Lorentzian model for nonlinear switching in a microresonator structure

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Abstract

We show that the optical response of two channel waveguides coupled by a microresonator is well described by a Lorentzian model. Using this model we derive a set of coupled differential equations that describe Kerr nonlinear optical pulse propagation and optical switching in systems coupled by a few microresonators. The equations are valid even when the pulse is within the gap of the structure.

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1. Introduction

In this paper we present a model for nonlinear optical pulse propagation in the system shown in Fig. 1(a), which consists of two channel waveguides that are side-coupled, via evanescent fields, to a set of periodically spaced, circular microresonators; we assume that the nonlinearity is of the Kerr type. Following the terminology in the literature [1,2], we call these two channel SCISSOR structures, where SCISSOR denotes 'side-coupled integrated spaced sequence of resonators'. For simplicity we consider the situation in which the strength of the coupling between the top channel and the microresonator is the same as the coupling between the bottom channel and the microresonator. Light travelling in the forward (backward) direction in the bottom (top) channel can couple, via the microresonators, to light travelling in the backward (forward) direction in the top (bottom) channel. In general, the coupling between the channel waveguides and the microresonators will be small. However, if the frequency of the light is close to an integer multiple of the resonant frequency of the microresonator itself ($\nu_{r}$), then the effective coupling between the two channels can become quite large, and a stop gap in the transmission spectrum arises [2–4].

It has recently been shown that nonlinear pulse dynamics in an infinite two channel SCISSOR
The structure is well described by a nonlinear Schrödinger equation (NLSE) if the pulse frequency is not “too deep” within a stop gap [2,5]. However, it has been numerically predicted that interesting nonlinear dynamics, such as optical switching, can occur in a two channel SCISSOR structures with a very small number of cells [5], where the validity of the NLSE would be called into question. Furthermore, when the pulse frequency is deep within the gap, the best description of pulse dynamics is as yet unclear. In Bragg systems, which exhibit similar switching, a set of nonlinear coupled mode equations (NCME) are used to describe nonlinear effects deep within the gap [7]; unfortunately, the usual CME do not obtain for the two channel SCISSOR structure [2].

In this paper we present a theoretical description of pulse propagation in the two channel SCISSOR structure that is valid for frequencies anywhere throughout the gap. We begin by considering the CW response of a single two channel SCISSOR cell, and, in an analysis very similar to that presented by Manolatou et al. [8], show that for frequencies near the resonant frequency of the system the linear transmission and reflection of a single two channel SCISSOR cell is well described by a Lorentzian model. We then extend this Lorentzian model to derive a set of differential equations that describe pulse propagation in a single cell in the presence of a Kerr nonlinearity. The differential equations can also be used to describe pulse propagation through a multi-cell structure, because for adjacent cells the input of one cell is the output of the other; thus, the differential equations that govern the individual cells in the structure couple together naturally. We use the model to simulate nonlinear switching in a multi-cell structure. To test the accuracy of the Lorentzian model, we compare its results to the results of a more exact numerical technique, which accounts more explicitly for light propagation in the structure [1,5]. When the structure is sufficiently short, and the pulses being simulated are sufficiently long, then the two techniques agree both qualitatively and quantitatively. For longer structures, or shorter pulses, the techniques agree qualitatively.
2. Light propagation – single cell

We consider a specific model [2] for light propagation in the single microresonator cell shown in Fig. 1(b). We assume that the channel waveguides and the microresonator both support a known mode profile with the electric field in the transverse direction \((E = Ey)\), and the magnetic field, \(H\), everywhere orthogonal to \(E\). Of course, in a physical system the light will not be exactly polarized in the \(y\) direction, but such an approximation makes the nonlinear interactions more tractable while maintaining the essential physics. We also assume that the coupling of light into and out of the microresonator occurs only at the points indicated by the large dots in Fig. 1(b), and that at each coupling point there is no reflection. There are two types of propagation modes: a bottom (top) mode, in which more intensity is contained in the bottom (top) channel than the top (bottom) channel. Each of these two modes can be associated with forward or backward propagation, so that there are a total of four types of modes in the system. However, in this paper we consider the simpler situation in which only one of the modes of the system is excited at the input.

We use a transfer-matrix technique [1,2,4] to describe light propagation within the system. We denote the electric field in the bottom channel \(I(r) = \eta S(x,y)l(z)\hat{y}\), and in the top channel \(U(r) = \eta S(x,y)u(z)\hat{y}\), where \(S(x,y)\) is the mode profile associated with the channel waveguides, and where \(\eta\) is a normalization constant which is defined such that \([l(z)]^{2}\) and \([u(z)]^{2}\) have units of intensity. We denote the electric field in the microresonator \(Q(y,R,\theta) = \eta T(y,R)q(\theta)\hat{y}\), where \(T(y,R)\) is the mode profile associated with the resonator, \(R\) is the radial variable, and \(\theta\) is the angle within the resonator, measured counter-clockwise from the bottom coupling point (see Fig. 1(b)). At the coupling points we define the self \((\sigma\) and cross \((\kappa\) coupling coefficients, and assume [2]

\[
\begin{bmatrix}
q(0_{+}) \\
l(1_{+})
\end{bmatrix}
= \begin{bmatrix}
\sigma & \kappa \\
\kappa & \sigma
\end{bmatrix}
\begin{bmatrix}
q(0_{-}) \\
l(1_{-})
\end{bmatrix},
\]

\[
\begin{bmatrix}
q(\pi_{+}) \\
u(\pi_{-})
\end{bmatrix}
= \begin{bmatrix}
\sigma & \kappa \\
\kappa & \sigma
\end{bmatrix}
\begin{bmatrix}
q(\pi_{-}) \\
u(\pi_{+})
\end{bmatrix},
\]

with \(a = d/2\), and where we have introduced the notation \(a_{\pm} = a \pm \delta a\), \(\pi_{\pm} = \pi \pm \delta \pi\), \(0_{\pm} = \pm \delta \pi\), where \(\delta a\) and \(\delta \pi\) are infinitesimal quantities. In order to conserve energy, the value of the coupling coefficients must be chosen such that \(|a_{+}|^{2} + |\kappa|^{2} = 1\) and \(\sigma \kappa = \sigma \kappa^{*}\). If \(\sigma\) and \(\kappa\) are real, then the second condition is automatically satisfied. In the remainder of the paper we assume that \(\sigma\) and \(\kappa\) are real, which simplifies the theoretical expressions without losing any of the essential physics of the situation. This corresponds to the assumption that coupling occurs only at a single point. A detailed derivation of the transfer matrix can be found elsewhere [6].

Away from the coupling points, the only effect of propagation is the accumulation of phase, which will be governed by the propagation constant associated with the mode profiles \(S(x,y)\) and \(T(y,R)\). We assume the linear propagation constant is equal for the channel guides and the microresonator, and denote it \(v = n_{\text{eff}}\omega/c\), where \(\omega\) is the frequency of the light, and where \(n_{\text{eff}}\) is the effective linear index of refraction associated with the waveguide. Strictly speaking, \(n_{\text{eff}}\) is a function of frequency, but in the following we consider sufficiently small frequency ranges that we can reasonably ignore its frequency dependence. We assume that light is travelling in the forward direction in the bottom channel, and in the backward direction in the top channel, so that for the channel guides \(l(a_{+}) = l(0)e^{ia}\) and \(l(\pi) = l(0)e^{i\pi}\) and \(u(a_{+}) = u(0)e^{-ia}\) and \(u(\pi) = u(0)e^{-i\pi}\). The microresonator, \(q(\pi_{+}) = q(0_{+})e^{i\rho\pi}, q(0_{-}) = q(0_{-}) e^{i\rho\pi}\), where \(\rho\) is the radius of the resonator. Combining these expressions for phase accumulation with the coupling matrices (1) we find

\[
\begin{bmatrix}
l(d) \\
u(d)
\end{bmatrix} = \frac{1}{r_{1}}
\begin{bmatrix}
t^{2} - r^{2} & r \\
-r & 1
\end{bmatrix}
\begin{bmatrix}
l(0) \\
u(0)
\end{bmatrix}.
\]

The transfer matrix is written in terms of the coefficients \(t(\omega)\) and \(r(\omega)\) – to be identified below as transmission and reflection coefficients – where

\[
t(\omega) = \frac{\sigma(e^{i\phi(\omega)} - 1)e^{i\rho\pi}}{(\sigma^{2}e^{i\phi(\omega)} - 1)},
\]

\[
r(\omega) = \frac{1 - \sigma^{2}e^{i\phi(\omega)}}{\sigma^{2}e^{i\phi(\omega)} - 1},
\]
and where we have introduced the quantity $\phi(\omega)$ to account for the phase accumulated by light propagating once around the resonator. We expand the phase, $\phi(\omega)$, as

$$\phi(\omega) = \phi_L(\omega) + \phi_{NL}(\omega),$$  

(4)

where $\phi_L(\omega)$ is the phase accumulated in the absence of nonlinearity, and $\phi_{NL}(\omega)$ is the extra phase accumulated due to the Kerr nonlinearity,

$$\phi_L(\omega) = 2\pi n_{eff} \omega / c,$$

$$\phi_{NL}(\omega) = 2\pi n_2 I_c \omega / c,$$  

(5)

where $I_c$ is the intensity of the light that circulates in the microresonator, and $n_2$ is the nonlinear index of refraction coefficient. We define the fundamental resonant frequency of the microresonator, $\omega_r$, as the frequency that makes $\phi_L = 2\pi$,

$$\omega_r \equiv c / (\rho n_{eff}).$$  

(6)

At integer multiples of this frequency we expect to observe a resonant coupling of light into the microresonator.

We are interested in four field amplitudes: the field entering the system at $l(0; \omega)$ and $u(d; \omega)$, and the field leaving the system at $l(d; \omega)$ and $u(0; \omega)$. We denote the incoming quantities $\tilde{l}_i(\omega)$ and $\tilde{u}_i(\omega)$, and the outgoing quantities $\tilde{l}_o(\omega)$ and $\tilde{u}_o(\omega)$, where the subscripts $i$ and $o$ stand for incoming and outgoing, respectively, and where the tildes are used to stress that these variables are dependent on frequency. In the following section, we take the Fourier transform of these quantities, and hence define $l_{i/o}(t)$ and $u_{i/o}(t)$. We denote the intensity circulating in the resonator at a given frequency $I_c(\omega)$. Using the transfer matrix (2) and the transmission coefficients (3) we find

$$\tilde{l}_o(\omega) = t(\omega) \tilde{l}_i(\omega) + r(\omega) \tilde{u}_i(\omega),$$

$$\tilde{u}_o(\omega) = r(\omega) \tilde{l}_i(\omega) + t(\omega) \tilde{u}_i(\omega).$$  

(7)

From these equations, the interpretation of $t(\omega)$ and $r(\omega)$ as transmission and reflection coefficients is clear: If the input to the system is in the lower (upper) channel only, then the transmitted light along the lower (upper) channel is given by $t(\omega)$, and the reflected light in the upper (lower) channel is given by $r(\omega)$.

We concentrate on light with frequency, $\omega$, close to a resonant frequency of the medium, $\omega = N\omega_r + \Delta$,

$$\omega = N\omega_r + \Delta,$$  

(8)

where $N$ is the resonance number, and $\Delta$ is the detuning from the $N$th resonance. We can then write

$$\phi(\omega) = 2\pi N + \delta \phi(\omega),$$

(9)

with

$$\delta \phi(\omega) = \frac{2\pi}{\omega_r} \left( \Delta + \frac{n_2 I_c}{n_{eff}} \omega \right).$$

(10)

In the following we assume that $\delta \phi \ll 1$, so that $e^{\phi} \approx 1 + i \delta \phi$; this amounts to assuming that $\Delta / \omega_r \ll 1$ and $n_2 I_c / \omega_c \ll 1$. This small $\delta \phi$ approximation gives a form for $r(\omega)$,

$$r(\omega) \simeq (-1)^{N+1} \frac{\gamma e^{\text{ind}}}{\gamma - i \left( \Delta + \tilde{\Omega}_{NL} \right)},$$

(11)

where we have defined

$$\gamma \equiv \frac{\omega_r}{2\pi} \frac{1 - \sigma^2}{\sigma^2}$$

(12)

and

$$\tilde{\Omega}_{NL}(\omega) = \frac{n_2 I_c}{n_{eff}} N \omega_r.$$  

In defining $\tilde{\Omega}_{NL}(\omega)$ we have approximated $\omega \simeq N\omega_r$, because the extra contribution from $\Delta$ would give a second-order contribution. We can, of course, perform the same small $\delta \phi$ expansion on the expression for $t(\omega)$ (3), but doing so results in an approximate expression for $t(\omega)$ that does not conserve energy, even if the approximation is carried out to second order. Therefore, we choose a model for $t(\omega)$,

$$t(\omega) \simeq -i \frac{\left( \Delta + \tilde{\Omega}_{NL} \right) e^{\text{ind}}}{\gamma - i \left( \Delta + \tilde{\Omega}_{NL} \right)},$$

(13)

that conserves energy, so $|r(\omega)|^2 + |t(\omega)|^2 = 1$; furthermore, this expression for $t(\omega)$ accurately estimates the phase lag induced by the resonator. We have chosen to expand $r(\omega)$ instead of $t(\omega)$ because expanding $r(\omega)$ gives us a closer resemblance to the Lorentzian lineshape under the same
approximation \( (\delta \phi(\omega) \ll 1) \). In the absence of nonlinearity \((n_2 = 0)\), (11) and (13) give a simple Lorentzian model for the transmission and reflection of the structure \[8\]. In Fig. 2(a) we plot the exact (solid line) and approximate (dashed line) transmission of the structure of Fig. 1(b), assuming that \( \sigma = 0.98, \ n_{\text{eff}} = 3, \ 2\pi\rho = 26 \mu m, \ d = 16 \mu m, \ n_2 = 0, \) and assuming that no light enters the system from the upper channel \((\tilde{u}_i = 0)\). We concentrate on the 51st resonance, which corresponds to a vacuum wavelength of 1.52941 \( \mu m \). The agreement between the exact and the approximate expressions is excellent. We have also verified that the phase of the transmitted and reflected light is excellently predicted by the Lorentzian model.

The presence of a Kerr nonlinearity complicates matters because the expression for \( \phi_{\text{NL}} \) (5) requires knowledge of the intensity of the light inside the resonator. Using the coupling matrices we find a cubic equation that relates the intensity of the light circulating in the resonator, \( \tilde{I}_c \), to the incident intensities \( |\tilde{I}_i|^2 \) and \( |\tilde{u}_i|^2 \)

\[
0 = \mu \tilde{I}_c^3 + \beta \tilde{I}_c^2 + \alpha \tilde{I}_c - (1 - \sigma^2)^2 |\tilde{I}_i + (-1)^N \tilde{u}_i|^2,
\]

where

\[
\begin{align*}
\alpha &= \left( 1 - \sigma^2 \right)^2 + \frac{4\pi^2 \sigma^2 A^2}{\omega_i^2}, \\
\beta &= \frac{8\pi^2 \sigma^2 NA}{\omega_i} \left( \frac{n_z}{n_{\text{eff}}^3} \right), \\
\mu &= 4\pi^2 N^2 \sigma^2 \left( \frac{n_z}{n_{\text{eff}}} \right)^2.
\end{align*}
\]

In deriving (14) we have neglected the difference between the intensity in the right half of the resonator \((0 < \theta < \pi)\) and the left half of the resonator \((\pi < \theta < 2\pi)\). This difference vanishes in the limit \( \sigma \to 1 \), and since \( \sigma \approx 1 \) for realistic resonator structures \[9\], this does not introduce any serious
error. Eq. (14) can be solved exactly for a given value of the incident intensities; it can also, of course, be used to describe optical bistability [10] when $\Delta < 0$. To simulate this bistability, we consider the situation where $\bar{u}_0 = 0$, so that all of the light enters from the lower channel ($\bar{I}_l \neq 0$). We use the same material parameters as in Fig. 2(a), but set $n_2 = 1.1 \times 10^{-13} \text{cm}^2/\text{W}$, which is consistent with Al$_{0.18}$Ga$_{0.82}$As [11], and fix $\lambda_{bh}$, the wavelength at which the bistability simulation is performed, at $\lambda_{bh} = 1.52985$. Defining $\delta_{res}$ as the full-width at half maximum transmission of the transmission spectrum, this value of $\lambda_{bh}$ corresponds to $\Delta/\delta_{res} = -1.16$. In Fig. 2(b) we plot the intensity of the reflected light, $|\bar{u}_0|^2$, as a function of the intensity of the input light, $|\bar{I}_l|^2$; for certain values of input intensity there are three allowable values of the reflected intensity. Note, however, that for a positive nonlinearity bistability only occurs when $\Delta < 0$. For $\Delta > 0$ no bistability occurs, and for a given value of $\bar{I}_l$, there is only one allowable value of $\bar{I}_r$ [10].

3. Coupled differential equations

We now consider a set of coupled resonators, such as shown in Fig. 1(a). The resonators are labelled 1, 2, ..., $n$, and each resonator can have a different coupling coefficient, $\sigma_i$. Optical pulse propagation in that system has been described qualitatively in terms of a NLSE [2,5], but the NLSE can not give an accurate quantitative picture of pulse propagation for two reasons. First, the NLSE is derived for an infinite set of resonators, while the systems we will discuss below have only a few resonators. Second, even if the system were infinite, the NLSE is unable to describe light propagation when the frequency of the pulse is deep within the stop band of the linear transmission spectrum [7]. In this section we use the Lorentzian model for the reflection (11) and transmission (13) coefficients, to derive a set of coupled differential equations that give excellent qualitative and quantitative predictions for optical pulse propagation through the coupled resonators, even when the carrier frequency of the pulse is near a given resonant frequency of a single resonator.

We start by using the model for $r(\omega)$ (11) and $t(\omega)$ (13) to re-write Eq. (7),

$$
\begin{align*}
\left(\gamma - i\left[\Delta + \Omega_{NL}\right]\right)\tilde{t}_n(\omega) \\
= e^{i\omega_d}\left( -i\left[\Delta + \Omega_{NL}\right]\tilde{t}_n(\omega) + ( -1)^{N+1}\gamma^m\bar{u}^m_n(\omega) \right), \\
\left(\gamma - i\left[\Delta + \Omega_{NL}\right]\right)\tilde{u}^m_n(\omega) \\
= e^{i\omega_d}\left( ( -1)^{N+1}\gamma^m\tilde{t}_n(\omega) - i\left[\Delta + \Omega_{NL}\right]\tilde{u}^m_n(\omega) \right),
\end{align*}
$$

where the superscript $n$ indicates that we are describing light in the $n$th cell of the system. We define the central frequency of the pulse of interest, $\bar{\omega}$, and assume that $e^{i\omega_d} \approx e^{i\omega_d^0}$, where $\omega = n_{eff}\bar{\omega}/c$, which amounts to ignoring the effect of the nonlinearity and the frequency dependence of the phase lag in the channel waveguides. We are justified in ignoring the effect of the nonlinearity in the channel waveguides because nonlinear switching in these short resonator structures occurs at such low intensities that the self-phase modulation (SPM) incurred by the nonlinearity in the channel guides is quite small, and over the propagation distances involved ($\approx 50 \mu m$) is almost negligible. Because of this the nonlinearity in the channel waveguides has little effect. Ignoring the frequency dependence of the phase lag in the channel waveguides has little effect on the results of the model, because the structures in question are so short such that $e^{i\omega_d} \approx e^{i\omega_d^0}$ for pulse widths that are normally associated with microresonator structure ($>5 \text{ ps}$). We have verified both assertions using the numerical code described later in this section.

We have assumed that our pulses are carried at a central frequency $\bar{\omega}$, using which we define a central detuning $\bar{\Delta} = \bar{\omega} - N\omega_i$. We then define envelope functions, centred about $\bar{\omega}$, that are associated with the fields $l_{i/o}(\omega)$ and $u_{i/o}(\omega)$, using

$$
\begin{align*}
\bar{l}_n^{i/o}(t) &= \int \tilde{l}_n^{i/o}(\omega)e^{-i\omega t}d\omega, \\
\bar{u}_n^{i/o}(t) &= \int \tilde{u}_n^{i/o}(\omega)e^{-i\omega t}d\omega.
\end{align*}
$$

(17)
Taking the Fourier transform of (16), and using the definitions (17), we arrive at the differential equations

\[
\frac{d I^m_\zeta(t)}{dt} = [\gamma^m + i \Gamma^m(t)] I^m_\zeta(t) + e^{\pi d} \left[ \frac{d}{dt} - i \Gamma^m(t) \right] I^m_\zeta(t) \\
+ (1)^{N+1} \gamma^m e^{\pi d} I^m_\zeta(t),
\]

\[
\frac{d n t(t)}{dt} = [\gamma^m + i \Gamma^m(t)] u^m_t(t) + e^{\pi d} \left[ \frac{d}{dt} - i \Gamma^m(t) \right] u^m_t(t) \\
+ (1)^{N+1} \gamma^m e^{\pi d} I^m_\zeta(t),
\]

where

\[
\Gamma^m(t) = \Delta + N \omega_i \frac{n_z I^m_\zeta(t)}{n_{\text{eff}}},
\]

and where \( I^m_\zeta(t) \) is calculated by solving the cubic equation

\[
0 = \mu (I^m_\zeta(t))^3 + \beta (I^m_\zeta(t))^3 + \alpha (I^m_\zeta(t)) \\
- (1 - \sigma^2) |I^m_\zeta(t)|^2 + (1)^{N+1} u^m_t(t),
\]

while taking the values of \( \mu, \beta \) and \( \alpha \) at \( \bar{\omega} \). The manner in which we determine \( I^m_\zeta(t) \) amounts to assuming that the build up of the intensity in the resonator is not a function of frequency. This is good approximation provided that the frequency width of the incident pulse is small relative to the frequency range over which \( \Gamma^m(\omega) \) varies appreciably. In particular, in a region of optical bistability, where \( \Gamma^m(\omega) \) is heavily dependent on frequency, the differential equations (18) would be valid only for long pulses. In the following section, we apply (18) to optical switching in a scheme similar to gap soliton formation. In this scheme the pulse to be switched is on the positive frequency side of the resonance (\( \Delta > 0 \)), where bistability does not occur for a positive nonlinearity. Because of this, the differential equations give a very good approximation to the switching characteristics of the system for pulses of width \( \approx 100 \text{ ps} \).

\[4. \text{Discussion and conclusion}\]

In order to test the accuracy of the differential equation (18), we compare its results to the results of a more accurate numerical technique [1,5] that simulates pulse propagation in the structure. The numerical technique takes advantage of the fact that we ignore the frequency dependence of \( n_{\text{eff}} \). This means that away from the coupling points the propagation of light over a distance \( \delta z \) merely leads, to good approximation, to the accumulation of a linear phase, \( \phi_L = n_{\text{eff}} \delta z / c \), and a nonlinear phase, \( \phi_{\text{NL}} = n_z c / c \), where \( \bar{\omega} \) is the carrier frequency of the light and \( I \) is its intensity. For definiteness we assume that light is travelling in the forward direction in the lower channel, backward in the upper channel, and counter-clockwise in the resonator. We then define a generalized length variable, \( \zeta \), with \( \delta \zeta = \rho \theta \) in the resonator, and \( \delta \zeta = \pm \delta z \) in the channel waveguides, where the \( \pm \) obtains when the light is in the lower and upper channels, respectively. We take \( \delta z \) to be positive, so that \( \delta \zeta \) will be negative in the upper channel. We define a field, \( A(\zeta, t) \), where \( |A(\zeta, t)|^2 \) represents the intensity in the field (so \( A(\zeta, t) = l(\zeta, t) \) in the lower channel, for example). We use the propagation equation,

\[
A(\zeta, t + \delta t) = A(\zeta - \delta \zeta, t) \\
\times \exp \left( i \left( n_{\text{eff}} + n_z |A(\zeta - \delta \zeta, t)|^2 \right) \right) \\
\times \omega (\delta \zeta)/c),
\]

to describe the evolution of the field within the system, where for a given value of \( \delta \zeta \) we have \( \delta t = \delta z n_{\text{eff}} / c \). Typically we use \( \delta \zeta = 1 \mu \text{m} \), which is sufficient for convergence. In order to simplify the numerics, we assume that there is no nonlinear phase accumulation in a small region about the coupling points. At the coupling points, the matrices (1) are used to determine field propagation. The boundary conditions that we use are as follows. We set the field in the lower channel waveguide at the leftmost end of the structure. We assume that there are no other fields injected into the system. We then collect the output field at the rightmost end of the lower channel waveguide – this is the transmitted light; we also collect the output field at the leftmost end of the upper waveguide – this is the reflected light. This numerical technique is valid both in the presence and absence of nonlinearity.
Before simulating optical switching we plot the linear transmission spectra (Fig. 3) for a two cell (solid line) and three cell (dashed line) structure, centred about the 51st resonance, using the material parameters given after (13). These spectra are obtained by repeated use of the transfer matrix (2) with the exact values of the transmission and reflection coefficients (3). The two cell structure has 
\[ r_n = 0.98, \]
while the three cell structure has \( r_1 = r_3 = 0.99 \) and \( r_2 = 0.98 \), which effectively apodizes the system; this apodization is necessary to remove the Fabry–Perot-type oscillations that appear in the transmission spectrum of a finite structure [12]. The high frequency edge of the transmission spectrum of the three cell structure is much sharper than the edges of the one and two cell structure, which means that the nonlinear switching curve for the three cell structure will be much more sharply defined [12].

In our simulations of nonlinear switching we use 100 ps pulses; since we are working near the 51th resonance they have a vacuum carrier wavelength \( \lambda \approx 1.529 \mu m \), and a carrier frequency \( \bar{\omega} = 2\pi c/\lambda \). The frequency width of these pulses is roughly 1/10 the frequency width of the transmission gap of the three cell structure. We choose the exact carrier frequency of the pulses by numerically simulating their propagation through the structures with a peak input intensity of 1 MW/cm², again using \( n_2 = 1.1 \times 10^{-4} \) cm²/GW, and finding the value of \( \bar{\tau} \) that corresponds to 1% transmission. In Fig. 4 we plot transmission as a function of incident intensity for the two structures. We plot both the results of the more exact numerical technique for the two cell (dashed line) and three cell (solid line) structures, and the results of the differential equation (18) for the two cell (square) and three cell (triangle) structures. In all cases the agreement is excellent, although in the three cell structure the differential equation slightly overestimates the transmission at some points. In Fig. 5 we show the transmitted pulse shape for the
three cell structure for an incident intensity of 16 MW/cm^2 predicted by the differential equation (dashed line) and the more exact numerical technique (solid line); both the qualitative and quantitative agreement is good. The slight deviation between the results can be explained in the following manner. The Lorentzian model for the transmission (13) and reflection (11) coefficients is less valid as one detunes from the resonant frequency of the system. These small differences between the Lorentzian approximation and the exact expressions (3) build up over a number of unit cells, so that even the linear transmission predicted by the differential equation (18) is slightly off. Because the strength of the SPM is intensity dependent, the effect of the nonlinearity is heavily dependent on how much light transmits through the structure, and small differences at the linear level of the equations will be amplified. When we apply (18) to structures with more unit cells, the switching curve is well described qualitatively, but the quantitative agreement suffers unless we reduce the value of σ'. A second reason for the slight difference is that our estimate for the intensity circulating in the resonator does not account for the frequency spread of the pulse. We have verified that the Lorentzian model works essentially perfectly for longer (≈1 ns) pulses when σ ≈ 0.98; conversely, we would expect the model to suffer for much shorter (1 ps) pulses, although its qualitative results would be sound.

To conclude, we have demonstrated that the optical response of a single cell of a two-channel SCISSOR structure is well described by the use of a Lorentzian model. Using this Lorentzian model we derived a set of coupled differential equations that are valid even when the frequency of the pulse is within the stop gap of the structure. The results of the differential equations agreed qualitatively and quantitatively with the results of a more accurate numerical technique, suggesting that they give an excellent description of the physics of pulse propagation in the structure. At this point very little is known about the nature of the pulses that can propagate in a two channel SCISSOR structure deep within the gap. In coupled-resonator optical waveguiding (CROW) systems which are similar to the two channel SCISSOR structure studied in this paper, a discrete solitary wave solution has been presented [13] that can exist within a photonic band gap. However, the CROW system can be accurately modelled using a tight-binding technique that includes only nearest-neighbour interaction [13], so that the solutions for the CROW system do not map directly onto the two channel SCISSOR structure. It is an open question whether or not the two channel SCISSOR structure, too, supports solitary wave solutions deep within the gap; we hope that an analysis of the differential equations presented in this paper will help to provide an answer to that question.

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