Resistive Network Analysis

The analysis of an electrical network consists of determining each of the unknown branch currents and node voltages.

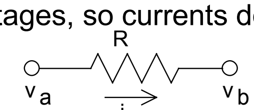
A number of methods for network analysis have been developed, based on Ohm's Law and Kirchoff's Law
- we will look at several of these.

General approach:

- Define all relevant variables in a systematic way.
- Identify the known and unknown variables.
- Construct a set of equations relating these variables.
- Solve the equations, using the smallest set of equations needed to solve for all the unknown variables.

The Node Voltage Method - 1

- This is the most general method for analysing circuits.
- Basis: define voltage at each node as an independent variable.
- One node is selected as a reference node (often, but not necessarily, ground).
- Each of the other node voltages is referenced to this node.
- Ohm's Law is applied between any two adjacent nodes to determine the voltage flowing in each branch.
- Each branch current is expressed in terms of one or more node voltages, so currents do not appear in the equations.



branch current: $i = \frac{v_a - v_b}{R}$

- Kirchoff's Current Law is applied at each of the n nodes.
- This gives n-1 independent linear equations for the n-1 independent node voltages (n^{th} = reference node).

The Node Voltage Method - 2

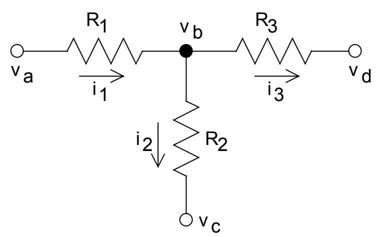
Methodology:

(1) Select a reference node (usually ground). All other node voltages are referenced to this node.

(2) Define the remaining n-1 node voltages as the independent variables.

(3) Apply Kirchoff's Current Law at each of the n-1 nodes, expressing each current in terms of the adjacent node voltages.

(4) Solve the linear system of n-1 equations in n-1 unknowns.



Example of applying KCL at one node (b):

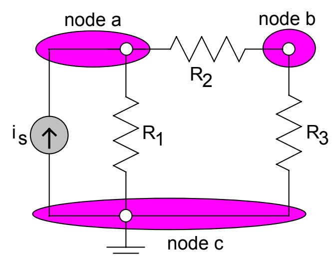
$$i_1 - i_2 - i_3 = 0$$

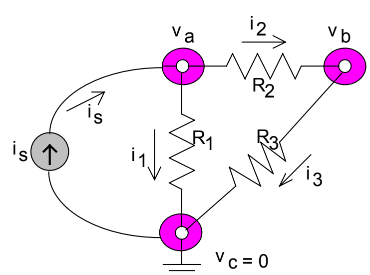
$$\frac{v_a - v_b}{R_1} - \frac{v_b - v_c}{R_2} - \frac{v_b - v_d}{R_3} = 0$$

The Node Voltage Method - 3

Example:

- Direction of current flow chosen assumes i_s is a positive current.
- Reference node v_c. Two other nodes: v_a and v_b. Solve for these.
- Application of KCL at node a: $i_S - i_1 - i_2 = 0$
- Application of KCL at node a: $i_2 - i_3 = 0$
- Application of KCL at node a: $i_1 + i_3 - i_S = 0$ (not independent!)





The Node Voltage Method - 4

Example continued:

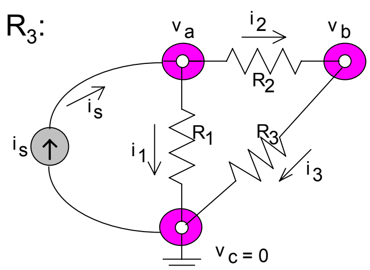
- Define currents as f(v,R): $i_1 = \frac{v_a - v_c}{R_1}$ $i_2 = \frac{v_a - v_b}{R_2}$ $i_3 = \frac{v_b - v_c}{R_3}$
- Substitute these in the two nodal equations:

$$i_S - \frac{v_a}{R_1} - \frac{v_a - v_b}{R_2} = 0 \qquad \frac{v_a - v_b}{R_2} - \frac{v_b}{R_3} = 0$$

- Solve for v_a, v_b in terms of i_s, R_1, R_2, R_3:

$$\left(\frac{1}{R_1} + \frac{1}{R_2}\right) v_a + \left(-\frac{1}{R_2}\right) v_b = i_S$$

$$\left(-\frac{1}{R_2}\right) v_a + \left(\frac{1}{R_2} + \frac{1}{R_3}\right) v_b = 0$$



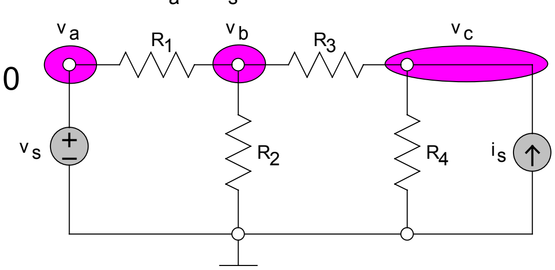
The Node Voltage Method - 5

- The node voltage method is easy to apply when current sources are present because they are directly accounted for by KCL.
- However, the node voltage method can also be applied when voltage sources are present, and this case is actually simpler.

Example: One node voltage is known: $v_a = v_s$.

$$b: \frac{v_S - v_b}{R_1} - \frac{v_b}{R_2} - \frac{v_b - v_c}{R_3} = 0$$

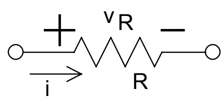
$$c: \frac{v_b - v_c}{R_3} + i_S - \frac{v_c}{R_4} = 0$$



Can solve for v_b and v_c in terms of i_s, and the four resistances.

The Mesh Current Method - 1

- This is an efficient and systematic method for analysing circuits because meshes are easily identified in a circuit.
- Basis: define current in each mesh as an independent variable.
- The current flowing through a resistor in a specified direction defines the polarity of the voltage across the resistor.



Current i, defined as flowing from left to right, determines the polarity of the voltage across R.

- To avoid confusion, define positive mesh currents as clockwise.
- The sum of the voltages around a closed circuit must equal zero by Kirchoff's Voltage Law.
- Apply KVL to each mesh to obtain a set of n equations, one for each mesh.
- Branch currents and voltages can be derived from mesh currents.

The Mesh Current Method - 2

Methodology:

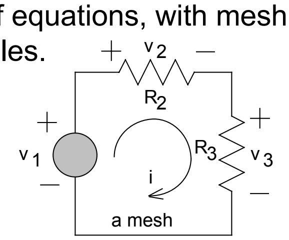
(1) Define each mesh current consistently. Generally define mesh currents clockwise, for convenience.

(2) Apply KVL around each mesh, expressing each voltage in terms of one or more mesh currents.

(3) Solve the resulting linear system of equations, with mesh currents as the independent variables.

Once the direction of current flow is selected, KVL requires:

$$v_1 - v_2 - v_3 = 0$$



- Note: an arbitrary direction can be assumed for any current in a circuit as long as signs are applied consistently. If the answer for the current is negative then the chosen reference direction is just opposite to the direction of actual current flow.

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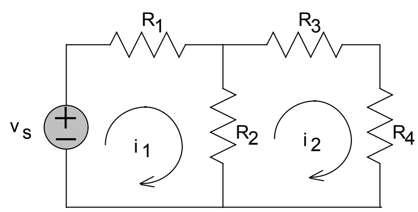
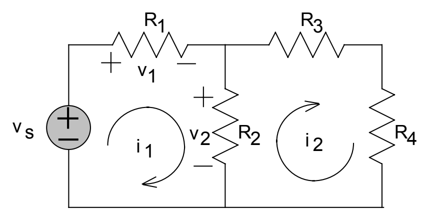
The Mesh Current Method - 3

Example:

- Two-mesh circuit with two unknowns: i_1 and i_2.
- Consider each mesh independently.
- Mesh 1
  - → Voltages v_1 and v_2 around the mesh have been assigned according the clockwise direction of mesh current i_1.
  - → Note: mesh current i_1 is flowing through R_1 (= branch current for R_1) but is not the branch current for R_2. This is i_1-i_2. So $v_2=(i_1-i_2)R_2$.
  - → Apply KVL for mesh 1: $v_s - i_1R_1 - (i_1 - i_2)R_2 = 0$





Consider mesh 1

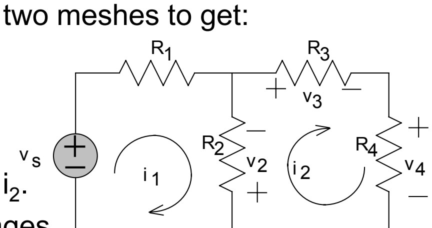
The Mesh Current Method - 4

Example continued:

- Mesh 2
  - → Voltages v_2, v_3, and v_4 around the mesh have been assigned according the clockwise direction of mesh current i_2.
  - → Note: mesh current i_2 is the branch current for R_3 and R_4, but not for R_2. This is i_2-i_1. So $v_2=(i_2-i_1)R_2$. This is the opposite to mesh 1 because the mesh currents flow through R_2 in opposing directions.
  - → Apply KVL for mesh 2: $(i_2 - i_1)R_2 + i_2R_3 + i_2R_4 = 0$
- Combine the equations for the two meshes to get:

$$(R_1 + R_2)\,i_1 - R_2 i_2 = v_S$$
$$-R_2 i_1 + (R_2 + R_3 + R_4)\,i_2 = 0$$

- Solve for mesh currents i_1 and i_2.
- Derive other currents and voltages.



Consider mesh 2

The Mesh Current Method - 5

- The mesh current method is very effective when applied to circuits that contain only voltage sources.
- However, it may also be applied to circuits containing both voltage and current sources - just be careful to identify the correct current in each mesh.

Example: Solve for unknown voltage v_x across the current source.

→ Presence of current source requires: $i_1 - i_2 = i_S = 2$ A

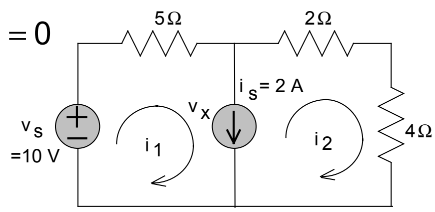
→ KVL for mesh 1: $10 - 5i_1 - v_x = 0$

→ KVL for mesh 2: $v_x - 2i_2 - 4i_2 = 0$

→ Solve to get: $i_1 = 2$ A

$i_2 = 0$ A

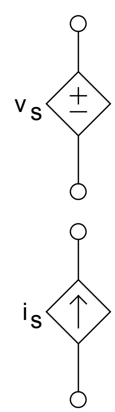
$v_x = 6i_2 = 0$ V



Dependent Sources

Dependent or controlled sources

- are sources whose current or voltage output is a function of some other voltage or current in a circuit (unlike ideal sources which are independent of any other element in a circuit)
- example: transistor amplifiers



Source type	Relationship
voltage-controlled voltage source (VCVS)	v_s=Av_x
current-controlled voltage source (CCVS)	v_s=Ai_x
voltage-controlled current source (VCCS)	i_s=Av_x
current-controlled current source (CCCS)	I_s=Ai_x

Circuit Analysis with Dependent Sources - 1

- The node voltage and mesh current methods can also be applied to dependent sources, with minor modification.
- When a dependent source is present in a circuit, it can be treated initially as an ideal source, with the node or mesh equations then written down as previously described.
- An additional constraint equation will also be needed, relating the dependent source to one of the circuit voltages or currents.
- This full set of equations can then solved.
- Note: once the constraint equation has been substituted into the initial set of equations, the number of unknowns remains the same.

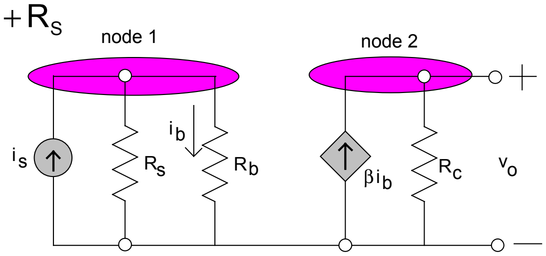
Circuit Analysis with Dependent Sources - 2

Example: simplified model of a bipolar transistor amplifier

- Two obvious nodes - apply node voltage analysis.
- KCL at node 1: $i_S = v_1\left(\frac{1}{R_S} - \frac{1}{R_b}\right)$ KCL at node 2: $\beta i_b + \frac{v_2}{R_C} = 0$
- Current i_b can be determined by considering a current divider:

$$i_b = i_S \frac{1/R_b}{1/R_b + 1/R_S} = i_S \frac{R_S}{R_b + R_S}$$

- Insert this into eqn 2 to get two equations that can be solved for v_1 and v_2.



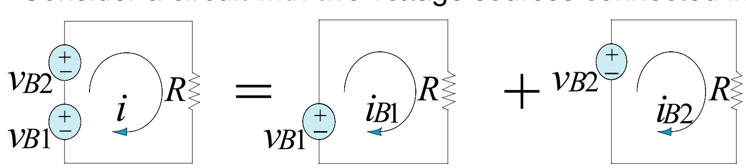
Principle of Superposition - 1

In a linear circuit containing N sources, each branch voltage and current is the sum of N voltages and currents, each of which may be computed by setting all but one source equal to zero and solving the circuit containing the single source.

- This is a conceptual aid rather than a precise analysis technique like the mesh current and node voltage methods.
- Useful in visualizing the behaviour of a circuit containing multiple sources.
- Applies to any linear system.
- While it can easily and sometimes effectively be applied to circuits with multiple sources, other methods are often more efficient.

Principle of Superposition - 2

- Consider a circuit with two voltage sources connected in series.



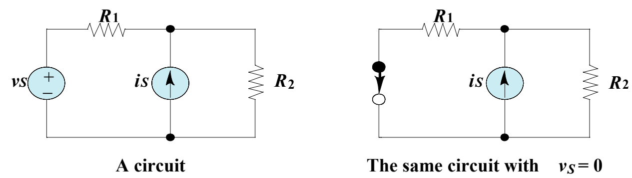
(from Rizzoni Figure 3.26)

- Net current through R = sum of individual source currents: $i = i_{B1} + i_{B2}$
- More formally: $i = \frac{v_{B1} + v_{B2}}{R} = \frac{v_{B1}}{R} + \frac{v_{B2}}{R} = i_{B1} + i_{B2}$
- This circuit is equivalent to the combination of two circuits, each containing a single source. A short circuit is substituted for the missing source in each subcircuit.
- A short circuit "sees" zero voltage across itself, so this is equivalent to zeroing the output of one of the voltage sources.

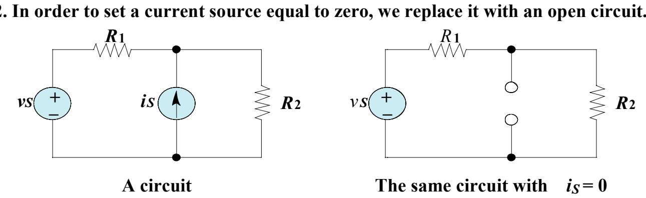
Zeroing Voltage and Current Sources

- When applying the Principle of Superposition
  - → voltage sources are zeroed by substituting short circuits
  - → current sources are zeroed by substituting open circuits (no current can flow through an open circuit, so this is equivalent to zeroing the output of the current source)

1. In order to set a voltage source equal to zero, we replace it with a short circuit.



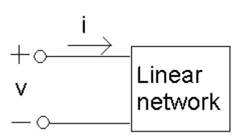
2. In order to set a current source equal to zero, we replace it with an open circuit.



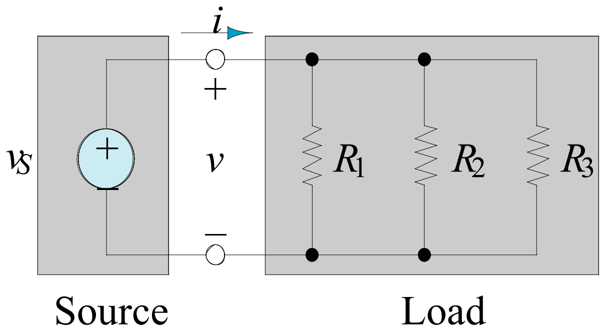
(from Rizzoni Figure 3.27)

Equivalent Circuits

- Because Ohm's Law and Kirchoff's Laws are linear, any DC circuit can be replaced by a simplified equivalent circuit.
  - → Applying Ohm's Law to a combination of resistors can give an equivalent resistor.
  - → Applying Kirchoff's Laws to a combination of circuit elements can give an equivalent circuit.
- It is useful to view each source or load as a two-terminal device described by an i-v curve.
  - → This configuration is called a one-port network.



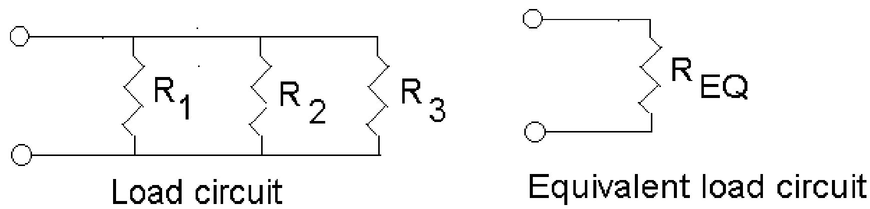
Example of an Equivalent Circuit



(from Rizzoni Figure 3.29)

$$R_{EQ} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

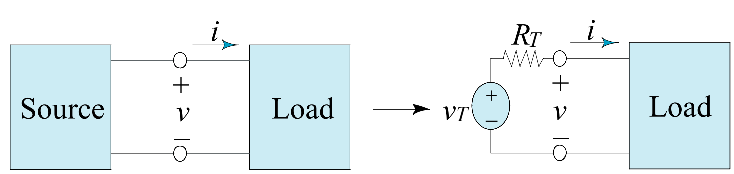
and $i = \frac{v_s}{R_{EQ}}$



Thevenin's Equivalent Circuit

The Thevenin Theorem

- As far as a load is concerned, any network composed of ideal voltage and current sources, and of linear resistors, may be represented by an equivalent circuit consisting of an <u>ideal voltage source, v_T, in series with an equivalent resistor, R_T</u>.

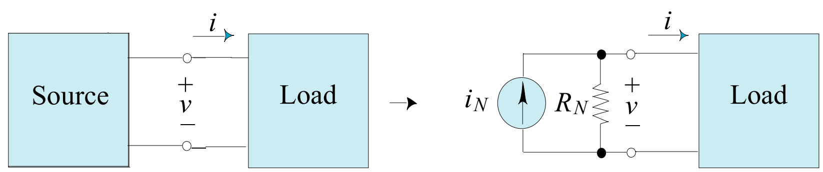


(from Rizzoni Figure 3.31)

Norton's Equivalent Circuit

The Norton Theorem

- As far as a load is concerned, any network composed of ideal voltage and current sources, and of linear resistors, may be represented by an equivalent circuit consisting of an ideal current source, i_N, in parallel with an equivalent resistor, R_N.



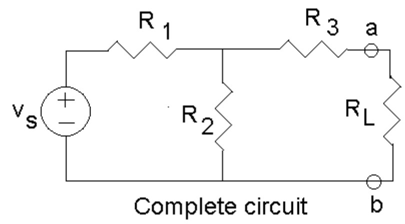
(from Rizzoni Figure 3.32)

Determining Thevenin & Norton Equivalent Resistance

- The first step in computing a Thevenin or Norton equivalent circuit is to find the equivalent resistance presented by the circuit at its terminals.
- This is done by setting all sources in the circuit equal to zero and calculating the effective resistance between the terminals.
- Voltage and current sources in the circuit are set to zero using the same approach as with the Principle of Superposition
  - → voltage sources are replaced by short circuits
  - → current sources are replaced by open circuits

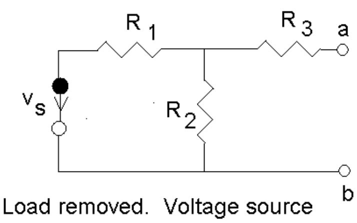
Example of Thevenin Resistance - 1

Example: What is the equivalent resistance that load R_L sees between terminals a and b?



Complete circuit

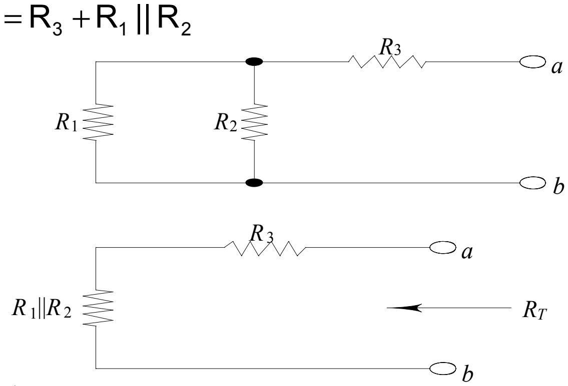
- Remove load resistance from the circuit and replace voltage source v_s by a short circuit.



Load removed. Voltage source replaced by short circuit.

Example of Thevenin Resistance - 2

- Equivalent resistance seen by the load:
  - → R_1 and R_2 are in parallel (connected between same two nodes)
  - → Let R_T be total resistance between terminals a and b.
  - → Then $R_T = R_3 + R_1 || R_2$



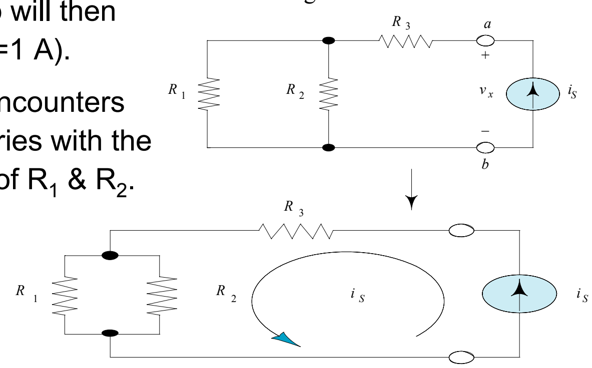
(from Rizzoni Figure 3.34)

Example of Thevenin Resistance - 3

Alternative method of determining the Thevenin resistance:

- Hypothetical 1-A current source is connected between a and b.
- Voltage v_x across a-b will then equal R_T (because i_s=1 A).
- The source current encounters R_3 as a resistor in series with the parallel combination of R_1 & R_2.

What is the total resistance the current i_S will encounter in flowing around the circuit?



$R_T = R_1 \parallel R_2 + R_3$

(from Rizzoni Figure 3.35)

Calculating Equivalent Resistance

Methodology for calculating equivalent resistance of a one-port network (Thevenin or Norton):

(1) Remove the load.

(2) Zero all independent voltage and current sources.

(3) Compute the total resistance between load terminals, with the load removed.

→ This resistance is equivalent to that which would be encountered by a current source connected to the circuit in place of the load.

Note that this procedure gives a result that is independent of the load. This is what we want, because once the equivalent resistance has been calculated for a source circuit, the equivalent circuit is unchanged if a different load is connected.

Determining the Thevenin Voltage - 1

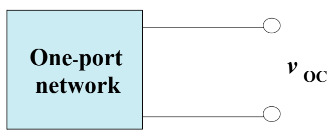
The equivalent Thevenin source voltage, v_T, is equal to the open circuit voltage present at the load terminals with load removed.

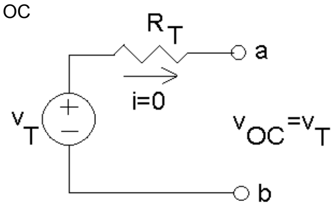
→ i.e., in order to calcuate v_T, it is sufficient to remove the load and to compute the open circuit voltage at the one-port terminals

- The open circuit voltage, v_{OC}, and the Thevenin voltage, v_T, must be the same if the Thevenin Theorem is true (see below).

→ This is because in the circuit containing v_T and R_T, the voltage v_{OC} must equal v_T because no current flows through R_T and so the voltage across R_T is zero.

→ From KVL: $v_T = R_T(0) + v_{OC} = v_{OC}$





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Determining the Thevenin Voltage - 2

Methodology:

(1) Remove the load, leaving the load terminals open-circuited.

(2) Define the open-circuit voltage v_{OC} across the open load terminals.

(3) Apply any preferred method (e.g., nodal analysis) to solve for v_{OC}.

(4) The Thevenin voltage is $v_T = v_{OC}$.

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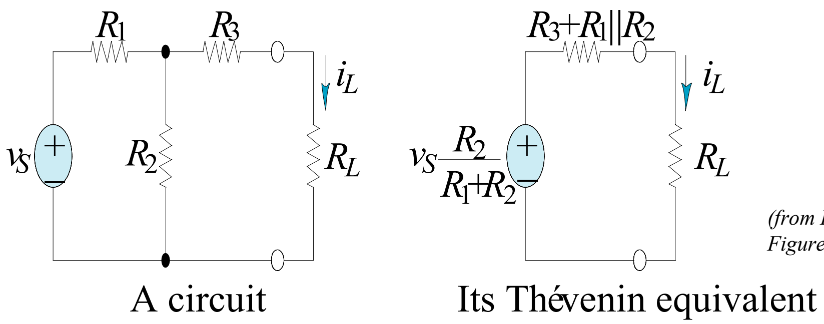
A Circuit and Its Thevenin Equivalent



(from Rizzoni Figure 3.49)

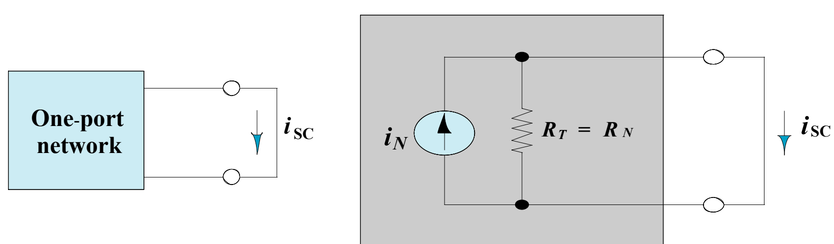
This is the circuit we considered earlier, along with its Thevenin equivalent. The two circuits are equivalent in that the current drawn by the load, i_L, is the same in both:

$$i_L = v_S \frac{R_2}{R_1 + R_2} \frac{1}{(R_3 + R_1 \,||\, R_2) + R_L} = \frac{v_T}{R_T + R_L}$$

Determining the Norton Current - 1

The Norton equivalent current, i_N, is equal to the short circuit current that would flow if the load were replaced by a short circuit.

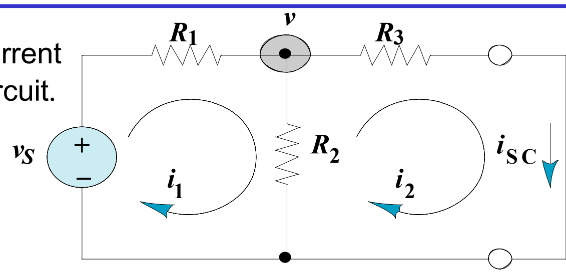
- Consider the one-port network and its Norton equivalent circuit:
  - → Current i_{SC} flowing through the short circuit replacing the load is the same as the Norton current i_N because all of the source current in this circuit must flow through the short circuit.



(from Rizzoni Figure 3.57)

Determining the Norton Current - 2

Let's find current i_{SC} in this circuit.



Short circuit replacing the load
(from Rizzoni Figure 3.58)

- Mesh Current Method:
  - → Let i_1 and i_2=i_{SC} be the mesh currents in the circuit.
  - → Two mesh equations (solve for i_{SC}): $(R_1 + R_2)i_1 - R_2 i_{SC} = v_S$
    $-R_2 i_1 + (R_2 + R_3)i_{SC} = 0$
- Node Voltage Method:
  - → Nodal equation (solve for v): $\frac{v_S - v}{R_1} = \frac{v}{R_2} + \frac{v}{R_3}$
  - → Thus: $i_N = \frac{v}{R_3} = \frac{v_S R_2}{R_1 R_3 + R_2 R_3 + R_1 R_2}$

Determining the Norton Current - 3

Methodology:

(1) Replace the load with a short circuit.

(2) Define the short circuit current i_{SC} to be the Norton equivalent current.

(3) Apply any preferred method (e.g., nodal analysis) to solve for i_{SC}.

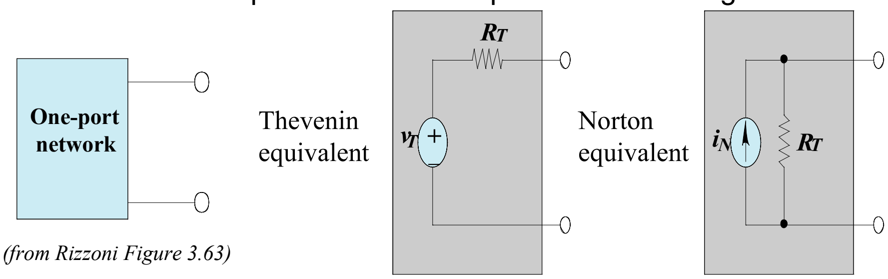
(4) The Norton current is i_N = i_{SC}.

Source Transformations - 1

Source transformations

- can be useful for determining equivalent circuits
- sometimes allow replacement of current sources with voltage sources and vice versa.

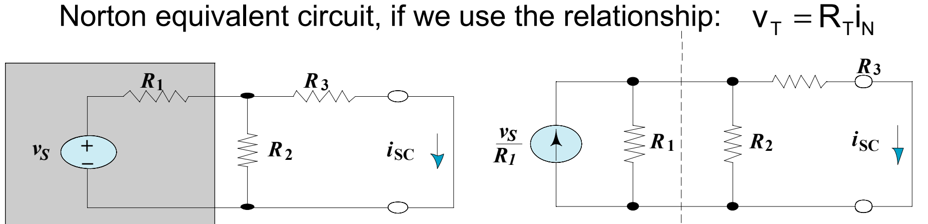
  The Thevenin and Norton Theorems state that any one-port network can be represented by a voltage source in series with a resistor, or by a current source in parallel with a resistor, and that either of these representations is equivalent to the original circuit.



(from Rizzoni Figure 3.63)

Source Transformations - 2

- Implication: any Thevenin equivalent circuit can be replaced by a Norton equivalent circuit, if we use the relationship: $v_T = R_T i_N$



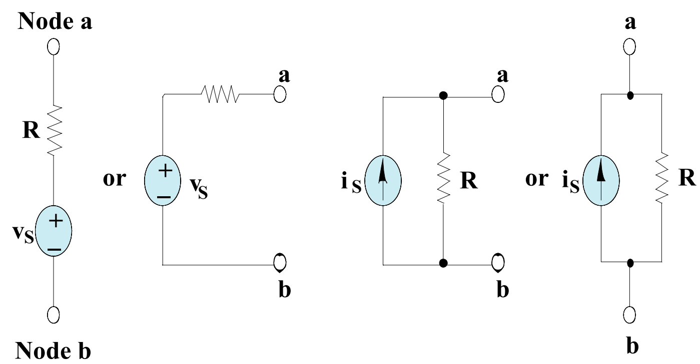
(from Rizzoni Figure 3.64)

- The subcircuit on the left of the dashed line can be replaced by its Norton equivalent, as shown.
- Current i_{SC} can be easily found because the three resistors are in parallel with the current source - use a simple current divider.
- i_{SC} flows through R_3, so:

$$i_{SC} = i_N = \frac{1/R_3}{1/R_1 + 1/R_2 + 1/R_3} \frac{v_S}{R_1} = \frac{v_S R_2}{R_1 R_3 + R_2 R_3 + R_1 R_2}$$

Source Transformations - 3

Subcircuits amenable to source transformation:



Thevénin subcircuits **Norton subcircuits**

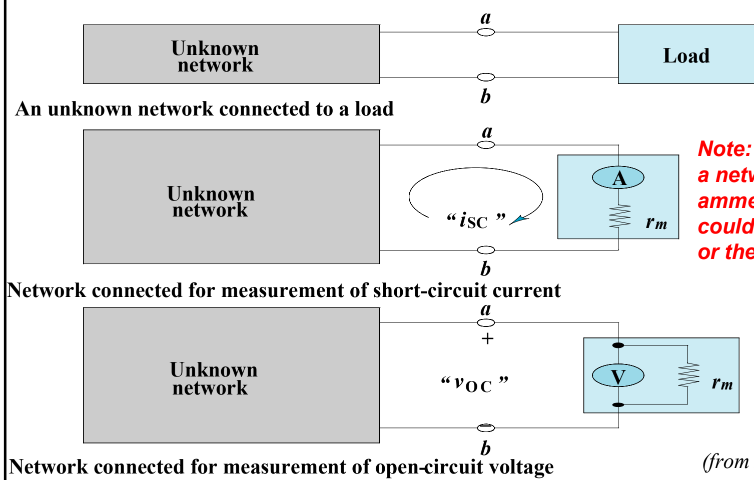
(from Rizzoni Figure 3.65)

Finding Thevenin & Norton Equivalents Experimentally 1

- Thevenin and Norton equivalent circuits can be evaluated experimentally using simple techniques.
- Basic idea:
  - → Thevenin voltage is an open-circuit voltage
  - → Norton current is a short-circuit current
- Therefore possible to make measurements to determine these quantities.
- Once v_T and i_N are known, the Thevenin resistance of the circuit can be found using $R_T = v_T / i_N$
- Need to measure v_T and i_N.

Finding Thevenin & Norton Equivalents Experimentally 2

Measurement of open-circuit voltage and short-circuit current for an arbitrary network connected to any load:



Note: Do not short circuit a network by inserting an ammeter in series - this could damage the circuit or the ammeter!

(from Rizzoni Figure 3.71)

Finding Thevenin & Norton Equivalents Experimentally 3

- These measurements require care because the measuring instruments are nonideal.
- In the presence of finite meter resistance r_m, this quantity must be taken into account when determining the open-circuit voltage and the short-circuit current.
- Quantities "v_{OC}" and "i_{SC}" have quotation marks to indicate that the measured values are affected by r_m and are not the true values.
- The true values can be calculated using (prove this to yourself!):

$$i_N = "i_{SC}"\left(1+\frac{r_m}{R_T}\right) \qquad v_T = "v_{OC}"\left(1+\frac{R_T}{r_m}\right)$$

where i_N = ideal Norton current, v_T = ideal Thevenin voltage, and R_T = true Thevenin resistance.

Finding Thevenin & Norton Equivalents Experimentally 4

$$i_N = "i_{SC}"\left(1 + \frac{r_m}{R_T}\right) \quad v_T = "v_{OC}"\left(1 + \frac{R_T}{r_m}\right)$$

- Recall
  - → For an ideal ammeter, r_m should approach zero (short circuit).
  - → For an ideal voltmeter, r_m should approach infinity (open circuit).
- So these two equations can be used to find the true Thevenin and Norton equivalent sources from an imperfect measurement of the open-circuit voltage and the short-circuit current, provided that the internal meter resistance r_m is known.
- In practice, the internal resistance of voltmeters is high enough to be considered infinite relative to the equivalent resistance of most circuits.
- However, it is impossible to build an ammeter with zero internal resistance: need to know r_m to determine the short circuit current.

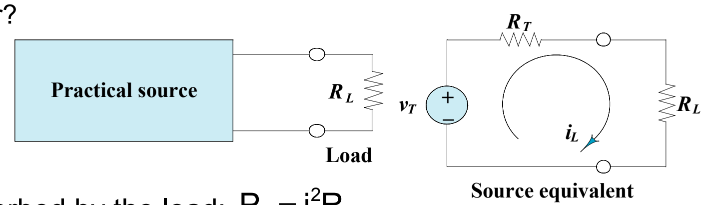
Maximum Power Transfer - 1

- The Thevenin and Norton models imply that some of the power generated by a source will be dissipated by the internal circuits within the source.
- Given this unavoidable power loss, how much power can be transferred to the load from the source under the most ideal conditions? Or, what is the load resistance that will absorb maximum power from the source?
  - → Maximum Power Transfer Theorem

Maximum Power Transfer - 2

- Power transfer between source and load:
  - → Practical source is represented by its Thevenin equivalent circuit
  - → Given v_T and R_T, what value of R_L will allow for maximum power transfer?

(from Rizzoni Figure 3.73)



- Power absorbed by the load: $P_L = i_L^2 R_L$
- Load current: $i_L = \dfrac{v_T}{R_L + R_T}$
- Combine to get load power: $P_L = \dfrac{v_T^2}{(R_L + R_T)^2} R_L$

Maximum Power Transfer - 3

- Differentiate P_L w.r.t. R_L to find find the value of R_L that maximizes the load power (assuming constant v_T and R_T).

$$\frac{dP_L}{dR_L} = \frac{v_T^2 (R_L + R_T)^2 - 2v_T^2 R_L (R_L + R_T)}{(R_L + R_T)^4} = 0$$

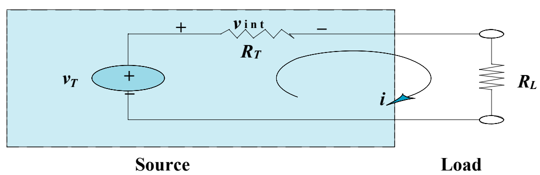
- Hence: $(R_L + R_T)^2 - 2R_L (R_L + R_T) = 0$
- For which the solution is: $R_L = R_T$

∴ To transfer the maximum power to a load, the equivalent source and load resistances must be matched (i.e., equal).

- Thus, in order to transfer maximum power to a load, given a fixed equivalent source resistance, the load resistance must match this equivalent source resistance.

Voltage Source Loading Effects

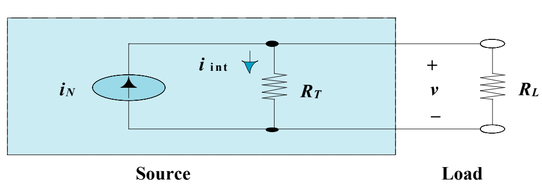
- Voltage source loading: When a practical voltage source is connected to a load, the current that flows from the source to the load will cause a voltage drop across the internal source resistance, v_{int}. As a result, the voltage seen by the load will be lower than the open-circuit (Thevenin) voltage of the source.
- Load voltage is then: $v_L = v_T - v_{int} = v_T - iR_T$
- Thus want small internal resistance in a practical voltage source.



(from Rizzoni Figure 3.74)

Current Source Loading Effects

- Current source loading: When a practical current source is connected to a load, the internal source resistance will draw some current, i_{int}, away from the load. As a result, the load will receive only part of the short-circuit (Norton) current available from the source.
- Load current is then: $i_L = i_N - \frac{v}{R_T}$
- Thus want large internal resistance in a practical current source.



(from Rizzoni Figure 3.74)