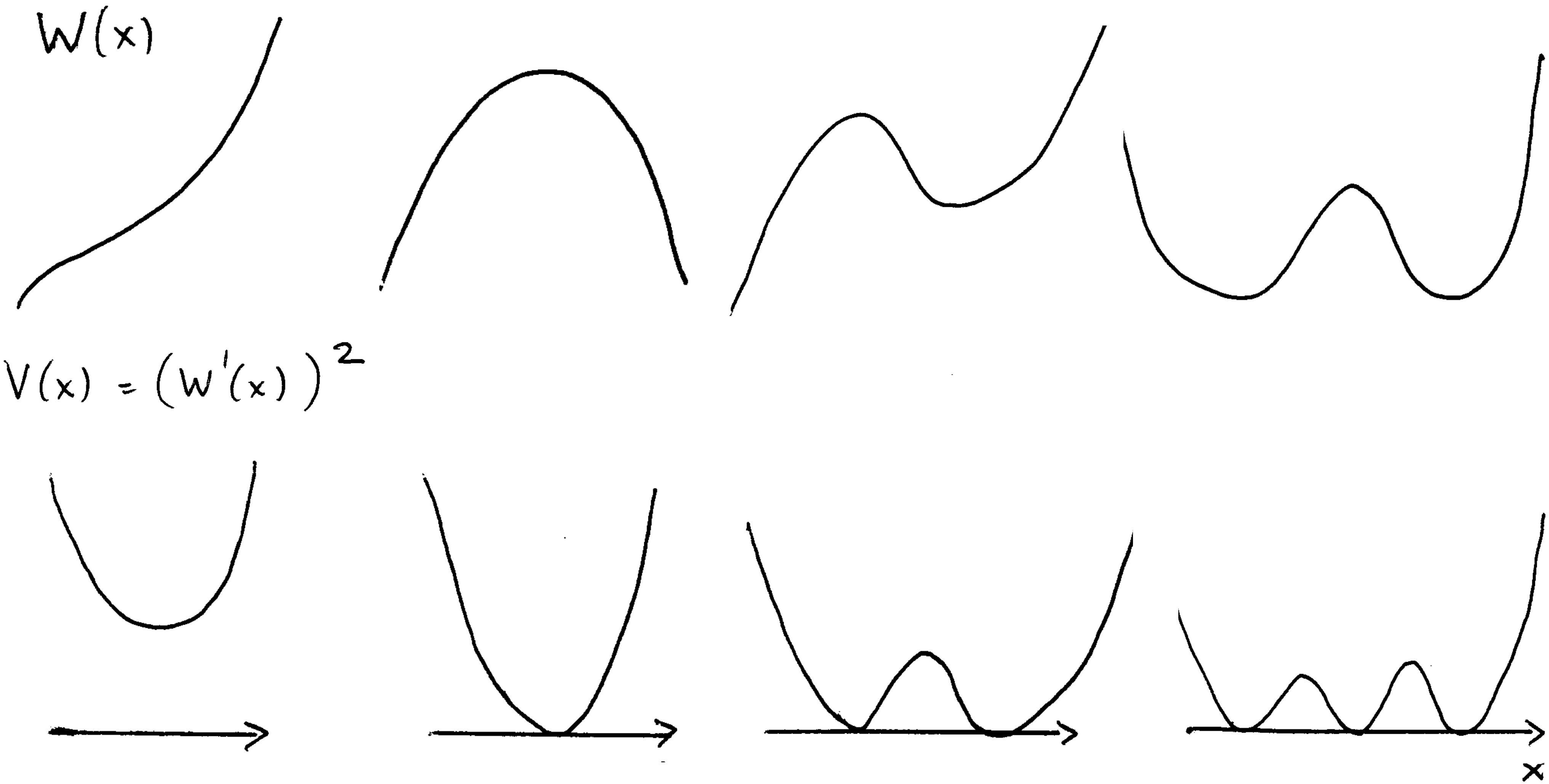
$c_2 = 0$, $W(x) \to -\infty$ as $|x| \to -\infty$

in other words supersymmetry is unbroken iff W(x) is "even at infinity"

equivalently, iff W(x) has odd number of extrema,

hence, V(x) has an odd number of zeros since V ~ (W')^2

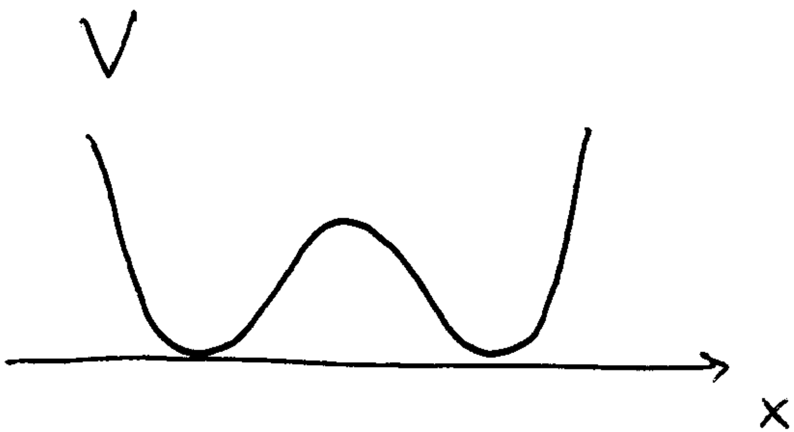
-in other words supersymmetry is unbroken iff $W(x)$ is "even at infinity"
-equivalently, iff $W(x)$ has odd number of extrema,
-hence, $V(x)$ has an odd number of zeros since $V \sim (W')^2$



broken | unbroken | broken (!) | unbroken

- even in QM, analysis of effect of tunneling on zero-energy "tree-level" ground state is complex!
- exact answer makes such analysis "unnecessary"... *imagine the bonus you get in difficult theories...*

(important side remark about third case:



supersymmetry unbroken classically, $E_0 = 0$

but broken by nonperturbative effects (tunneling = instantons):

$$E_0^{WKB} = \langle 0 | H | 0 \rangle$$

$$\sim \hbar\omega \, e^{-\frac{1}{\hbar}\int dx \sqrt{2V(x)}} \lll \hbar\omega$$

VERY relevant for particle theory models and the hierarchy problem.

scales:

gravity 10^{19} GeV
"UV cutoff scale"

weak interactions 10^{2} GeV

hierarchy problem:

how does a theory with a characteristic length scale of 10^{-33} cm give rise to structure (the electro-weak interactions + all the stuff we are made of) 17 orders (and more) of magnitude larger?

previous experience in physics makes one think that there should be a dynamical reason

typical answers: supersymmetry, or strong dynamics...,
or, these days, (what amounts to) fine tuning

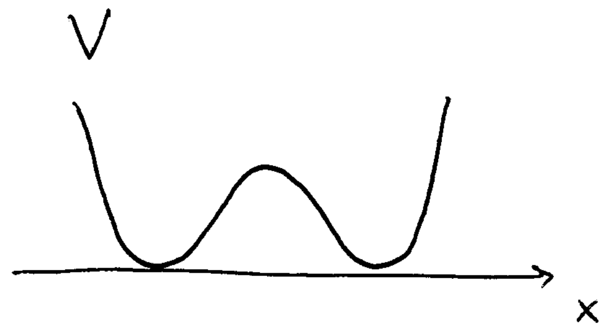
how does supersymmetry breaking by quantum nonperturbative effects help?

suppose supersymmetry breaking scale and the electroweak scale are somehow related

then ***selectron, photino,... neutralino, chargino,...*** masses are all "of order" the electroweak scale

supersymmetry breaking also responsible for W-,Z-boson masses

- supersymmetry breaking ultimately responsible for physics as we see it



analogy: gravity (Planck) scale - frequency near bottom of one well

superparticle masses & weak scale - ground state energy

$$E_0^{WKB} = \langle 0 | H | 0 \rangle \sim \hbar\omega \, e^{-\frac{1}{\hbar}\int dx \sqrt{2V(x)}} \lll \hbar\omega \quad \Longrightarrow \quad M_{\substack{\text{selectron} \\ \text{photino} \\ \cdots}} \sim e^{-\frac{4\pi}{\alpha}} M_{\text{Planck}}$$

$(\sim 10^2 \text{ GeV}?)$ $\qquad (10^{19} \text{ GeV})$

is $M_{\text{superpartner}} \sim M_{\text{electroweak}}$? ... ask the LHC more on this can be learned from my review for non-experts:

"Dynamical supersymmetry breaking: why and how?" arXiv: hep-ph/9710274

during the ten years past the subject has evolved
- LEP-2 data indicate supersymmetry more fine-tuned than hoped for
- newer theoretical ideas about supersymmetry

both sane ..."s p l i t supersymmetry," "splat supersymmetry"...

and otherwise...

Supersplit Supersymmetry

Patrick J. Fox,[1] David E. Kaplan,[2] Emanuel Katz,[3,4] Erich Poppitz,[5]
Veronica Sanz,[6] Martin Schmaltz,[4] Matthew D. Schwartz,[7] and Neal Weiner[8]

[1] *Santa Cruz Institute for Particle Physics, Santa Cruz, CA, 95064*
[2] *Dept. of Physics and Astronomy, Johns Hopkins University, Baltimore, MD 21218*
[3] *Stanford Linear Accelerator Center, 2575 Sand Hill Rd. Menlo Park, CA 94309*
[4] *Dept. of Physics, Boston University, Boston, MA 02215*
[5] *Department of Physics, University of Toronto, 60 St George St, Toronto, ON M5S 1A7, Canada*
[6] *Universitat de Granada, Campus de Fuentenueva, Granada, Spain*
[7] *University of California, Dept. of Physics, Berkeley, CA 94720-7300*
[8] *Center for Cosmology and Particle Physics, Dept. of Physics, New York University, New York , NY 10003*

(Dated: April 1, 2005)

The possible existence of an exponentially large number of vacua in string theory behooves one to consider possibilities beyond our traditional notions of naturalness. Such an approach to electroweak physics was recently used in "Split Supersymmetry", a model which shares some successes and cures some ills of traditional weak-scale supersymmetry by raising the masses of scalar superpartners significantly above a TeV. Here we suggest an extension - we raise, in addition to the scalars, the

- still important to work on, to answer

is $M_{superpartner} \sim M_{electroweak}$? ... ask the LHC

end important side remark)

most important lesson from QM example was that the criterion whether supersymmetry breaks or not is topological:

existence of zero energy ground state depends only on the properties of W(x) "at infinity"... independent on details in between, so long as smooth function!

this can be viewed as an example of a topological index:

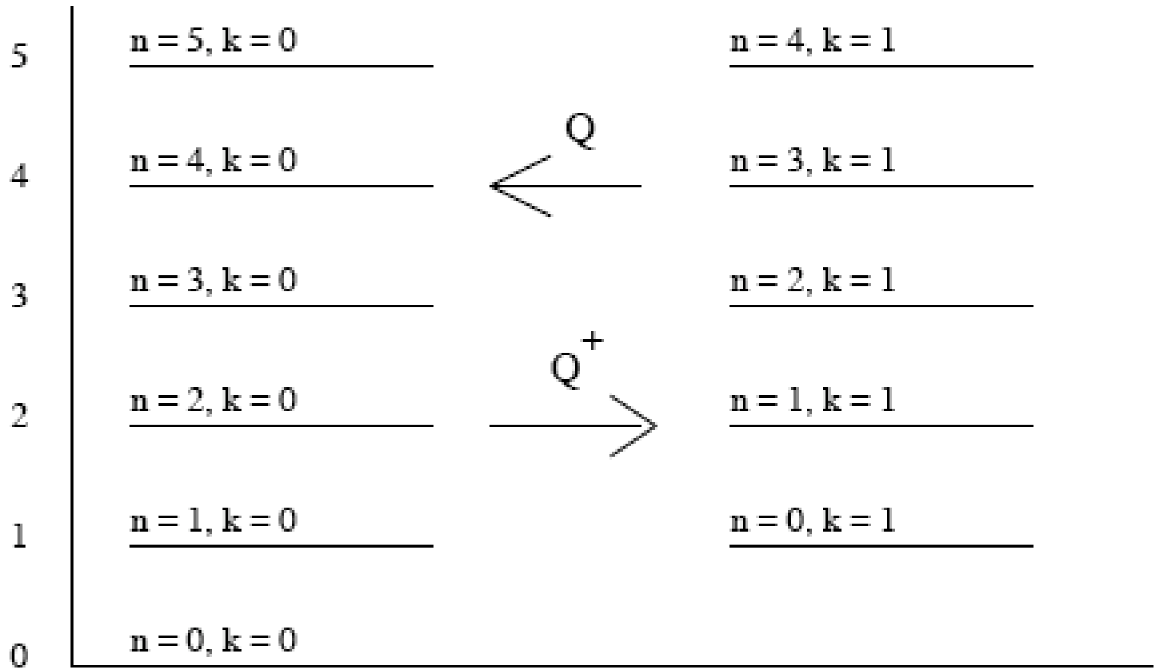
Witten index

$$\mathrm{Tr}\,(-1)^F \equiv \sum_E n_B(E) - n_F(E) = n_B(0) - n_F(0) .$$

Witten index is invariant under deformations that do not change the asymptotics of the potential at infinity; *if nonzero, then supersymmetry definitely unbroken!*

allowed deformations include changes of potential (i.e., of coupling constants!), ***so can be calculated at weak coupling but true at any coupling***

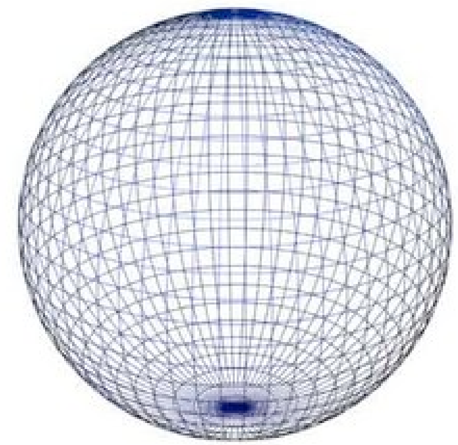
- a theorist's dream!



$$\mathrm{Tr}\,(-1)^F \equiv \sum_E n_B(E) - n_F(E) = n_B(0) - n_F(0)\,.$$

generic spectrum not equidistant, but always degenerate for all E>0...
upon deformations of the theory (change W, volume, coupling)
states can leave E=0 or become E=0 states, but only in B-F pairs,
however, such changes do not affect the index... Q.E.D.

our example was, of course, a very simple one

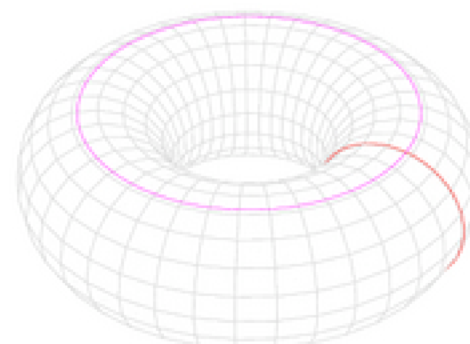


more examples are obtained when one considers QM of particles moving on, e.g. general Riemann manifolds

then one can show that the Witten index is equal to the Euler characteristic

χ = 2 - 2g for a genus g Riemann surface

g= number of handles: sphere χ=2, torus χ=0...

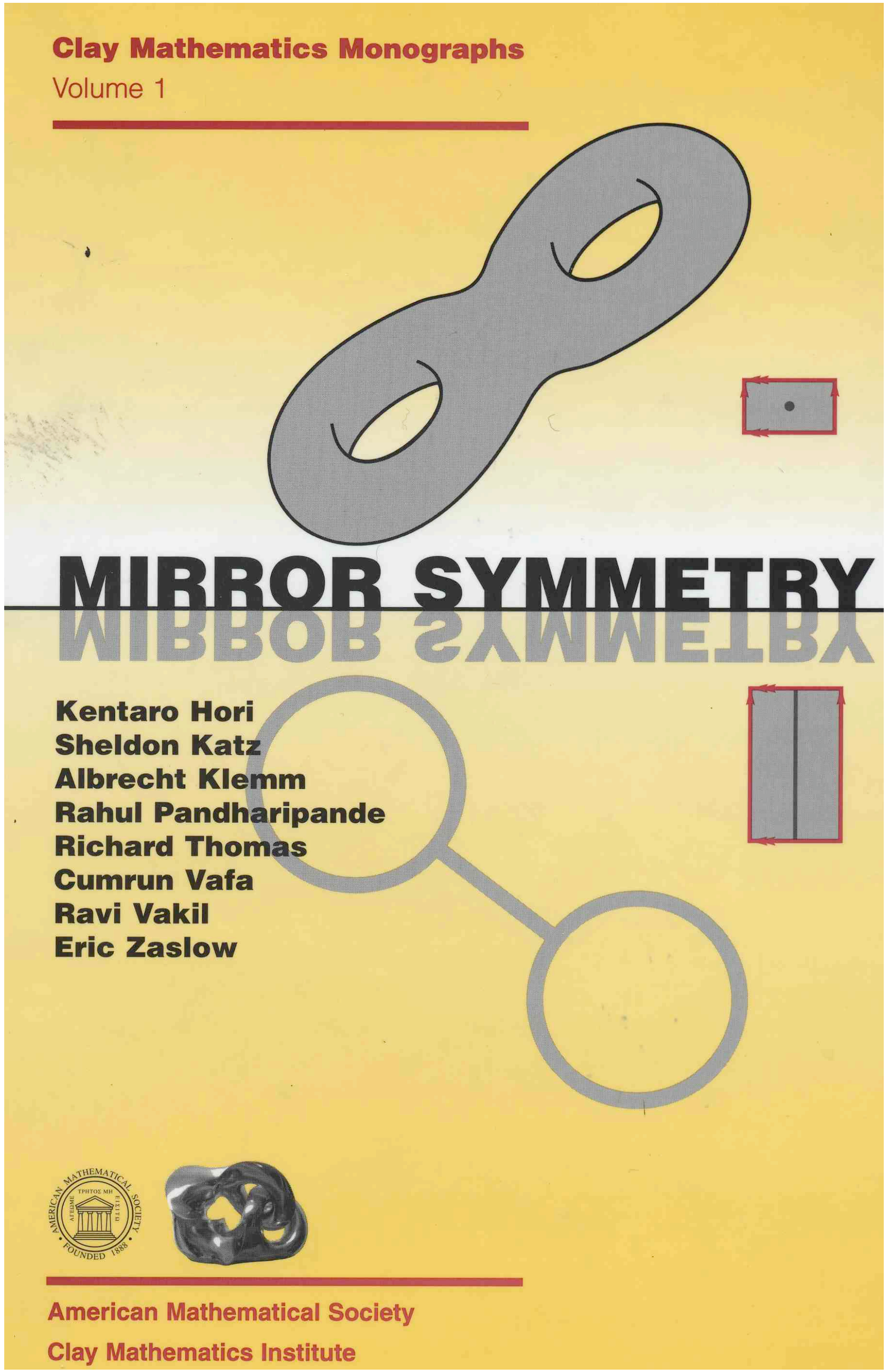


moreover analytic formulae for the Euler characteristic can be obtained most simply - for a physicist, at any rate - by studying the partition function of the supersymmetric QM:

$$\int_M \mathrm{Pf}\Omega = (2\pi)^n \chi(M) \qquad \mathrm{Pf}\Omega \sim \epsilon^{i_1 \ldots i_n} R_{i_1 i_2} \ldots R_{i_{n-1} i_n}$$

Gauss-Bonnet theorem, classic result in mathematics

thus: physics $\xleftrightarrow{\text{supersymmetry}}$ mathematics physics $\xleftrightarrow{\text{supersymmetry}}$ mathematics



beware: 929 pages!

physics $\xleftrightarrow{\text{supersymmetry}}$ mathematics

indices of massless Dirac operators in external gauge fields

$$\mathrm{ind}(D) = n_L - n_R$$

to study, one engineers a supersymmetric QM whose Witten index = index of the Dirac operator of interest

...fastest way (for physicists, for sure!) to obtain index theorems

...not how all index theorems were first obtained, but some new ones

physical relevance?

weak interactions: B+L violation in the SM due to nonperturbative effects, role in generating baryon asymmetry of the universe

strong interactions: U(1) problem, axion mass (strong CP problem)

chiral fermions - L/R asymmetry of nature: chiral fermions in 4d are zero modes of Dirac operators in extra dimensions; nontrivial index guarantees chirality

the only way we understand UV origin of chirality!

- nonzero index is the ultimate reason why we exist!

MORAL:

- **supersymmetry has led to some very interesting, beautiful, and important developments in mathematical physics, so**
- **supersymmetry is of interest whether or not weak-scale supersymmetric particles exist**
- **particle physicists hope to prove or disprove the existence of weak-scale supersymmetry at the LHC**

supersymmetry also shows up, somewhat intriguingly, in another place:

the theory of motion driven by random noise and the relaxation to equilibrium high-viscosity limit equation of motion of a "Brownian particle"

$$\dot{x} = -\frac{\epsilon}{2} W'(x) + \eta(t)$$ - Langevin equation

drift force   noise $\epsilon = \pm 1$

regular "drift force" (W') + random, Gaussian, delta-correlated "noise"

$$\langle \eta(t)\, \eta(t') \rangle \sim \delta(t - t')$$

Langevin equation is a stochastic differential equation - has only a probabilistic solution (probability P(x,t|0,0) that at (x,t) if at (0,0)):

substitution $P(x,t \mid 0,0) = e^{-\frac{\epsilon W(x)}{2}} \rho(x,t)$

yields Fokker-Planck equation $$\frac{d\rho}{dt} = - H_{FP}\, \rho$$

with positive Fokker-Planck "Hamiltonian"

$$H_{FP} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{8}\left(W'(x)\right)^2 - \frac{1}{4}\epsilon W''(x)$$

note the similarity of supersymmetric quantum mechanics from a few slides back:

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\left(\frac{dW(x)}{dx}\right)^2 + \frac{1}{2}\sigma_3 \frac{d^2W(x)}{dx^2}$$

to the Fokker-Planck Hamiltonian

$$H_{FP} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{8}(W'(x))^2 - \frac{1}{4}\epsilon W''(x)$$

Now, as t goes to infinity, lowest eigenvalue dominates, so equilibrium probability distribution = "ground state wave function"

$$P(x,t)\Big|_{t\to\infty} = \frac{e^{-\epsilon W(x)}}{\int dx\, e^{-\epsilon W(x)}}$$

limiting P(x) exists iff W(x) is "even at infinity," hence

unbroken "supersymmetry" = existence of equilibrium distribution

---

for the experts:

- fermions are "ghosts" introduced to enforce Langevin equation as a constraint while noise-averaging
- what really happens is that the BRS symmetry of path-integral representation of probability distribution gets enlarged to a quantum-mechanical supersymmetry

--- this example of

unbroken "supersymmetry" = existence of equilibrium distribution

illustrates how supersymmetry appears rather generally in problems involving random noise averages

this "Parisi-Sourlas supersymmetry (1979)" arises in many problems involving averaging over random noise (also, slightly different, in averaging over disorder in condensed matter systems) as well as in stochastic quantization

it is a "quantum-mechanical" supersymmetry (rather than d+1-dim) even in d>1 theories with random noise what is supersymmetry?

it is a beautiful new quantum mechanical symmetry

it is rich of properties

it helps physicists better understand complex math, and even contribute to its development!

what is it good for?

it appears strikingly in all sorts of problems, not all quantum-mechanical (random noise averages & topology & classical critical phenomena)

why we have studied it for so long?

it's fun

also, it is of great interest in particle physics, but details of that are a subject to a separate talk and **near-future experimental developments!**