Prescription for beam design: optimizing power transport to a target

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A prescription is derived for determining the beam that delivers the maximum power to a specified target for a given allowed aperture.

The recent rediscovery of Bessel function solutions of the Helmholtz equation has led to their experimental investigation in optics. Although these Bessel function solutions are nominally diffraction free, strictly speaking they can never be constructed since, like plane-wave solutions of the Helmholtz equation, their total energy is infinite. Yet truncated versions of these beams have been compared with Gaussian beams with respect to beam divergence and power-transport efficiency.

Such investigations lead one to the more general question: For a specified target and allowed beam aperture, what field distribution across the aperture produces the maximum power transport to the target? In this Letter we show that in general this question can be answered by finding the eigenfunction associated with the largest eigenvalue of a Hermitian operator. We also consider a simple special case but defer more detailed analytic and numerical work to later publications.

For simplicity we consider the scalar problem, in which the complex field amplitude $\psi(r)$ satisfies the Helmholtz equation

$$\left[\nabla^2 + k^2\right]\psi(r) = 0. \quad (1)$$

From this equation it is easy to demonstrate that the field amplitude at all points with $z > 0$ is determined by specifying both the field amplitude at all points on an object plane $z = 0$ and the outgoing radiation condition as $|r| \to \infty$ for $z > 0$. If we put $\psi(x', y', z' = 0) = \phi(\rho')$, where $\rho' = (x', y')$, the solution is

$$\psi(r) = \int g(r; \rho') \phi(\rho') d\rho' \quad (2)$$

(see Fig. 1), where

$$g(r; \rho') = \frac{k}{2\pi i} \frac{e^{ikR}}{R} \left(1 + \frac{i}{kR}\right) \frac{2}{R} \cdot R. \quad (3)$$

Here $R = |R|$ and $R = r - \rho'$. The integral in Eq. (2) goes over the allowed aperture, i.e., over the region of the object plane $z = 0$ where we are allowed to have $\phi(\rho') \neq 0$.

As usual in the scalar theory, we assume that if a target were present the energy deposition at a position $r$ on it would be proportional to

$$A(r) \equiv |\psi(r)|^2, \quad (4)$$

the absorption intensity. If we specify our target by a nonnegative real target function $f(r)$, which is largest where we most want to deliver power, then we would like to maximize the quantity

$$P = \int A(r)f(r) dr. \quad (5)$$

Here the integral can go over the entire half-space $z > 0$, since $f(r)$ can be set equal to zero, where we are not interested in delivering power. By using Eqs. (2) and (4), Eq. (5) can be written as

$$P = \int \phi^*(\rho_1) H(\rho_1, \rho_2) \phi(\rho_2) d\rho_1 d\rho_2, \quad (6)$$

where the integrals in Eq. (6) go over the allowed aperture range and

$$H(\rho_1, \rho_2) = \int g^*(r; \rho_1)f(r)g(r; \rho_2) dr. \quad (7)$$

Note that $H(\rho_1, \rho_2)$ is Hermitian: $H^*(\rho_1, \rho_2) = H(\rho_2, \rho_1)$. We want to choose the object function $\phi(\rho)$ such that $P$ is maximized. But of course $P$ scales with any overall factor that multiplies $\phi(\rho)$, so we maximize $P$ subject to the constraint that the total power through the aperture is held constant. In many treatments of scalar diffraction theory in optics the quantity $A(r)$ is identified as the intensity; this would lead us to maximize

$$Q = P - \mu F_0, \quad (8)$$

where

$$F_0 = \int |\phi(\rho)|^2 d\rho \quad (9)$$

and where $\mu$ is a Lagrange multiplier.

Note, however, that in the scalar theory leading to Eq. (1) a proper energy flux vector can be defined, and its time average is proportional to $S(r) = \text{Im}[\psi^*(r) \nabla \psi(r)]$. Thus the total power through the aperture is, more correctly, proportional to
infinity without introducing significant errors. Using the Fraunhofer limit the diffraction is restricted to

\[ J_0(kr) \]

where \( J_0 \) is the zeroth-order Bessel function of the first kind and the integrals over \( \rho_1 \) and \( \rho_2 \) range over the aperture. Returning to Eq. (11), we assume that in the Fraunhofer limit the diffraction is restricted to small enough angles that we can approximate \( \cos \theta \approx 1 \), \( \sin \theta \approx \theta \) and extend the \( \theta \) integral from zero to infinity without introducing significant errors. Using

these approximations in Eq. (12) along with some Bessel function identities, we find that

\[ F_1 \rightarrow \int |\phi(\rho)|^2 d\rho = F_0. \tag{13} \]

Thus within these approximations expression (10) is indeed simply proportional to \( F_0 \), and we are in fact correct in maximizing the \( Q \) of Eq. (8). These approximations are of course essential in most if not all applications of scalar diffraction theory to problems in optics, where the \( \phi(\rho) \) of Eq. (1) is identified as one of the Cartesian components of \( E(\rho) \). For if those approximations were not satisfied, it would not be valid to identify \( A(\rho) \) with the dot product \( E(\rho) \cdot E^*(\rho) \). It is because of this that in such problems \( A(\rho) \) is and can generally be considered to be the intensity.\(^6\)

Turning now to the problem of maximizing \( Q \), we first find the functions \( \phi \) that produce extrema; that is, we find the functions \( \phi \) such that small variations around them leave \( Q \) unchanged, \( \delta Q = 0 \). The solutions to problems of this sort are well-known results of functional analysis;\(^10\) this particular problem will be familiar to most physicists because of its similarity to the quantum-mechanical problem of deriving the time-independent Schrödinger equation from a variational principle.\(^11\) As usual, we can treat either \( \text{Re}[\phi(\rho)] \) and \( \text{Im}[\phi(\rho)] \) or \( \phi(\rho) \) and \( \phi^*(\rho) \) as our variables. When we choose the latter, \( \delta Q = [\delta Q/\delta \phi(\rho)]\delta \phi(\rho) + [\delta Q/\delta \phi^*(\rho)]\delta \phi^*(\rho) = 0 \) yields \( \delta Q/\delta \phi(\rho) = \delta Q/\delta \phi^*(\rho) = 0 \). Both of those yield the same equation,

\[ \int H(\rho_1, \rho_2)\phi(\rho_1)\phi(\rho_2)d\rho_1d\rho_2 = \mu\phi(\rho). \tag{14} \]

That is, the functions \( \phi \) that we seek are eigenfunctions of \( H \). We label the normalized eigenfunctions \( \phi_n \) and their eigenvalues \( \mu_n \):

\[ \int H(\rho_1, \rho_2)\phi_n(\rho_1)\phi_n(\rho_2)d\rho_1d\rho_2 = \mu_n\phi_n(\rho_1). \tag{15} \]

If a normalized \( \phi(\rho) \) is chosen to be one of the \( \phi_n(\rho) \), then \( P \) is an extremum. The value of \( P \) if \( \phi(\rho) = \phi_n(\rho) \) is

\[ P_n = \int \phi_n^*(\rho_1)H(\rho_1, \rho_2)\phi_n(\rho_2)d\rho_1d\rho_2 \]

\[ = \mu_n \int \phi_n^*(\rho_1)\phi_n(\rho_1)d\rho_1 \]

\[ = \mu_n. \tag{16} \]

The general prescription for maximizing power deposition to a target is therefore clear: For the chosen target function \( f(\rho) \) construct \( H(\rho_1, \rho_2) \) [Eq. (7)], where \( \rho_1 \) and \( \rho_2 \) range over the area of the allowed aperture. The field distribution across the aperture should be set proportional to the eigenfunction of \( H \) with the largest eigenvalue.

We close with a simple example. Suppose that we are interested in maximizing the power delivered to a small region in the neighborhood of a point \( r_o \). Then we can take \( f(\rho) = \delta(\rho - r_o) \), and we have...
\[ H(p_1, p_2) = g^*(r_0; p_1)g(r_0; p_2). \] (17)

This \( H \) is so simple that one can immediately spot a normalized eigenfunction:

\[ \phi(p) = \sigma^{-1}g^*(r_0; p), \] (18)

where

\[ \sigma^2 = \int g^*(r_0; p)g(r_0; p)dp. \] (19)

The eigenfunction [Eq. (18)] has a nonzero eigenvalue, equal in fact to \( \sigma^2 \). But the \( H \) of Eq. (17) is also proportional to a projection operator. That is, defining

\[ p(p_1, p_2) = \sigma^{-2}H(p_1, p_2), \] (20)

we find that

\[ \int p(p_1, p_2)p(p_2, p_3)dp_2 = p(p_1, p_3). \] (21)

Since the eigenvalues of a projection operator are zero and unity, we see that all other eigenfunctions besides Eq. (18) have eigenvalue zero. Thus \( \phi(p) \) should be set to \( \phi(p) \) to maximize power deposition in the immediate neighborhood of \( r_0 \). When we write out \( \phi(p) \), it is clear that the wave front that it describes across the aperture is just one that is focused toward \( r_0 \). This special case has been dealt with by Anderson\(^1\) in a slightly different context, but the approach introduced here can of course be applied to target functions much more general than the delta function that he implicitly considered.

Finally, we note that if we take \( r_0 = (0, 0, z_0) \) and let \( z_0 \) be so large that the Fraunhofer approximation can be used for \( g(r_0; p) \), we find that \( \phi(p) \) is independent of \( p \). Thus, to maximize the power delivered along the axis in the far field, the object function should be uniform over the aperture, regardless of the shape of the aperture.

In summary, we have shown how to design a beam so that the maximum power is delivered to a target for a given allowed aperture. Only a simple example has been given here; we plan to turn to the detailed discussions of the optimum beam profiles for targets in both the Fraunhofer and Fresnel regions in future publications.

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References

7. The scalar theory is proper for acoustical waves in a homogeneous medium; see, e.g., P. M. Morse, *Vibration and Sound*, 2nd ed. (McGraw-Hill, New York, 1948), Chap. 7, p. 294.
8. See, e.g., Ref. 5, p. 131.
9. Reference 5, Sec. 9.8, p. 427.