Dynamical Model of Directly Modulated Semiconductor Laser Diodes

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Abstract—We present a new dynamical model for a directly modulated semiconductor diode, applicable to systems in which the dynamical time scales of interest are longer than the round-trip time of light in the diode. Employing a multiple scales analysis to simplify the familiar phenomenological equations, we find that the dynamical response of the diode can be described by time-dependent reflection and transmission coefficients for the electric field and one ordinary differential equation for the integrated carrier density. We do not assume that the photon and carrier densities are uniform along the diode and do not need to calculate them explicitly at each point. Additionally, we need not restrict ourselves to only a small-signal response. We justify the multiple scales analysis for parameters corresponding to typical structures through a comparison of the numerical solution of our results and a direct numerical integration of the original phenomenological equations.

Index Terms—Dynamic response, modulation, numerical analysis, semiconductor device modeling, semiconductor lasers.

I. INTRODUCTION

The semiconductor diode depicted schematically in Fig. 1, with antireflective (AR) coating on one end and arbitrary reflectivity at the other, is an element in many optical device structures. Recently, extended cavity lasers that include a diode with AR coating, such as the fiber grating router laser (MFL) [5], [6], have been considered for application in telecommunication systems. Both the FGL and MFL have cavities on the order of centimeters and can involve the direct modulation of the current across the diode. In addition, the diode structure shown in Fig. 1 can function as a simple amplifier, where the light input is coupled into the diode at the AR coated end and the output signal exits at the other, or as an amplifier within more complicated structures [7]. Our aim is to develop a model of the semiconductor diode that can be applied to a study of the dynamics of composite systems such as extended cavity lasers and amplifier structures.

To date, much of the theoretical understanding of the nonlinear dynamics of modulated semiconductor lasers and amplifiers has been based on detailed numerical analyses, typically employing a model based on traveling wave equations for the electric field and a phenomenological equation for the carrier density throughout the diode [8]. The direct numerical integration of these equations is straightforward but can require significant computation time. Furthermore, the underlying physics of diode behavior is not always clear from the numerical solutions. Several methods involving the simplification of the phenomenological equations have already been developed. Many studies are based on single rate-equation models that ignore the longitudinal spatial dependence of the photon and carrier densities [9]. This is inadequate for many systems, even under typical CW operation. This approach can be extended to models in which multiple rate equations are employed to study the dynamics of side modes in laser cavities [10], [11], but becomes complicated when many mode equations must be considered. Analytic studies of directly modulated semiconductor diodes have mainly focused on small-signal linear analysis [12], [13]. We achieve a simpler, more analytic understanding of the dynamics of the semiconductor diode that is general enough to account for arbitrarily large current modulation and yet does not involve an assumption of uniform fields and carrier density along the diode.

Due to the nonlinear nature of the equations describing the field and carrier density in the semiconductor diode, even the steady-state solutions must be found numerically. At first sight, therefore, it would appear unlikely that the dynamics of this system could be described without a full numerical treatment. However, the problem can be greatly simplified if the characteristic dynamical time scales of interest—such as the round-trip time in an external cavity, the rise and fall times of current modulation, and the rise and fall times of any input electric field intensity—are typically much longer than the round-trip time of light in the semiconductor diode.
itself. In this limit, a multiple scales analysis can be used to reduce the description of the dynamical response of the laser diode to the solution of a single nonlinear ordinary differential equation for the total carrier density in the diode, involving only the fields at the boundaries of the diode and the injected current. The resulting reflected and transmitted fields of the diode are then easily calculated, since they can be expressed as a product of the incoming field and time-dependent reflection and transmission coefficients that depend on the total carrier density in the diode (and its derivative) and the input current. The complicated dynamical evolution of the fields and carrier density within the diode thus does not need to be determined explicitly.

The phenomenological semiconductor equations which form the basis of our theoretical model are presented in Section II and are there reduced to dimensionless form for convenience. In Section III, we present typical time-independent solutions of these equations for the appropriate boundary conditions in terms of effective reflection and transmission coefficients of the diode for a given input electric field and current density. We use steady-state solutions to form the basis of the dynamical solutions and perform a multiple scales analysis in Section IV, which contains our main result: a new formalism that provides a simple description of the dynamical response of the semiconductor diode. Section V justifies the assumptions made for the multiple scales analysis by presenting a comparison of the numerical simulations based on this formalism and those obtained by the usual approach of directly solving the phenomenological equations numerically. The limitation to an AR coated device is done purely for simplicity and could easily be relaxed. Conclusions are given in Section VI.

II. PHENOMENOLOGICAL EQUATIONS

We begin our analysis with the usual phenomenological equations that describe the electric field and carrier density in semiconductor active media [10], [14], [15]

\[ \frac{\partial E_\pm}{\partial t} \pm v_g \frac{\partial E_\pm}{\partial z} = \frac{1}{2} \gamma_1 g_0 \left[ \frac{(N - N_t)}{1 + \tilde{c}S} - i\beta_1 N \right] E_\pm \]  

(1)

\[ \frac{\partial N}{\partial t} = \frac{J}{\epsilon d} - \frac{N}{T_n} - B N^2 - C N^3 \]

\[ - v_g \frac{\partial (N - N_t)}{1 + \tilde{c}S} S \]

(2)

where \( N(z, t) \) is the electron–hole (carrier) density with transparency value \( N_t \) and \( E_+(z, t) \) and \( E_-(z, t) \) are slowly varying envelope functions describing the forward- and backward-propagating field amplitudes, respectively, where the total electric field is written as

\[ E(z, t) = E_+(z, t)e^{i(k_o z - \omega_o t)} + E_-(z, t)e^{-i(k_o z + \omega_o t)} + c.c. \]

(3)

where \( \omega_o \) is the frequency of the incoming field and \( k_o \) is the corresponding wavenumber in the passive structure. The group velocity in the absence of pumping is denoted by \( v_g \), \( a \) is the differential gain, \( \Gamma \) is the fraction of the mode volume occupied by the gain medium, and \( \tilde{c} \) is the parameter describing gain compression, an effect usually attributed to carrier heating or spectral hole burning [14], [16]. The linewidth enhancement factor \( \beta_1 \) describes the refractive index change in the medium associated with electron–hole pair injection, \( J \) is the pump current density across the semiconductor diode, \( e \) is the electronic charge, and \( d \) is the effective thickness of the gain medium. The carrier density recombination is controlled by a carrier relaxation time \( T_n \), the quadratic recombination coefficient \( B \), and the Auger coefficient \( C \) [10], [15]. We denote the length of the gain medium by \( L_g \). Since in this paper we are focusing on field and carrier density dynamics of the diode only above the gain medium threshold, we have neglected noise sources in (1) and (2). We define effective photon densities associated with the forward- and backward-propagating beams as

\[ S_\pm(z, t) = \frac{n_o^2 |E_\pm(z, t)|^2}{2\pi} \]

where \( n_o \) is the background index of refraction in the semiconductor. Equation (1) then yields the usual photon density equations

\[ \frac{\partial S_\pm}{\partial t} \pm v_g \frac{\partial S_\pm}{\partial z} = \Gamma v_g o \frac{(N - N_t)}{1 + \tilde{c}S} S_\pm. \]

(4)

The total effective photon density \( S(z, t) \) is given by \( S_+(z, t) + S_-(z, t) \).

Equations (1) and (2) serve as the basic equations of our analysis and must be supplemented by a boundary condition at the back end of the semiconductor diode

\[ \frac{E_+(\pm L_g, t)}{E_-(\pm L_g, t)} = R e^{i\chi_o^R} e^{i\chi_o L_g} \]

(5)

where \( R^2 \) is the reflectivity of the back end and \( \chi_o^R \) is the appropriate phase factor. Alternatively, for describing transmission through the interface (important for single-pass devices), we use

\[ \frac{E_{\text{oout}}(t)}{E_{\text{oout}}(-L_g, t)} = \Gamma e^{i\chi_o^T} \]

(6)

where \( E_{\text{oout}}(t) \) is the outgoing field at \( z = \pm L_g \), \( T^2 \) is the transmissivity of the interface, and \( \chi_o^T \) is again the appropriate phase factor. For this paper, we assume an AR coated device with no back-reflections at the front end of the diode, so that \( E_{\text{oout}}(0, t) \) is the input field and \( E_{\text{oout}}(0, t) \) is the output field at \( z = 0 \). This assumption can be lifted by considering that the input field includes some fraction of \( E_{\text{oout}}(0, t) \).

The analysis of these equations can be greatly simplified by the introduction of dimensionless variables that respect the length scales and time scales relevant to the problem. Three important time scales can be identified. In addition to the carrier relaxation time \( T_n \), there is time for light to propagate from the front end of the semiconductor diode to the back end and back out again

\[ T_g \equiv \frac{2L_g}{v_g} \]

\[ 1 \text{The linewidth enhancement factor, sometimes called the antiguiding parameter [10], is often denoted by } \alpha. \]
and a scale identifying the shortest characteristic time over which the changes of interest in the semiconductor diode occur, \( T_p \). This time could be, for example, the round-trip time in an extended cavity or the rise and fall times of the electronic current in directly modulated systems. It is useful to use the time \( T_p \) to define two important ratios
\[
\lambda \equiv T_n/T_p, \\
\mu \equiv T_B/T_p.
\]
(7)

For a diode length of 400 \( \mu \)m and an effective refractive index of \( n_o \approx 3.7 \), \( T_g \) is about 10 ps. For a carrier relaxation time \( T_n \approx 1 \) ns and \( T_p \gg 10 \) ps, then \( \lambda \approx 5 \) and \( \mu \ll 1 \).

We describe the dynamics in terms of scaled time and length variables
\[
\hat{t} \equiv \frac{t}{T_p}, \\
\hat{z} \equiv \frac{z}{2L_g}.
\]

As \( t \) ranges over the time \( T_p \), \( \hat{t} \) ranges from zero to unity and, as a beam enters one end of the gain medium, propagates to the back end, reflects, propagates back through the medium and exits again, a scaled distance of unity is involved. We take the front end of the diode to be at \( \hat{z} = 0 \) and the back end at \( \hat{z} = -1/2 \). It is also convenient to write the electric field amplitudes \( E_{\pm} \) and the carrier density \( N \) in terms of dimensionless variables \( c_{\pm} \) and \( n \), respectively. We reference the carrier density to the transparency density and normalize it by
\[
N_o \equiv \frac{1}{2L_o}.
\]

We determine
\[
n(\hat{z}, \hat{t}) = \frac{N(\hat{z}, \hat{t}) - N_t}{N_o}.
\]

Similarly, we set
\[
\hat{E}(\hat{z}, \hat{t}) \equiv \frac{E(\hat{z}, \hat{t})}{E_o} = c_{\pm}(\hat{z}, \hat{t}) e^{i(\hat{z}/2)(N_t/N_o)\hat{z}}
\]

where \( E_o \) is a (positive) reference electric field amplitude, given in terms of a reference photon density \( S_o \) defined by
\[
S_o = \frac{\hbar^2 E_o^2}{2 \pi n_o \omega_o} = \frac{1}{\nu_o a T_n}.
\]

In terms of the dimensionless variables \( c_{\pm} \) and \( n \), (2) becomes
\[
\lambda \frac{\partial n}{\partial \hat{t}} = (j(\hat{t}) - a_0) - a_1 n - a_2 n^2 - a_3 n^3 - \frac{ns}{1 + \epsilon s}.
\]
(9)

where \( s_{\pm} \equiv |c_{\pm}|^2 = S_{\pm}/S_o \) and \( s = s_+ + s_- \). We define a dimensionless function associated with the current density
\[
j(\hat{t}) = \frac{J(\hat{t})}{e \nu_0 a T_n} \frac{T_n}{N_o}
\]

and a dimensionless gain compression factor
\[
\epsilon = \frac{\hat{\epsilon}}{\nu_0 a T_n}
\]

TABLE I

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Differential gain</td>
<td>( a )</td>
<td>( 3 \times 10^{-16} \text{ cm}^2 )</td>
</tr>
<tr>
<td>Linewidth enhancement factor</td>
<td>( \beta )</td>
<td>4.0</td>
</tr>
<tr>
<td>Background index of refraction</td>
<td>( n_o )</td>
<td>3.7</td>
</tr>
<tr>
<td>Gain compression factor</td>
<td>( \hat{\epsilon} )</td>
<td>( 1 \times 10^{-17} \text{ cm}^{-3} )</td>
</tr>
<tr>
<td>Transparency carrier density</td>
<td>( N_t )</td>
<td>( 1.5 \times 10^{18} \text{ cm}^{-3} )</td>
</tr>
<tr>
<td>Carrier relaxation time</td>
<td>( T_n )</td>
<td>1.0 ns</td>
</tr>
<tr>
<td>Quadratic recombination</td>
<td>( B )</td>
<td>( 1.0 \times 10^{-10} \text{ cm}^3 \cdot \text{s}^{-1} )</td>
</tr>
<tr>
<td>Auger recombination</td>
<td>( C )</td>
<td>( 3.0 \times 10^{-29} \text{ cm}^6 \cdot \text{s}^{-1} )</td>
</tr>
<tr>
<td>Mode confinement factor</td>
<td>( \Gamma )</td>
<td>0.34</td>
</tr>
<tr>
<td>Effective waveguide thickness</td>
<td>( d )</td>
<td>0.15 ( \mu )m</td>
</tr>
<tr>
<td>Effective waveguide width</td>
<td>( w )</td>
<td>1.0 ( \mu )m</td>
</tr>
</tbody>
</table>

These are only meant to be typical values, see, e.g., [15].

typically, \( \epsilon \ll 1 \). The parameters
\[
a_o = \frac{N_t}{N_o} [1 + T_n N_t (B + CN_t)]
\]

\[
a_1 = 1 + T_n N_t (2B + 3CN_t)
\]

\[
a_2 = T_n N_o (B + 3CN_t)
\]

\[
a_3 = CT_n N_o^2
\]

are dimensionless coefficients arising from the various carrier recombination terms. Typical values for device parameters, such as those listed in Table I, yield \( a_o \approx 15, a_1 \approx 1.5, a_2 \approx 0.03, a_3 \approx 0.0005 \), and \( \epsilon \approx 0.004 \). As we show later, we can neglect \( a_2 n^2 \) and \( a_3 n^3 \) in (9).

With these considerations, we can write the final dimensionless versions of (1), (2):
\[
\mu \hat{c}_{\pm} = \frac{\partial \hat{c}_{\pm}}{\partial \hat{z}} = \frac{1}{2} \left[ \frac{1}{1 + \epsilon s} - i \frac{\hat{\epsilon}}{\hat{c}} \right] n \hat{c}_{\pm},
\]
(10)

\[
\lambda \frac{\partial n}{\partial \hat{t}} = (j(\hat{t}) - a_0) - a_1 n - \frac{ns}{1 + \epsilon s}.
\]
(11)

From (10), we find
\[
\mu \hat{s}_{\pm} = \frac{\partial \hat{s}_{\pm}}{\partial \hat{z}} = \frac{ns_{\pm}}{1 + \epsilon s},
\]
(12)

which are (4) in dimensionless form. Equations (10)–(12), along with the appropriate normalized boundary condition, are the starting point of our analysis in the following sections.

III. THE STEADY STATE

The steady-state properties of semiconductor laser diodes have been well studied theoretically [10]–[12]. Since the approximate solutions we identify when we consider the full dynamics in the next section will be referenced to a time-independent solution of (10)–(12), we outline in this section a particular method of characterizing such a steady-state solution. For a given incident field and pumping level, unique effective reflection and transmission coefficients of the diode are defined which, in general, must be found numerically.
The (normalized) steady-state fields \( \mathcal{E}_\pm(z) \) and carrier density above transparency \( \mathcal{N}(z) \) must satisfy the equations

\[
\pm \frac{d\mathcal{E}_\pm}{dz} = \left[ \frac{1}{1+c\mathcal{S}} - i\beta_c \right] \mathcal{N} \mathcal{E}_\pm
\]

\[
0 = (\overline{J} - a_o) - \mathcal{N} \left[ \alpha_1 + \frac{\mathcal{S}}{1+c\mathcal{S}} \right]
\]

where the “hats” on the dimensionless variables \( \mathcal{\hat{t}}, \mathcal{\hat{z}} \) have been dropped (as they are from this point on), and where \( \mathcal{S}(z) = \mathcal{S}_+(z) + \mathcal{S}_-(z) = |\mathcal{E}_+(z)|^2 + |\mathcal{E}_-(z)|^2 \). The corresponding steady-state equations for the normalized photon densities are

\[
\pm \frac{d\mathcal{S}_\pm}{dz} = \frac{\mathcal{N} \mathcal{S}_\pm}{1+c\mathcal{S}}.
\]

The boundary conditions (5) and (6) to which these equations are subject become

\[
\mathcal{E}_+(z=-1/2) = R e^{i\chi_R} \mathcal{E}_-(-1/2)
\]

\[
\mathcal{E}_{\text{ext}}(z=-1/2) = T e^{i\chi_T} \mathcal{E}_-(-1/2)
\]

where \( \chi_R = \chi_0^R + 2\mu L + \beta \mathcal{N}_L / N_0 \) and

\[
\mathcal{E}_{\text{ext}}(z=-1/2) = T e^{i\chi_T} \mathcal{E}_-(-1/2)
\]

where \( \chi_T = \chi_0^T + \beta \mathcal{N}_L / (2N_0) \). For a given incident field \( \mathcal{E}_-(0) \), the effective steady-state reflection and transmission coefficients of the (front-end AR-coated) device can be found by constructing a self-consistent relation for the electric field from (13). Substituting (14) into (13) and formally integrating, we obtain

\[
\mathcal{E}_\pm(z) = \mathcal{E}_\pm(-1/2) \times \exp \left[ \pm \frac{\mathcal{S}}{2} \int_{-1/2}^{z} \frac{1-i\alpha [1+c\mathcal{S}(z')]}{\alpha_1 + [1+c\mathcal{S}(z')]^2} dz' \right].
\]

The reflection and transmission coefficients of the semiconductor diode are then given by

\[
\mathcal{R} \equiv \frac{\mathcal{E}_+(-1/2)}{\mathcal{E}_-(0)}
\]

\[
\mathcal{T} \equiv \frac{T e^{i\chi_T} \mathcal{E}_-(-1/2)}{\mathcal{E}_-(0)}
\]

Note from (18) that, in general, \( |\mathcal{R}|^2 R = T^2 |\mathcal{T}|^2 \), even if \( T^2 + R^2 \neq 1 \).

The coefficients \( \mathcal{R} \) and \( \mathcal{T} \) can now be calculated numerically for a given pump parameter \( \mathcal{J} \) and input electric field \( \mathcal{E}_-(0) \). Assuming a reasonable value for \( \mathcal{E}_-(0) \), and using the boundary condition, we integrate forward along the length of the diode to find \( \mathcal{E}_\pm(z) \). To calculate the successive \( \mathcal{E}_\pm(z+\Delta z) \), we implement a numerical quadrature algorithm using (18), the right-hand side of which only depends upon the field values at previous points. We use our calculated values of \( \mathcal{E}_\pm(0) \) to obtain \( \mathcal{R} \) and \( \mathcal{T} \), and, performing the calculation for a range of \( \mathcal{E}_-(1/2) \), we then map out the space of steady-state solutions. Alternately, to find the response for a specified \( \mathcal{E}_-(0) \), we begin with a guess for \( \mathcal{E}_-(1/2) \) and then iterate this integration procedure, using \( \mathcal{R} \) to get a better estimate for \( \mathcal{E}_-(1/2) \).

Fig. 2. Steady-state profiles of the carrier density above transparency (dashed line) and the effective powers associated with the forward (solid line) and backward (dotted line) propagating electric field amplitudes for an input power of 1.0 mW, bias current of 41.12 mA, and a back face reflectivity of 1.0. The diode parameters are specified in Table I. Clearly, the carrier density and total power are far from uniform. The dimensionless parameter values for this particular steady-state solution are \( |\mathcal{E}_-(0)|^2 = 0.5306 \), \( \mathcal{J} = 34.91 \), and \( |\mathcal{R}|^2 = 15.85 \).

From (15), it is obvious that the product \( (S_+ S_-) \), and not the total normalized photon density \( (S_+ + S_-) \), is uniform in \( \mathcal{Z} \). For a typical value of the pumping parameter \( \mathcal{J} \), this nonuniformity, as well as that of \( \mathcal{N} \), is clearly seen in Fig. 2, which shows the profiles of the carrier density above transparency and the effective powers associated with the field amplitudes, defined by [7]

\[
\mathcal{P}_\pm(z) = \left( \frac{\nu d}{1} \right) v_g \omega_0 \mathcal{S}_\pm(z) = \left( \frac{\nu d}{1} \right) \frac{\nu \omega_0}{\alpha_n} |\mathcal{E}_\pm(z)|^2
\]

where \( \mathcal{P} \) is Planck’s constant over \( 2\pi \), \( c \) is the speed of light in vacuum, and \( w \) is the effective width of the semiconductor diode waveguide. This illustrates that the often-made simplifying assumption of uniform carrier and photon densities is not valid for some typical systems.

Before considering the dynamical equations, it is useful to rewrite the solutions of (13)–(15) in the following way. First, we note that we can solve for \( \mathcal{S} \) as a function of \( \mathcal{N}(z) \)

\[
\mathcal{S}(\mathcal{N}) = \frac{(\mathcal{J} - a_o) - a_1 \mathcal{N}}{\mathcal{N} - \epsilon (\mathcal{J} - a_o - a_1 \mathcal{N})}
\]

which is valid provided that we do not cross the threshold of the gain medium, where \( \mathcal{J} = a_o \) and \( \mathcal{N} = 0 \). This will hold for our systems of interest, as discussed later. Using (19), we can make the simplification

\[
\left( \frac{\mathcal{N}}{1+\epsilon} \right) \simeq \frac{\mathcal{N} - \epsilon (\mathcal{J} - a_o - a_1 \mathcal{N})}{\mathcal{N} - \epsilon (\mathcal{J} - a_o - a_1 \mathcal{N})}
\]

since \( \epsilon \ll 1 \), and then (13) gives

\[
\mathcal{E}_\pm(z) = \mathcal{E}_\pm(-1/2) e^{i\chi(z)} \int_{-1/2}^{z} \mathcal{S}(z') dz' \times e^{i\chi(z')} (\mathcal{J} - a_o) (z+1/2).
\]

We restrict ourselves to the lowest order in \( \epsilon \) throughout this paper, both with respect to our steady-state solutions in this section and the dynamical solutions in Section IV. An important quantity for the dynamical equations will be the (normalized) carrier density above transparency encountered...
by a beam as it enters the diode, propagates through to the back end, reflects, and propagates back out again

\[ \sigma(t) \equiv 2 \int_{-1/2}^{0} n(z,t) \, dz \]

and we define its steady-state counterpart as

\[ \bar{\sigma} = 2 \int_{-1/2}^{0} n(z) \, dz \]

which appears in (21). In terms of \( \bar{\sigma} \), the steady-state reflection and transmission coefficients of the diode can be rewritten as

\[ R = R_0 e^{\chi^R_c(1/2)(-1-\Delta) \bar{\sigma}} \]
\[ T = T_0 e^{\chi^T_c(1/4)(-1-\Delta) \bar{\sigma}} \]

and the diode reflectivity and transmissivity are

\[ |R|^2 = R_0^2 e^{-\chi^R_c(1-\Delta) \bar{\sigma}} \]
\[ |T|^2 = T_0^2 e^{-\chi^T_c(1-\Delta) \bar{\sigma}}. \]

We see from (24) and (25) that the gain level, not surprisingly, depends primarily on the (normalized) integrated carrier density above threshold \( \bar{\sigma} \). Since \( \epsilon \ll 1 \), there is only a small correction due to gain compression. Note that, from the steady-state carrier density equation (14), the stimulated emission loss term \( -\bar{\sigma} \bar{\sigma} \) is equal to

\[ \lambda \frac{\partial \bar{n}}{\partial t} = j(t) - \bar{n} \left( \bar{\sigma}_0 + \frac{s}{1+\epsilon s} \right) \frac{1}{\bar{\sigma}} \frac{\partial \bar{n}}{\partial t}. \]

The integral of (26) must also be negative so the effect of gain compression above transparency is to decrease the gain by the factor \( e^{-\chi^R_c(1/2)(1-\Delta) \bar{\sigma}} \) in (24), or \( e^{-\chi^T_c(1/4)(1-\Delta) \bar{\sigma}} \) in (25). Since we want to consider the chirp induced in the output electric field by the semiconductor diode when we look at the full dynamical system, it is also useful to note from (22) and (23) that the phases of the output reflected and transmitted fields relative to the input field are given by \( \beta \bar{\sigma}_c \) and \( \beta \bar{\sigma}_T \), respectively. Of course, these are in addition to the phase shifts \( \chi^R_c \) and \( \chi^T_c \).

IV. DYNAMICAL EQUATIONS

We now present the multiple scales analysis that we use to obtain a new description of the dynamical response of the semiconductor diode under modulation of the input field and/or current density. We proceed by seeking a solution of the electric field amplitudes and photon densities in the diode of the form

\begin{align*}
\mathcal{E}_\pm(z,t) &= \mathcal{E}_\pm(z)e^{\theta_\pm(z,t) + (1/2)e_\pm(z,t)} \\
\mathcal{S}_\pm(z,t) &= \mathcal{S}_\pm(z)e^{\xi_\pm(z,t)}
\end{align*}

where \( \mathcal{E}_\pm(z) \) and \( \mathcal{S}_\pm(z) \) are the time-independent solutions corresponding to a steady-state reference characterized by an incident field \( \mathcal{E}_0(z) \), pumping parameter \( \bar{j} \), and either the resulting reflection coefficient \( R \) or transmission coefficient \( T \). This “reference” solution can be any steady-state solution that is convenient to use. Since the fields \( \mathcal{E}_\pm(z,t) \) satisfy the same boundary condition at \( z = -1/2 \) as required for the steady-state fields \( \mathcal{E}_\pm(z) \) [see (16)], \( \theta_\pm \) and \( \xi_\pm \) must satisfy the boundary conditions

\[ \theta_\pm(-1/2,t) = \theta_\pm(-1/2,t) \]
\[ \xi_\pm(-1/2,t) = \xi_\pm(-1/2,t). \]

Substituting (27) into (10), making the approximation

\[ \frac{n}{1+\epsilon s} \approx n - \epsilon \left( j(t) - \alpha_0 - \alpha_0 n - \lambda \frac{\partial \bar{n}}{\partial t} \right) \]

by first solving for \( s(z,t) \) in (11) in the same spirit as (20) in Section III, and using the fact that the steady-state solutions \( \mathcal{E}_\pm(z) \) satisfy (13), we find that \( \theta_\pm \) and \( \xi_\pm \) must satisfy

\[ \mu \frac{\partial \theta_\pm}{\partial t} \pm \frac{\partial \theta_\pm}{\partial z} = \frac{1}{2} \beta \bar{n} \]
\[ \mu \frac{\partial \xi_\pm}{\partial t} \pm \frac{\partial \xi_\pm}{\partial z} = \bar{n} - \epsilon \left( j(t) - \bar{j} - \alpha_0 \bar{n} - \lambda \frac{\partial \bar{n}}{\partial t} \right) \]

where

\[ \bar{n}(z,t) \equiv n(z,t) - \bar{n}(z). \]

Equation (11) then leads to the following dynamical equation for \( \bar{n}(z,t) \), where \( \alpha_0 \) has been substituted from (14):

\[ \lambda \frac{\partial \bar{n}}{\partial t} = j(t) - \bar{j} - \bar{n} \left( \bar{\sigma}_0 + \frac{s}{1+\epsilon s} \right) \frac{1}{\bar{\sigma}} \frac{\partial \bar{n}}{\partial t} \]
\[ - \bar{\sigma} \left[ \frac{s}{1+\epsilon s} - 1 \right] \]

In order to simplify the last term, we make the approximation

\[ \frac{s}{1+\epsilon s} \approx \frac{s}{1+\epsilon s} \left[ 1 - \epsilon (s - \bar{s}) + O(\epsilon^2) \right] \]

which, unlike the direct Taylor expansion of \( 1/(1+\epsilon s) \), explicitly allows the last term of (33) to vanish in the steady state. We then rewrite this last term as

\[ - \bar{n} \left[ \frac{s - \bar{s}}{1+\epsilon s} \right] \frac{1}{1-\epsilon s} = \left[ u_\pm \frac{\partial \bar{\sigma}_\pm}{\partial z} - u_\pm \frac{\partial \bar{\sigma}_\pm}{\partial z} \right] \left( 1-\epsilon s \right) \]

where, in the last step, we have used the form for \( s_{\pm}(z,t) \) given by (28), used (15) to substitute \( \bar{\sigma}_{\pm}(1+\epsilon s) \), and where we have defined

\[ u_\pm(z,t) = 1 - \epsilon \xi_\pm(z,t). \]

Finally, noting that

\[ \frac{\partial}{\partial z} \left[ \mathcal{E}_\pm \right] = u_\pm \frac{\partial \mathcal{E}_\pm}{\partial z} + \bar{\sigma}_\pm \epsilon \xi_\pm \frac{\partial \mathcal{E}_\pm}{\partial z} \]

we use (31) to write \( \partial \xi_\pm/\partial z \) in terms of \( \partial \xi_\pm/\partial t \). Substituting (34) back into (33), keeping only the terms up to first order in \( \epsilon \), then dividing both sides by \( (1-\epsilon s) \), we obtain the final form of the dynamical equation for \( \bar{n}(z,t) \):

\[ \lambda \frac{\partial \bar{n}}{\partial t} - j(t) + \bar{j} + \alpha_0 \bar{n} \]
\[ = \left( \mu \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) [\mathcal{E}_\pm \mathcal{S}_\pm] + \left( \mu \frac{\partial}{\partial t} - \frac{\partial}{\partial z} \right) [\mathcal{S}_\pm \mathcal{E}_\pm]. \]
This equation, along with (30), (31), and the boundary conditions (29), define the dynamical problem we now address.

Our dynamical variables are now \( \tilde{n}(z, t), \tilde{\theta}_\pm(z, t), \) and \( \xi_\pm(z, t) \). We perform a standard multiple scales analysis [17] by seeking solutions of the form
\[
\tilde{n} = \tilde{n}^{(0)}(z; t_0, t_1, \cdots) + \mu \tilde{n}^{(1)}(z; t_0, t_1, \cdots) + \cdots
\]
\[
\theta_\pm = \theta^{(0)}_\pm(z; t_0, t_1, \cdots) + \mu \theta^{(1)}_\pm(z; t_0, t_1, \cdots) + \cdots
\]
\[
\xi_\pm = \xi^{(0)}_\pm(z; t_0, t_1, \cdots) + \mu \xi^{(1)}_\pm(z; t_0, t_1, \cdots) + \cdots
\]
and
\[
w_\pm = w^{(0)}_\pm(z; t_0, t_1, \cdots) + \mu w^{(1)}_\pm(z; t_0, t_1, \cdots) + \cdots
\]
\[
= (1 - e^{(0)}_\pm) - \mu e^{(1)}_\pm + \cdots
\]
where the functions \( f^{(m)}(z; t_n) \) are all assumed to be, at most, of order unity and vary significantly at most only as each of their temporal arguments \( t_n \equiv \mu t \) varies over a range of unity. For functions of the indicated form, we have
\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \mu \frac{\partial}{\partial t_1} + \cdots.
\]

We now insert (37) and (38) into (30), (31), and (36), expand the time derivatives on their different scales, and collect the terms in the equations that depend on different powers of \( \mu \). The philosophy of the approach is that satisfying these sets of equations to higher and higher orders in \( \mu \) should give (asymptotically) better and better approximations of the exact solutions. Note that, if we have \( \lambda > 1 \), the multiple scales analysis is still valid because, by simply dividing (36) through by \( \lambda \), each term multiplied by a factor \( \mu^m \) is made smaller, thus ensuring they are of order unity or less. We will restrict ourselves here to lowest order in \( \mu \) by considering only the terms of order \( \mu^0 \) since \( \mu \ll 1 \); the validity of this approximation will be examined in detail in Section V. To this order, (30) and (31) yield
\[
\pm \frac{\partial \tilde{n}(0)}{\partial z} = -\frac{\beta_c}{2} \tilde{n}(0)
\]
\[
\pm \frac{\partial \xi_\pm(0)}{\partial z} = \tilde{n}(0) - \epsilon \left[ j - \tilde{j} - a_\lambda \tilde{n}(0) - \lambda \frac{\partial \tilde{n}(0)}{\partial t_0} \right]
\]
\[
\lambda \frac{\partial \tilde{n}(0)}{\partial t_0} = j - \tilde{j} - a_\lambda \tilde{n}(0) + \frac{\partial}{\partial z} \left[ \tilde{s}_+ w_+ + \tilde{s}_- w_- \right].
\]

The first and second equations of (39) can be formally integrated from \( z = -1/2 \) to \( z = 0 \) to obtain
\[
\theta^{(0)}_\pm(0; t_n) = \theta^{(0)}_\pm(-1/2; t_n) + \frac{\beta_c}{4} \tilde{\sigma}(0)
\]
\[
\xi^{(0)}_\pm(0; t_n) = \xi^{(0)}_\pm(-1/2; t_n) + \frac{1}{2} \tilde{\sigma}(0)
\]
\[
\tilde{n}(0) = 2 \int_{-1/2}^{0} \tilde{n}(z; t_n) \, dz.
\]

Recalling the forms of \( e_\pm(z, t) \) and \( s_\pm(z, t) \) from (27) and (28), and noting (40) and (41), we see that an equation for the time evolution of this function \( \tilde{\sigma}(0) \) is required if we wish to determine the dynamical reflection and transmission coefficients of the diode to lowest order in \( \mu \). Since the right-hand side of the third equation of (39) involves only a derivative with respect to \( z \), by integrating that equation in \( z \) we obtain a dynamical equation for \( \tilde{\sigma}(0) \) that only involves the electric field intensities at the front and back ends of the diode
\[
\lambda \frac{d \tilde{\sigma}(0)}{d t_0} = (j(t_n) - \tilde{j}) - a_\lambda \tilde{\sigma}(0) + 2 \tilde{s}_+ w_+(0; t_n)
\]
\[
-2 \tilde{s}_-(-1/2) w_+(-1/2; t_n)
\]
\[
-2 \tilde{s}_-(0) w_-^{(0)}(0; t_n) + 2 \tilde{s}_-(-1/2) w_-^{(0)}(-1/2; t_n).
\]

Since we are stopping the analysis at this level (\( \mu^0 \) of approximation), we put \( \tilde{\sigma}(0) \approx \tilde{n}, \theta^{(0)}_\pm \approx \theta^{(0)}_\pm, \xi^{(0)}_\pm \approx \xi^{(0)}_\pm, \) and \( w_\pm(0) \approx w_\pm = (1 - e^{(0)}_\pm) \) and neglect any time dependence on a scale higher than \( t_0 \). Then we can take
\[
\tilde{\sigma}(0)(t_n) \approx 2 \int_{-1/2}^{0} [n(z, t) - n(z)] \, dz \equiv \tilde{\sigma}(t).
\]

Respecting the boundary conditions (16) and (29), which must hold to each order of \( \mu \) in the expansions of (37), and using (21), (41) and the steady-state reflection coefficient (24), we rewrite all the photon densities in (42) in terms of \( s_-(0, t), \) the normalized incoming field intensity. Then, recalling that \( |e^{(0)}_-(0, t)|^2 = s^{(0)}_-(0) e^{(0)}_-(0, t) \), we get a nonlinear ordinary differential equation for \( \tilde{\sigma}(t) \)
\[
\lambda \frac{d \tilde{\sigma}(t)}{dt} = (j(t) - \tilde{j}) - a_\lambda \tilde{\sigma}(t) + 2 |e^{(0)}_-(0, t)|^2 \left[ |t|^2 + \frac{1 - R^2}{R} |t| - 1 \right]
\]
\[
-2 |e^{(0)}_-(-\frac{1}{2}, t)|^2 \left[ \tilde{\sigma}(t) \right]^2 + \frac{1 - R^2}{R} \tilde{\sigma}(t) - 1 \right]
\]
where the constants \( e^{(0)}_-(0, t), j, \) and \( \tilde{\sigma} \) characterize the reference steady-state solution, where \( \tilde{\sigma}(t) \) is defined in (45) as the time-dependent reflection coefficient of the diode, and where we are assuming \( R \neq 0 \) (i.e., the diode is not perfectly transmitting at the back end). Using (40), (41), and (27), we obtain
\[
\tilde{\sigma}(t) \equiv \frac{e^{(0)}_+(0, t)}{\tilde{\sigma}_-(0)}
\]
\[
= \tilde{\sigma}_-(0)(1/2, j, \tilde{\sigma}(t) - e^{(0)}_-(-\frac{1}{2}, t)|\tilde{\sigma}(t)| - N(d\tilde{\sigma}(t)/dt)].
\]

Equations (44) and (45) together give the time-dependent response of the semiconductor. Alternatively, in terms of the steady-state transmission coefficient, (44) and (45) become
\[
\lambda \frac{d \tilde{\sigma}(t)}{dt} = (j(t) - \tilde{j}) - a_\lambda \tilde{\sigma}(t)
\]
\[
+ 2 |e^{(0)}_-(-\frac{1}{2}, t)|^2 \left[ \frac{R^2}{T^2} |t|^4 + (1 - R^2) \frac{R}{T^2} |t|^2 - 1 \right]
\]
\[
-2 |e^{(0)}_-(-\frac{1}{2}, t)|^2 \left[ \frac{R^2}{T^2} |t|^4 + (1 - R^2) \frac{R}{T^2} |t|^2 - 1 \right]
\]
where now $\bar{t}(t)$ is the time-dependent transmission coefficient defined by

$$\bar{t}(t) = \frac{e_{out}(t)}{e_{in}(0,t)} = e^{(1/4)(1 - i \beta_{c}) \sigma(t) - (\epsilon/4)[(j(t) - j) + \sigma(t) - N d\sigma(t)/dt]}. \quad (47)$$

We used $[\bar{R}(t)]^2 R = T^2 [\bar{r}(t)]$ and $[\bar{R}(t)]^2 R = T^2 \bar{r}$ to derive (46) and assumed $T \neq 0$, i.e., there is some transmitted light at the back end; these expressions hold even if $T^2 + \bar{R}^2 \neq 1$. Equation (44) must be used if $T = 0$, and (46) if $R = 0$. Otherwise, either can be used, and they are, of course, equivalent. This is our main result in this section, and we refer hereafter to the new characterization of the semiconductor diode dynamical response given by (44) and (45) or by (46) and (47) as the “ODE model.”

Although both the reference steady-state fields $(\bar{r}_+(z), \bar{r}_-(z))$ and the time-dependent fields $(e_+(z,t), e_-(z,t))$ can vary greatly throughout the diode, the problem of finding the time-dependent reflectivity or transmissivity of the diode has been reduced to one involving only one ordinary differential equation. At this level of the multiple scales analysis, with respect to determining the reflectivity and transmissivity of the diode, the inhomogeneities in the fields are “integrated over” and it is only the function $\bar{\sigma}(t)$ that appears. From these time-dependent diode reflection and transmission coefficients, a physical picture of the overall dynamics of the response of the system emerges that is clearer than what is apparent from the original partial differential equations. We see from (45) that the relative time-dependent phase shift of the incident and outgoing electric fields is $-\beta_c \bar{\sigma}(t)/2$. The instantaneous frequency of the output field is then given by

$$\frac{\beta_c s}{4\pi} \bar{\sigma}(t) \quad (48)$$

which follows the time rate of change of the integrated carrier density $\bar{\sigma}$. The linewidth enhancement factor $\beta_c$ enters into the phenomenological equations (1) and (2) because they are derived with the assumption that the index of refraction varies linearly with the local carrier density, with a constant proportional to $\beta_c$ [10]. From this assumption, it follows that the average index of refraction in the medium, integrated over $z$, depends upon the average of the carrier density, with a constant proportional to $\bar{\sigma}(t) + \text{constant}$. Thus, the rate of change of the average index of refraction directly follows the rate of change of $\bar{\sigma}(t)$; it is no surprise, then, that the instantaneous frequency is (48).

Note that (48) is the rate of change of the relative phase of input and output fields of the semiconductor diode. If we substitute (22) into (45) and (23) into (47), we get

$$\frac{e_+(0,t)}{e_{in}(0,t)} = T e^{\beta_c \sigma(t)} (1/2)(1 - i \beta_{c}) \sigma(t) - (\epsilon/4)[(j(t) - j) + \sigma(t) - N d\sigma(t)/dt] \quad (49)$$

$$\frac{e_{out}(t)}{e_{in}(0,t)} = T e^{\beta_c \sigma(t)} (1/4)(1 - i \beta_{c}) \sigma(t) - (\epsilon/4) j + \sigma(t) - N d\sigma(t)/dt \quad (50)$$

where, recall from (43), $\sigma(t) = \bar{\sigma}(t) + \sigma$. Analogous to the time-independent case, we see that gain is dominated by $\sigma(t)$. The effect of gain compression is to decrease the gain by the factor $e^{-\beta_c \sigma(t)}$, which is proportional to $\beta_c$. From (11), we see that, as in the steady-state reflection and transmission coefficients, this corresponds to the total loss of carriers per unit time due to stimulated emission.

We emphasize that the particular steady-state solution chosen as a reference is arbitrary; there is no assumption here that $\sigma(t)$ is small. The only condition required for the validity of (44) and (45) or (46) and (47), besides the assumption that the dimensionless quantities are of order unity or less, is that the actual electric fields throughout the diode vary over a time scale much greater than the round-trip time in the semiconductor. In terms of the dimensionless time $t$, in these equations, the condition is that $e_+(z,t)$ and $e_-(z,t)$ vary little or, more correctly, their logarithms vary little, as $t$ varies on the order of unity. Finally, note that to employ (44) and (45) or (46) and (47), it is not even necessary to know the full solution of the reference steady-state fields $(\bar{r}_+(z), \bar{r}_-(z))$ in the diode; all that appears in those equations is $\bar{t}$ or $\bar{r}$.

Thus, the ODE model is valid for any reference steady-state solution, and it is generalizable to any system where the time scale of the dynamics is much greater than the round-trip time. This is valid even for semiconductor lasers, which have a much faster time scale than the semiconductor diode, because the gain bandwidth is limited by the semiconductor material characteristics, and not by the diode. The ODE model is a powerful tool for understanding the dynamics of semiconductor devices and systems, and it is widely used in the analysis of optical communication systems.
the steady-state diode reflection or transmission coefficient of the reference state, respectively, and the particular incident intensity $|\mathcal{E}_0(0)|^2$ and pumping parameter $\mathcal{J}$ that establishes that steady-state reference.

V. NUMERICAL RESULTS

In order to examine the validity of our termination of the multiple scales analysis at order $\mu^0$, we compare the numerical solution of (44) and (45) with a standard direct numerical integration of the original partial differential equations (PDE’s) given by (1) and (2) subject to the boundary condition (5). We find that our approximate treatment is indeed valid for the parameter space corresponding to many real physical systems of interest, and it is thus possible to use (44) and (45) to understand and address the underlying physics. We also consider under what conditions the multiple scales analysis breaks down, and our ODE model does not adequately capture the dynamics.

Compared to the integration of the partial differential equations (1) and (2), solving (44) numerically for $\dot{\sigma}(t)$ is uncomplicated. We employed a simple Runge–Kutta algorithm and, although an iterative procedure had to be used to find $\dot{\sigma}/dt$ which appears on both sides of (44), convergence is reached quickly due to the small effect of $\epsilon \lambda_d \dot{\sigma}/dt$. Once $\dot{\sigma}(t)$ is known, the output electric field can be found using (45).

The solution of the PDE’s (1) and (2) is a common method of obtaining diode response (see, e.g., [8], [15]), and it was accomplished in this paper using a known fourth-order finite-difference scheme that employs a collocation algorithm and that solves the field equations along the characteristic lines $t \pm z/v_g$ [20], [21]. On a Pentium Pro 180 MHz machine, simulations performed using the PDE model take about 30 min, whereas the corresponding simulations using the ODE model derived in this paper require only a few seconds.

The simulations that we present involve either the modulation of the current density with the input electric field kept constant, or the modulation of the input field at a fixed current level. In both cases, the system is initially in the steady state, and then either the current density or the incoming electric field is lowered within time $T_p$ and, after a certain time, raised within time $T_p$ to its former value. We select the initial steady state to be the reference steady state and determine the parameters $\mathcal{G}, \mathcal{G}_0(0)$, and $\mathcal{J}$ such that the diode has a gain of 12 dB when it is in the “on” state. We choose the lowered level of the current density so that the output power decreases by 10 dB, and our parameters are such that this never drops the diode below medium threshold. In a directly modulated laser system, lowering the current below threshold would force the system to build up again from spontaneous emission noise, which is not described by the original PDE’s (1) and (2). The rise and fall of the normalized current density $j(t)$ and input electric field $e_-(0,t)$ is modeled by a ninth-order polynomial. For simplicity, the phase of the input field is held constant and, thus, the instantaneous frequency is simply proportional to the derivative of the phase of the output field. For all of our simulations, we use the semiconductor parameters listed in Table I, with an input light wavelength of 1550 nm and $R = 1$.

We confirm first that (44) and (45) are indeed valid for $\mu \ll 1$, $\lambda \approx 1$, and when the quantities involved in the multiple scales analysis, namely $\mathcal{G}, \theta \pm , \xi \pm = \ln(s_\pm /\overline{s}_\pm)$, and $w_\pm = (1 - s_\pm /\overline{s}_\pm)$, are of order unity. Choosing the length of the diode to be 400 $\mu$m and the rise time to be 350 ps, we find that $\mu = 0.028$ and $\lambda = 2.9$. We set the reference (and initial) input power to be 1 mW. Fig. 3 shows the time evolution of the output power and instantaneous frequency for current density modulation using both the “exact” PDE model and the ODE model; clearly, there is quite good agreement between the two calculations. There is a slight discrepancy in the steady-state output powers that results from, and justifies, the approximation made in Section II, where $\alpha 2\pi^2$ and $\alpha 3\pi^3$ were dropped from the carrier density equation (9) since $\alpha_2$ and $\alpha_3$ were both small.

We can use (44) and (45) to explain the dynamics resulting from the lowering of the current, already well known for semiconductor lasers from previous studies [22]. Initially, $\dot{\sigma}/dt = 0$, and, as the current is reduced, $\dot{\sigma}/dt$ will become negative, causing a decrease in $\sigma$ and thus in the total number of carriers. Since $\epsilon$ is small and the input field remains fixed, the main contributions to $\dot{\sigma}/dt$ are from $\dot{\sigma}(t)$ and $\mathcal{J}(t)$. As $\sigma$ decreases, it in turn causes an increase in $\dot{\sigma}/dt$, which eventually leads the system to a new steady state. The instantaneous frequency, as given by (48), is directly proportional to $\dot{\sigma}/dt$, and indeed the numerical solution of (1) and (2) shows that it does follow the changes in $\dot{\sigma}/dt$ essentially adiabatically, becoming negative when the current is first turned down, then increasing again to zero. Equation (45) predicts that the output electric field should follow the decrease in $\dot{\sigma}(t)$ adiabatically; the PDE model calculation shows that this, too, is a good approximation.
The dynamics resulting from input electric field modulation with constant current density are illustrated in Fig. 4. Equation (44) predicts that, as $\eta$ decreases, $\delta\sigma/dt$ becomes positive, leading to an increase in the number of carriers, $\tilde{\sigma}$, which then reduces $\delta\sigma/dt$ until a new steady state is reached. The output electric field depends on both $c_-(0, t)$ and $\tilde{\sigma}(t)$. Since the relaxation of the number of carriers is governed roughly by $T_n = 1$ ns, which is much slower than the time scale of change of $c_-(0, t)$, $T_p$, there is an initial overshoot in the decrease of output field as it follows the input field. As the carriers continue to increase to reach the steady state, $c_+(0, t)$ increases to its new steady-state value. Once again, we find from the simulations that the instantaneous frequency essentially follows the behavior of $\delta\sigma/dt$. We see from Fig. 4 that there is good agreement between the PDE model and ODE model calculations. Notice, however, that in the ODE model there is no delay time between the variation of $c_-(0, t)$ and that of $c_+(0, t)$, giving an effectively instantaneous response of the diode. In Fig. 4, the slight time offset evident between the two calculations is a manifestation of this.

Generally, we would like to be able to model composite systems where the time scales of interest are much faster than 350 ps. Setting $T_p = 50$ ps, where now $\mu = 0.2$, we still find good agreement between the PDE and ODE models. Indeed, as shown in Figs. 5 and 6, even if we take the rise time to be as small as 1 ps, where $\mu = 10$, our multiple scales approximation works surprisingly well. For small $T_p$, not only does $\mu$ get large, but $\lambda$ gets large; if $T_p = 1$ ps then $\lambda = 1000$. Reexamining our multiple scales analysis, we see that the quantities in the carrier density equation (36) would be divided by this large $\lambda$, so that if $\mu/\lambda$ is very small, the resulting approximate equation for $\tilde{n}$ would be accurate, even if $\mu$ itself is not small. However, $\lambda$ does not appear in the equations for $\theta_{\pm}$ and $\xi_{\pm}$ [see (30) and (31)], so that, in fact, for larger $\mu$ values, the time-dependent reflection coefficient of the diode derived from these equations may not be as accurate. Thus, although the ratio $\mu/\lambda$ is the same for simulations with different rise times (in our simulations $T_p = 1, 50, \text{and} 350$ ps), the agreement between the PDE and ODE models is slightly worse for the smaller $T_p$ values.
In order to demonstrate the effect on the multiple scales approximation of changing the ratio $\mu/\lambda$, we consider artificially long diodes while keeping the input power and gain the same as before. Keeping $T_p$ fixed at 350 ps, we find that, for $L_D = 4$ mm, where $\mu/\lambda = 0.1$, there is good agreement between the two calculations, but, as is clear from Figs. 7 and 8, qualitative differences do occur when we increase the length to 16 mm, where $\mu/\lambda = 0.4$. Note that the time offset described above is large for this diode length. We see that the multiple scales approximation breaks down as the complicated spatial variation of the carrier density and field amplitudes become important. For a rise time of 50 ps, this breakdown is evident at $L_D = 4$ mm, but not yet at $L_D = 1.6$ mm.

Lastly, we consider the effect of increasing the input power to 3 mW, again adjusting the current so that a 12–dB gain is maintained for the “on” state. For a diode length of 400 $\mu$m, there are still no qualitative differences between the PDE and ODE models for $T_p \geq 50$ ps. Considering an artificial diode of length 1.6 mm and $T_p = 50$ ps, qualitative differences between the two calculation methods do appear, unlike for 1-mW input power. As the power level rises, some of the quantities involved in the multiple scales analysis become farther away from unity so that if, in addition, $\mu/\lambda$ is not small, the multiple scales approximation begins to break down.

VI. CONCLUSIONS

We have developed a semi-analytic dynamical model for the semiconductor diode structure shown in Fig. 1. This model is based on the usual phenomenological equations for the electric field and carrier density, but requires neither the uniformity of those fields along the diode, nor small-signal variations. Our main assumption is that the dynamical time scales of interest are longer than the round-trip time of light in the diode, and this allows us to perform a multiple scales analysis to reduce the nonlinear coupled partial differential equations to one ordinary differential equation for the total carrier density, involving only the input electric field and the current density. Time-dependent reflection and transmission coefficients for the diode can be written explicitly, eliminating the need to solve for the fields and carrier density at each point along the diode. Our “ODE model” is much simpler to solve numerically, requires far less computation time, and allows more insight into the dynamics of the response of the system than is apparent from the original partial differential equations. Indeed, we derive simple expressions for the relative instantaneous frequency and the gain. In the future, we intend to use this formalism to describe systems in which the semiconductor diode is only a composite part.

By comparing the numerical solution of the ODE model outlined in this paper to a direct numerical integration of the full PDE model, we justify our multiple scales approximation for regions of parameter space that correspond to real physical systems. We find that approximation is valid where expected, that is, for $\mu < 1, \lambda \approx 1$, and multiple scales quantities of order unity. Also, we find that, typically, if either $\mu$ is increased with a fixed $\mu/\lambda$ ratio (by decreasing $T_p$), or if instead $\mu/\lambda$ is increased (by increasing $L_D$), or if instead the power is increased such that the multiple scales quantities are larger than unity, the multiple scales approximation, and thus our formalism, is valid. This is essentially what we wanted to show: that the dynamics of the semiconductor diode are accurately described by (44) and (45) or, alternatively, (46) and (47) so that our ODE model can be used for the study of
more complicated structures, where the diode can be treated quite simply, but accurately, as a “black box.”

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