PHY305F
Electronics Laboratory I

Section 2

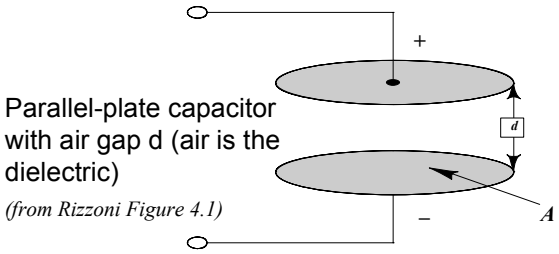
AC Circuit Basics: Passive and Linear Components and Circuits

Basic Concepts

- Alternating current (AC) circuit analysis
  - → deals with (sinusoidally) time-varying current and voltage signals whose time average values are zero
  - → particularly important because much of our electrical power is in the form of AC voltages and currents
- Energy-storage components
  - → capacitors
  - → inductors

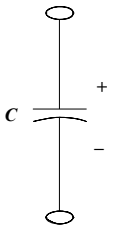
Ideal Capacitors - 1

- A capacitor is a device that can store energy in the form of a charge separation when appropriately polarized by an electric field (i.e., a voltage).
- Simplest capacitor = two parallel conducting plates of cross-sectional area A separated by air or another dielectric material.
  - → *Note: dielectric material is not an electrical conductor, but contains a large number of electric dipoles which become polarized in the presence of an electric field.*



Parallel-plate capacitor with air gap d (air is the dielectric)

(from Rizzoni Figure 4.1)



Circuit symbol

$$C = \frac{\varepsilon A}{d}$$

ε = permittivity of air = 8.854×10^{-12} F/m

Ideal Capacitors - 2

- Capacitors act as open circuits for DC currents because the insulating dielectric will not allow the current to flow.
- If the voltage at the capacitor terminals changes with time, so will the charge accumulated at the two capacitor plates.
- Charge separation caused by the polarization of the dielectric is proportional to the applied electric field and hence to the voltage

$$Q = CV \quad \text{or} \quad q(t) = Cv(t)$$

- C is the capacitance of the element and is a measure of the ability of the capacitor to store charge.
- The SI unit of capacitance is the farad (F): 1 F = 1 C/V.
- More common: microfarads (1 μF = 10^{-6} F) or picofarads (1 pF = 10^{-12} F)
- To increase capacitance, real capacitors are often made of tightly rolled sheets of metal film with a dielectric sandwiched between.

Ideal Capacitors - 3

- For applied voltage v(t), the change with time of the stored charge is analogous to a current.
- Differentiate equation for q(t) to get the defining circuit law for a capacitor (defines the i-v relationship):

$$\boxed{i(t) = \frac{dq(t)}{dt} = C\frac{dv(t)}{dt}}$$

- Integrate to get the voltage across a capacitor:

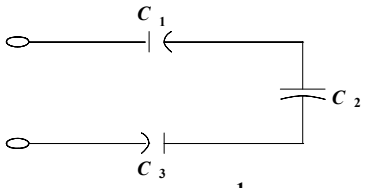
$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t')\,dt' = \frac{1}{C}\int_{t_o}^{t} i_C(t')\,dt' + V_o \qquad \text{for} \quad t \geq t_o$$

$$\text{where} \quad V_o = v_C(t_o) = \frac{1}{C}\int_{-\infty}^{t_o} i_C(t')\,dt'$$

- Initial voltage V_o just indicates that some charge is stored in the capacitor at time t_o.

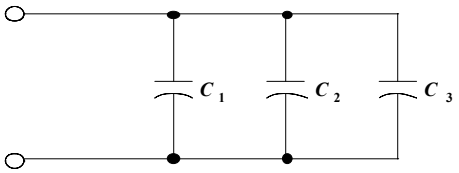
Combining Capacitors

- Capacitors connected in series combine in the same way as resistors connected in parallel.



$$C_{EQ} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}$$

- Capacitors connected in parallel add.



(from Rizzoni Figure 4.2)

$$C_{EQ} = C_1 + C_2 + C_3$$

Energy Storage in Capacitors

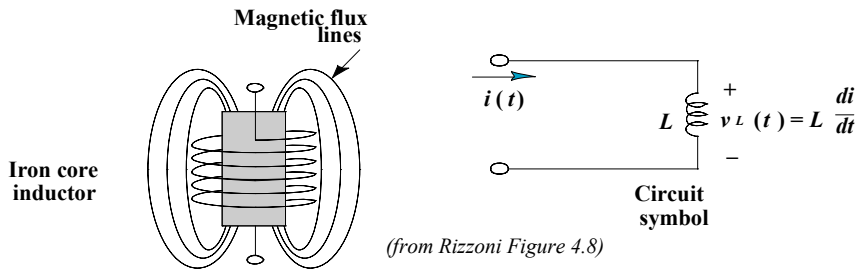
- The energy stored in a capacitor can be readily derived:

$$W_C(t) = \int P_C \, dt'$$
$$= \int v_C(t') \, i_C(t') \, dt'$$
$$= \int v_C(t') \frac{dv_C(t')}{dt'} dt'$$

$$\boxed{W_C(t) = \frac{1}{2} C [v_C(t)]^2}$$

Ideal Inductors - 1

- An inductor is a device that can store energy in a magnetic field.
- Usually made by winding a coil of wire around an insulating core.
  - → As current flows through the coil, a magnetic field is set up.
- In an ideal inductor, the resistance of the wire is zero, so a constant current through the inductor flows without causing a voltage drop.
- Thus, an ideal inductor acts as a short circuit for DC currents.



(from Rizzoni Figure 4.8)

Ideal Inductors - 2

- Application of a time-dependent voltage across the inductor will generate a current, defined by: $\boxed{v_L(t) = L\frac{di_L(t)}{dt}}$
- L is the inductance of the coil.
- The SI unit of inductance is the henry (H): 1 H = 1 V-s/A.

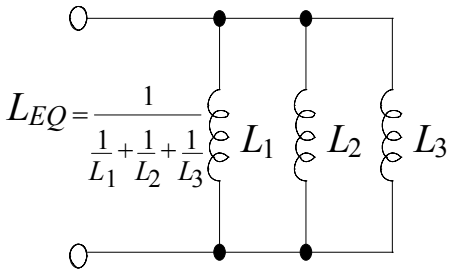
- Note the duality with the i-v relationship for capacitors: $i(t) = C\frac{dv(t)}{dt}$
  → the roles of i and v are reversed between the two
- Integrate to get the current across an inductor:

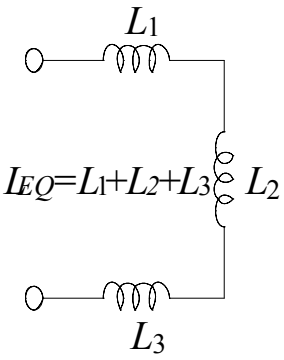
$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t')\,dt' = \frac{1}{L}\int_{t_o}^{t} v_L(t')\,dt' + I_o \quad \text{for} \quad t \geq t_o$$

$$\text{where} \quad I_o = i_L(t_o) = \frac{1}{L}\int_{-\infty}^{t_o} v_L(t')\,dt'$$

Combining Inductors

- Inductors connected in parallel combine in the same way as resistors connected in parallel.
- Inductors connected in series add.





(from Rizzoni Figure 4.9)

Energy Storage in Inductors

- The energy stored in an inductor can be derived by first determining the instantaneous power in the inductor:

$$P_L(t) = i_L(t)\, v_L(t) = i_L(t)\, L \frac{di_L(t)}{dt} = \frac{d}{dt}\left\{\frac{1}{2} L [i_L(t')]^2\right\}$$

- Hence the total stored energy is:

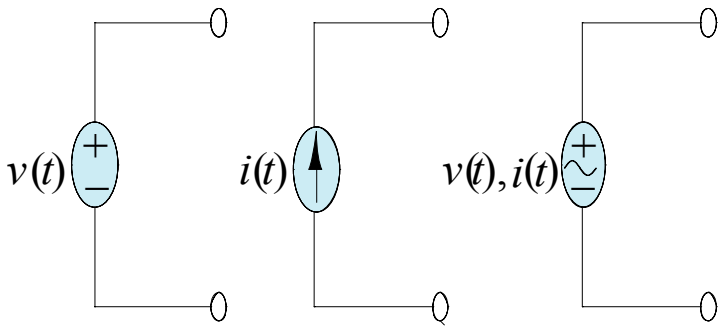
$$W_L(t) = \int P_L(t')\, dt'$$

$$= \int \frac{d}{dt'}\left\{\frac{1}{2} L [i_L(t')]^2\right\} dt'$$

$$\boxed{W_L(t) = \frac{1}{2} L\, [i_L(t)]^2}$$

Time-Dependent Signal Sources

Convention for indicating time-dependent signal sources:



Generalized time-dependent sources Sinusoidal source

(from Rizzoni Figure 4.18)

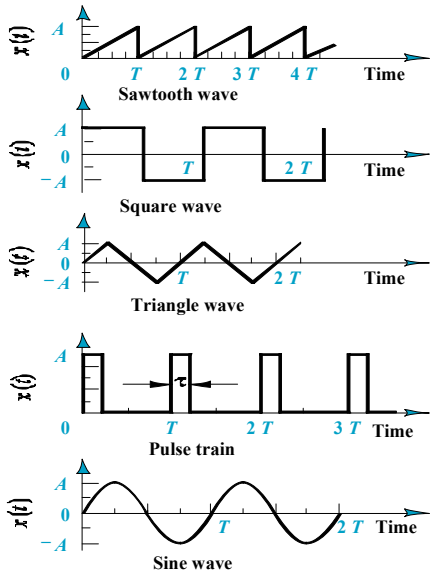
Periodic Signal Waveforms

- Periodic time-dependent signals appear frequently in practical applications and are useful approximations of many phenomena.
- A periodic signal satisfies the equation:

$$x(t) = x(t + nT) \quad n = 1, 2, 3$$

where T = period of x(t).

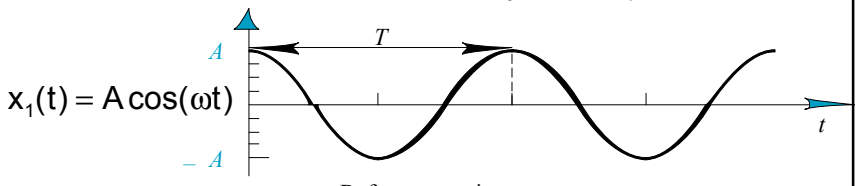
- Signal or waveform generators can provide periodic voltages (or currents) of variable period and amplitude.

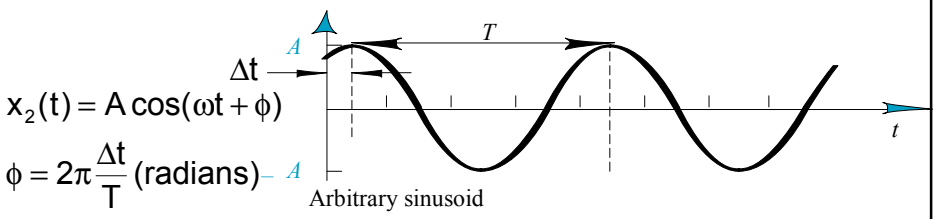


(from Rizzoni Figure 4.19)

Sinusoidal Waveforms

- Sinusoidal waveforms are the most important type of time-dependent signals. General equation: $x(t) = A\cos(\omega t + \phi)$ where A = amplitude, ω = radian frequency (=$2\pi f$), ϕ = phase.





(from Rizzoni Figure 4.20)

Average and RMS Values

Two ways to quantify strength of a time-varying electrical signal:

(1) Time-averaged (or DC) value

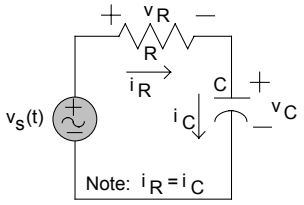
- Measurement of mean voltage or current over some time period.
- Defined as $\langle x(t) \rangle = \frac{1}{T}\int_0^T x(t')\,dt'$ where T=period of integration
- Average value of a sinusoidal signal is always zero, but average power is not! Need another way to quantify AC signal strength.

(2) Root-mean-square (or RMS) value

- Accounts for fluctuations of the signal about its average value.
- Defined as $x_{rms} = \sqrt{\frac{1}{T}\int_0^T [x(t')]^2\,dt'}$
  - → square root of the average (mean) of the square of the AC signal
- RMS of a sinusoidal signal is always 0.707 times its peak value.

Circuits with Dynamic Elements

Consider this circuit containing a capacitor.



- Apply KVL: $v_S(t) = v_R(t) + v_C(t)$
- Hence: $v_S(t) = Ri_C(t) + \frac{1}{C}\int_{-\infty}^{t} i_C(t')dt'$
- Integrate:

$$\frac{di_C}{dt} + \frac{1}{RC}i_C = \frac{1}{R}\frac{dv_S}{dt}$$

Note that $\frac{d}{dt}\left(\int_{-\infty}^{t} i_C(t')dt'\right) = i_C(t)$

- We could also apply KCL: $i_R = \frac{v_S - v_C}{R} = i_C = C\frac{dv_C}{dt}$

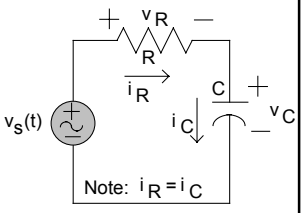
$$\frac{dv_C}{dt} + \frac{1}{RC}v_C = \frac{1}{RC}v_S$$

- Solution of either of these differential equations will determine all voltages and currents in the circuit.

Sinusoidal Forcing of Circuits - 1

- Consider the same circuit with a sinusoidal voltage source: $v_S(t) = V\cos(\omega t)$
- Substitute this in the previous diff'l equation:

$$\frac{d}{dt}v_C + \frac{1}{RC}v_C = \frac{1}{RC}V\cos\omega t$$



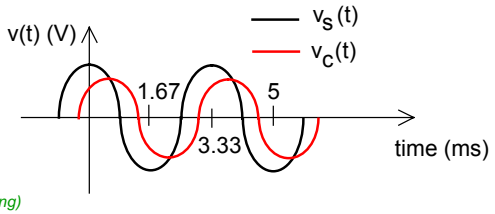
- The forcing function is sinusoidal, so assume the solution is too:

$$v_C(t) = A\sin\omega t + B\cos\omega t = C\cos(\omega t + \phi)$$

- Substituting this into the equation for $v_c(t)$, grouping coefficients of like terms, and noting that the coefficients of cos & sin must = 0:

$$A = \frac{V\omega RC}{1+\omega^2(RC)^2}$$

$$B = \frac{V}{1+\omega^2(RC)^2}$$



Sinusoidal Forcing of Circuits - 2

- This solution method works for simpler circuits but can require solution of higher-order differential equations if many circuit elements are present, particularly capacitors and inductors.
- However, it leads to the following summary of the key points in AC circuit analysis.

  In a sinusoidally forced linear circuit, all branch voltages and currents are sinusoids at the same frequency as the forcing signal.

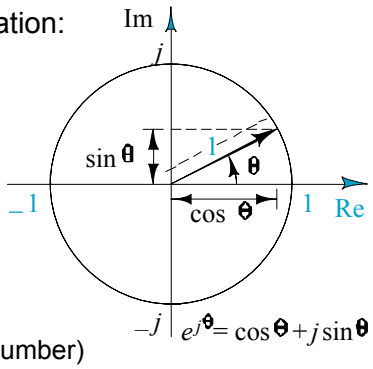
  The amplitudes of these voltages and currents are a scaled version of the excitation amplitude

  The voltages and currents may be shifted in phase with respect to the forcing signal.

Euler's Identity

- It is possible to represent sinusoidal functions using only the frequency, amplitude, and phase, with help of complex numbers.
  - → Allows complex algebra to replace solution of differential equations.
- Euler's Identity is the basis for this notation: It defines the complex exponential $e^{j\theta}$ as a point in the complex plane that can be represented by real and imaginary components:

$$e^{j\theta} = \cos\theta + j\sin\theta$$



  - → this identity is just a trig. relation in the complex plane (equates polar and rectangular forms of a complex number)
  - → vector of magnitude $\left|e^{j\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

Phasors

- Apply Euler's Identity to a generalized sinusoid:

$$A\cos(\omega t + \phi) = \mathrm{Re}\left[Ae^{j(\omega t+\phi)}\right] = \mathrm{Re}\left[Ae^{j\omega t}e^{j\phi}\right]$$

- This follows from:

$$\mathrm{Re}\left[Ae^{j(\omega t+\phi)}\right] = \mathrm{Re}\left[A\cos(\omega t+\phi) + jA\sin(\omega t+\phi)\right] = A\cos(\omega t+\phi)$$

- So we can express a generalized sinusoid as the real part of a complex number whose argument or angle is given by ($\omega t+\phi$) and whose length or magnitude is the peak amplitude of the sinusoid.
- The complex phasor that corresponds to sinusoidal signal $A\cos(\omega t+\phi)$ is defined as the complex number $Ae^{j\phi}$:

$$Ae^{j\phi} = \text{complex phasor notation for } A\cos(\omega t+\phi) = A\angle\phi$$

- This definition is a simplification, as it removes the operator "Re" and the term $e^{j\omega t}$ from the full expression above.

Phasor Methodology

(1) Any sinusoidal signal may be mathematically represented in one of two ways:

→ a time-domain form: $v(t) = A\cos(\omega t + \phi)$

→ a frequency-domain or phasor form: $\mathbf{V}(j\omega) = Ae^{j\phi} = A\angle\phi$

Note the $j\omega$ in the notation $V(j\omega)$, which indicates the $e^{j\omega t}$ dependence of the phasor. Bold uppercase indicates phasor voltages or currents.

(2) A phasor is a complex number, expressed in polar form, consisting of a magnitude equal to the peak amplitude of the sinusoidal signal and a phase angle equal to the phase shift of the sinusoidal signal referenced to a cosine signal.

(3) When using phasor notation, it is important to note the frequency ω of the sinusoidal signal, as this is not explicit in the phasor eqn.

Superposition of AC Signals - 1

Example: superposition of two sinusoidal sources of different phase and amplitude but of the same frequency.

Given: $v_1(t) = A_1\cos(\omega t + \phi_1) = 15\cos(377t + \frac{\pi}{4})$

$v_2(t) = A_2\cos(\omega t + \phi_2) = 15\cos(377t + \frac{\pi}{12})$

Write the voltages in phasor form:

$\mathbf{V}_1(j\omega) = A_1e^{j\phi_1} = A\angle\phi_1 = 15\angle\frac{\pi}{4}$ V

$\mathbf{V}_2(j\omega) = A_2e^{j\phi_2} = A\angle\phi_2 = 15\angle\frac{\pi}{12}$ V

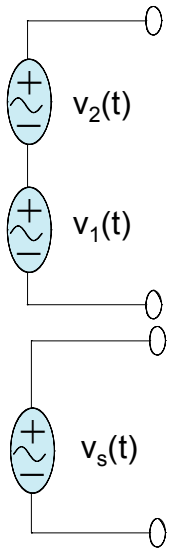
Convert to rectangular form: $\mathbf{V}_1(j\omega) = 10.61 + j10.61$ V

$\mathbf{V}_2(j\omega) = 14.49 + j3.88$ V

Add:

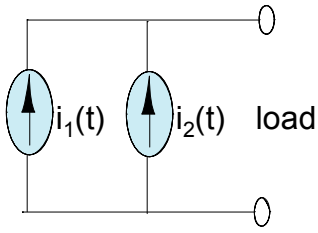
$\mathbf{V}_S(j\omega) = 25.10 + j14.49 = 28.98e^{j\pi/6} = 28.98\angle\frac{\pi}{6}$ V

$v_S(t) = 28.98\cos(377t + \frac{\pi}{6})$ V



Superposition of AC Signals - 2

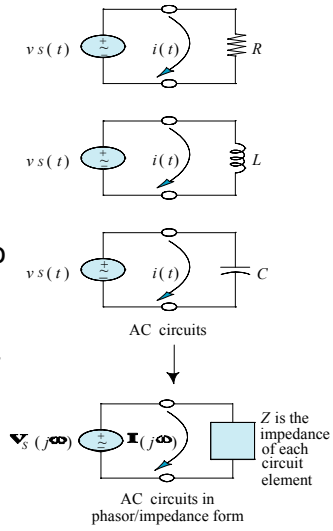
- The previous approach cannot be used in the case of two sinusoidal signals that are not at the same frequency.
- Consider a load forced by two current sources in parallel, with
  $i_1(t) = A_1 \cos(\omega_1 t) \quad i_2(t) = A_2 \cos(\omega_2 t)$
- Load current:
  $i_L(t) = i_1(t) + i_2(t) \quad \text{or} \quad \mathbf{I}_L = \mathbf{I}_1 + \mathbf{I}_2$



- Here we cannot use addition of phasors because the term $e^{j\omega t}$ is implicitly present. This is clear from the full expresssions for the phasor currents $\mathbf{I}_1 = A_1 e^{j0}, \mathbf{I}_2 = A_2 e^{j0}$
  $\mathbf{I}_1 = \mathrm{Re}\left[A_1 e^{j0} e^{j\omega_1 t}\right], \; \mathbf{I}_2 = \mathrm{Re}\left[A_2 e^{j0} e^{j\omega_2 t}\right]$
- The equation for $i_L(t)$ is the only unambiguous eqn. for load current.
- To analyse a circuit with multiple sinusoidal sources at different frequencies using phasors, the circuit must be solved separately for each signal. Then the results for each source are added.

Impedance

- Impedance is a parameter defined for resistors, capacitors, and inductors that can be regarded as a complex resistance.
- It allows theorems for DC circuits to be extended to AC circuits.
- The concept of impedance is equivalent to stating that capacitors and inductors act as frequency-dependent resistors, i.e., as resistors whose resistance is a function of the frequency of the sinusoidal excitation.
- Generalizing Ohm's Law to AC circuits gives: $\vec{v}(\omega, t) = Z(\omega)\vec{i}(\omega, t)$
  where Z is the impedance (unit = Ohms).



(from Rizzoni Figure 4.33)