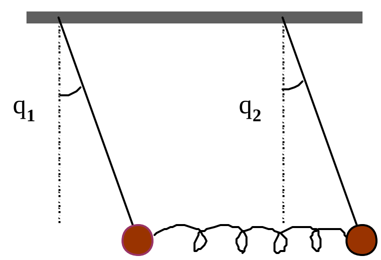
COUPLED OSCILLATORS

A real physical object can be regarded as a large number of simple oscillators coupled together (atoms and molecules in solids). The question is: how does the coupling affect the behavior of each of the individual oscillators?

Two identical pendulums

We have two identical pendulums (length L) for which we consider small oscillations. In order to find what is the simplest motion, we imagine two experiments:

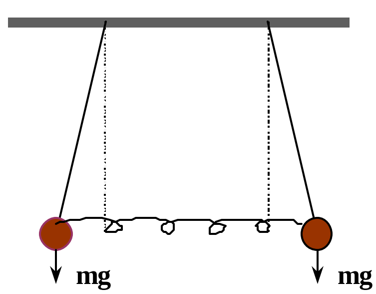
1) If we draw the two masses aside some distance and release them simultaneously from rest, **they will swing in identical phase with no relative change in position. The spring will remain unstretched (or uncompressed) and will exert no force on either mass.**



We call this vibration pattern the first mode of vibration of the system.

2) The other obvious way of starting a symmetric oscillation will be to stretch the spring from both ends. If we release the masses from rest simultaneously, we may notice that:

a) The spring now exerts forces during motion
b) From symmetry of motions of A and B, their positions are mirror images of each other



We call this vibration pattern the second mode of vibration of the system.

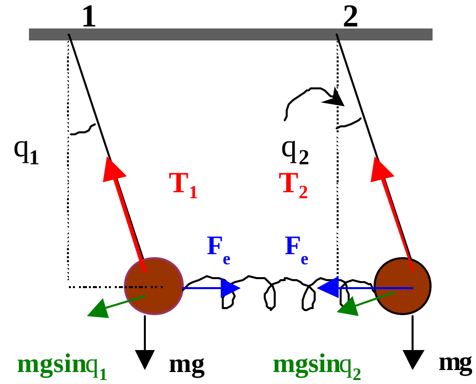
Note: each pendulum in the one of the modes above oscillates with the same frequency: the normal oscillation frequency.

The two oscillating patterns are called <u>normal modes</u>.

Both are SHM of constant angular frequency and amplitude.

General motion as superposition of normal modes

We take two coupled pendulums, identical, each starting from rest. Any motion of the system, showing no special symmetry may be described as a combination of the two normal modes of oscillation.



We assume small displacements from equilibrium: x_1, x_2.
Each pendulum swings because of the combined force of gravity **mg** and the string tension **T**. The combined force is:

$$\mathbf{mg\ sin}\ q_1 \ @\ \mathbf{mg}\ q_1 = \frac{mg}{L}x_1 \quad \text{for mass 1,}$$

but force along the spring exerted by mass 1 is $\mathbf{F_1} \sim \mathbf{mg}q_1\mathbf{cos}q_1 \sim \mathbf{mg}q_1 \sim \mathbf{mgx_1/L}$

$$\mathbf{mg\ sin}\ q_2 \ »\ \mathbf{mg}\ q_2 = \frac{mg}{L}x_2 \quad \text{for mass 2,}$$

but force along the spring exerted by mass 2 is $\mathbf{F_2} \sim \mathbf{mg}q_2\mathbf{cos}q_2 \sim \mathbf{mg}q_2 \sim \mathbf{mgx_2/L}$

Note Displacements from equilibrium are very small, angles q_1 and q_2 are very small:
sin$q_{1,2} \simeq q_{1,2}$; **cos**$q_{1,2} \simeq$ **1.** Displacements are given by: $x_1 \cong L\theta_1$, $x_2 \cong L\theta_2$

If we consider $x_1 \neq x_2$, the spring is stretched by $x_2 - x_1$ and the elastic restoring force in the spring will be $\mathbf{F_e = k(x_2 - x_1)}$

The total restoring force on mass 1 is $\quad \mathbf{-[mgx_1/L - k(x_2 - x_1)]}$
The total restoring force on mass 2 is: $\quad \mathbf{-[mgx_2/L + k(x_2 - x_1)]}$

Equations of motion can be written for each of the masses by using Newton's second law:

$$\begin{cases} m\dfrac{d^2x_1}{dt^2} = -\dfrac{mg}{L}x_1 + k(x_2 - x_1) & (1) \\ m\dfrac{d^2x_2}{dt^2} = -\dfrac{mg}{L}x_2 - k(x_2 - x_1) & (2) \end{cases}$$

The Symmetry Method to solve the system of second order differential equations

The two equations are symmetric. We add and subtract them:

$$\begin{cases} m\dfrac{d^2(x_1 + x_2)}{dt^2} = -\dfrac{mg}{L}(x_1 + x_2) \\ m\dfrac{d^2(x_1 - x_2)}{dt^2} = -m\left(\dfrac{g}{L} + \dfrac{2k}{m}\right)(x_1 - x_2) \end{cases} \quad \textbf{(3)}$$

We obtained equations that look like the SHM. We have obtained **independent** oscillations in $(x_1 + x_2)$ and $(x_1 - x_2)$. We solve for $(x_1 + x_2)$, $(x_1 - x_2)$:

$\mathbf{x_1 + x_2 = A \cos w_p t}$

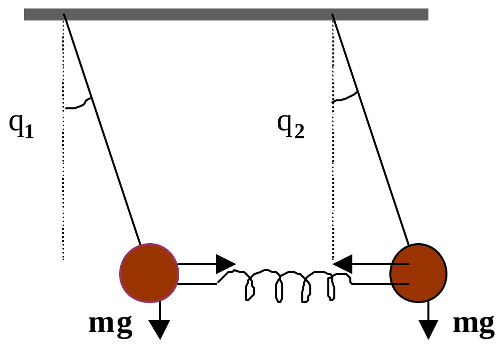
$\mathbf{x_1 - x_2 = B \cos w_s t}$

where $\omega_p = \sqrt{\dfrac{g}{L}}$ and $\omega_s = \sqrt{\dfrac{g}{L} + 2\dfrac{k}{m}}$ (4)

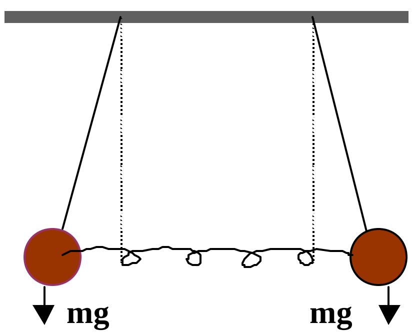
ω_p and ω_s are called the **normal** frequencies.

In writing the solutions, we need to apply the initial conditions. For this we have to distinguish between the two oscillating patterns:

(1) parallel oscillation:



and (2) symmetric oscillation:



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1) Parallel oscillation:

Let B = 0, $x_1 - x_2 = 0$

The two pendulums are moving in parallel. The spring does **nothing (as if it didn't exist).** They both oscillate with ω_p = the natural frequency of free pendulum without coupling. **This is the first (lower) normal mode of oscillation.**

2) Symmetric Oscillation:

Let A = 0, $x_1 + x_2 = 0$

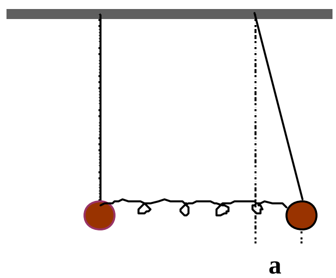
The spring gets expanded/shrunk by **twice** the movement of each pendulum.

$\omega_s = \sqrt{\frac{g}{L} + 2\frac{k}{m}}$ is determined by **both the pendulum and spring**. Each pendulum oscillates with frequency ω_s but they are out of phase by π. **This is the second (higher) normal mode.**

General solution of the system of differential equations is a linear combination of normal modes:

$$
\begin{aligned}
x_1(t) &= \frac{A}{2}\cos\omega_p t + \frac{B}{2}\cos\omega_s t \\
x_2(t) &= \frac{A}{2}\cos\omega_p t - \frac{B}{2}\cos\omega_s t
\end{aligned}
\qquad (5)
$$

We need a solution that satisfies an initial condition. Let's use an **example**:



Initial conditions for the two masses are: $\begin{cases} x_1(0) = a \\ \dot{x}_1(0) = \frac{dx_1}{dt}\Big|_{t=0} = 0 \end{cases}$ and $\begin{matrix} x_2(0) = 0 \\ \dot{x}_2(0) = 0 \end{matrix}$

$$
\begin{cases} a = \dfrac{A}{2} + \dfrac{B}{2} \\ 0 = \dfrac{A}{2} - \dfrac{B}{2} \end{cases} \rightarrow \begin{rcases} A = a \\ A = B \end{rcases} \Rightarrow \quad A = B = a
$$

which results if the following solution:

$$
\begin{aligned}
x_1(t) &= \frac{a}{2}(\cos\omega_p t + \cos\omega_s t) \\
x_2(t) &= \frac{a}{2}(\cos\omega_p t - \cos\omega_s t)
\end{aligned}
$$

The two normal modes are:

$x_1 + x_2 = a\cos\omega_p t$ first (lower) mode

$x_1 - x_2 = a\cos\omega_s t$ second (higher) mode

We define by $q_1 = (x_1 + x_2)$ and $q_2 = (x_1 - x_2)$ the normal coordinates of the system

If we use: $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$x_1(t) = a\cos\left(\frac{\omega_p + \omega_s}{2}t\right)\cos\left(\frac{\omega_p - \omega_s}{2}t\right)$$

$$x_2(t) = -a\sin\left(\frac{\omega_p + \omega_s}{2}t\right)\sin\left(\frac{\omega_p - \omega_s}{2}t\right)$$

are the two individual oscillations

If $|\omega_p - \omega_s| << \omega_p + \omega_s$, the spring constant is very small and the coupling between the two pendulums is very weak.

$$\omega_p = \sqrt{\frac{g}{L}}; \quad \omega_s = \sqrt{\frac{g}{L} + 2\frac{k}{m}}$$

are the two normal frequencies.

We notice that in each normal mode, the individual oscillators oscillates with the same normal frequency

Observation. Up to now, we have studied only coupled oscillations of the **same angular frequency**. If the two frequencies are different, we obtain **beats**: modulations of amplitude produced by two oscillations of **slightly different frequencies.**