PHY293 Lecture #3

October 30, 2017

1. Last of four simple examples

   IV Length Contraction

   - Consider light emitted from one end of the train car (length L) and reflected back to detector at same end of car
   - How long does it take for return trip? For the moving observer (Anna) this is easy: $t_t = 2L_t/c$.
   - Careful L_t is the length of the train car for the moving observer (Anna)
   - For the stationary observer (Bob) things a little more involved
     - Light travels a longer distance on the way 'out' (front of the car will have moved a distance vt_g before light arrives)
     - But it will have to travel less far on the way back – does this cancel ... no, not quite
     - Outbound we have $ct_{out} = L_g + vt_{out}$, Coming back we have $ct_{back} = L_g - vt_{back}$
     - Can solve these to get $t_{out} = L_g/(c-v)$ and $t_{back} = L_g/(c+v)$
     - Total time is $t_{out} + t_{back} = L_g[\frac{1}{c-v} + \frac{1}{c+v}] = L_g\frac{2c}{c^2-v^2} = \frac{2L_g}{c}\frac{1}{1-v^2/c^2} = \frac{\gamma^2 2L_g}{c}$
     - This is the total time observed by the stationary observer t_g, but we already saw that $t_g = \gamma t_t$ and $t_t = 2L_t/c$
     - So putting these two together we find $t_g = \frac{\gamma^2 2L_g}{c} = \frac{\gamma 2L_t}{c}$ from which we can extract $L_g = \frac{L_t}{\gamma}$
     - To an observer on the ground the moving train car is shorter (by a factor γ) than to the observer on the train.
     - All that is affected is the dimension parallel to the relative velocity: This is known as Lorentz contraction

   - Have seen that time intervals grow by a factor γ from moving to stationary frame
   - Distance intervals shrink by a factor γ from moving to stationary
   - Now find actual coordinates transform to achieve these two – very real – effects

2. Lorentz Transformations

   - If the speed of light is measured/observed to be the same in all inertial reference frames
   - A pulse of light emerging from the origin (at $t = t' = 0$) becomes a sphere, growing at the same rate for all observers.
   - In particular this requires that the wavefront satisfy both: $x^2 + y^2 + z^2 = (ct)^2$ for the stationary observer, but also $x'^2 + y'^2 + z'^2 = (ct')^2$ for the moving observer.
   - Can see (almost by inspection) that this is **not true** for the Galilean coordinate transformations of normal mechanics. Viz:

$$x' = x - vt, y' = y, z' = zt' = t \Rightarrow x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2 - 2xvt + v^2t^2 = (ct)^2$$

   - There are clearly some extra terms $v^2t^2 - 2xvt$ which vanish for $v = 0$ but then that's just the same reference frame.
   - Look for a **linear** transformation that does give the same light-sphere for both observers.
   - Expect both space and time coordinates to be affected (but not y or z) so consider:

$$x' = Ax + Bt, \quad y' = y, \quad z' = z, \quad \text{and} \quad t' = Cx + Dt$$

   - First constraint comes from considering velocity of the origin of the moving frame: $x' = 0 \Rightarrow dx/dt = v \Rightarrow -B/A = v$
   - Similarly for $x = 0 \Rightarrow dx'/dt' = v \Rightarrow -B/D = v$: conclude D=A
   - With this simplification plug back into full sphere equation:

$$x' = Ax + Bt, t' = Cx + At \quad \text{into} \quad x'^2 + y'^2 + z'^2 = (ct')^2$$

   - Gives: $A^2x^2 + 2ABxt + B^2t^2 + y^2 + z^2 = c^2(C^2x^2 + 2ACxt + A^2t^2)$
   - To make this equal to $1x^2 + y^2 + z^2 = c^2t^2$ results in the constraints:

$$2AB = 2ACc^2 \qquad A^2 - c^2C^2 = 1 \quad \text{and} \quad -B^2 + c^2A^2 = c^2$$

   - Three equations in three unknowns. Replace B with $-Av$ (a few lines up) and use third constraint to get : $A = \frac{1}{\sqrt{1-v^2/c^2}}$

- Then easy to extract the other constants: $B = -\frac{v}{\sqrt{1-v^2/c^2}} = -\gamma v$, $C = -\frac{v/c^2}{\sqrt{1-v^2/c^2}} = -\gamma v/c^2$ and $D = A = \frac{1}{\sqrt{1-v^2/c^2}}$.
- Thus the full Lorentz transformations become:

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z \quad t' = \gamma(t - \frac{v}{c^2}x) \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

- This is the same factor γ that we saw last time in our physical examples. Known as the Lorentz boost factor.
- Often you will see $\beta \equiv v/c$ so $\gamma = \frac{1}{\sqrt{1-\beta^2}}$
- As $\beta \to 0$ for $v \ll c$ recover non-relativistic situation. Convince yourselves that Galilean Transformations re-appear.
- Known as Lorentz Transformations. Lorentz had already shown that this 'math' would explain the Michelson-Morley experiment. Actually said that the arms of the interferometer were 'contracting' by an amount γ and showed how this would produce their observation, but he did not have a physical explanation for why... Einstein came up with that.

3. Simultaneity – from Lorentz Transform perspective

- Use Lorentz transformations to predict the time difference between two simultaneous events in a moving frame.
- Event1: x'_1, y'_1, z'_1, Event2: (x'_2, y'_2, z'_2) occur at t' (same for both events)

$$t_1 = \gamma(t' + \beta/cx'_1) \qquad t_2 = \gamma(t' + \beta/cx'_2)$$

$$t_2 - t_1 = \gamma[(t' + \beta/cx'_2) - (t' + \beta/cx'_1)] = (\gamma\beta/c)\ [x'_2 - x'_1]$$

- Two events simultaneous in S' (both occurred at t') are **not simultaneous** in S, unless at the same location in S'.
- After one of the lectures I got the question: Why does the inverse transformation (the one used here gives t as a function of x' and t') still have multiplication by γ. The, not very convincing, answer I gave was that 'it just works out like that'. I double checked (Google "inverse Lorentz transformations if you want to see more) and it does work out like the formulas given here. The trick/challenge is that you have to invert both the x' and t' parts from the derivation we did in the other direction. So it's not just as simple as inverting 'only the t' piece or 'only the x piece.
- Revisit textbook variant of simultaneity thought experiment, where two light pulses arrive at a single detector in the middle of the train car
  - Two light sources at opposite ends of the train car, detector in the middle
  - Moving observer perspective: the two light sources flash at the same time and detector sees the light at same time
- In this case the stationary observer clearly sees the light pulses arrive at the detector at **the same time** – happens at the same place from their perspective (also at the same place from the perspective of the moving observer)
- Both observers agree that light pulses arrive at the detector at the same time – simultaneous event at a single point in space
- What they can't agree on is when the light was **emitted**
  - Simultaneous in S' – moving frame
  - But back pulse emerges sooner (and front pulse later) in S – stationary observer. Arrange just-so the two can propagate at c and arrive at detector at the same time

4. Length (Lorentz) Contraction – from Lorentz Transform perspective

- Time difference between two events (at different locations in space) are different
- What about the distance between the points?
- Suppose two events in previous example where the two ends of a stick, stationary relative to the moving observer in S'
- $x'_1 - x'_2$ is a measure of the stick length in S'
- An observer in S measures the length by measuring x_1 and x_2 **at the same time** in S:

$$L_0 = x'_1 - x'_2 = \gamma(x_1 - \beta ct) - \gamma(x_2 - \beta ct) = \gamma(x_1 - x_2) = \gamma L$$

- The length of an object in the frame where it is at rest is the "proper length" (or "rest length")
- In a frame where it is moving it will be shortened (only in the direction of the velocity) by a factor γ:

$$L = \frac{L_0}{\gamma}$$

5. Time Dilation – from Lorentz Transform perspective

   - Now consider two events that happen at the same location but at two different times
   - What is relationship between the elapsed times in S' and S?

$$t_2 - t_1 = \gamma(t_2' + \frac{\beta}{c}x') - \gamma(t_1' + \frac{\beta}{c}x') = \gamma(t_2' - t_1')$$

   - Remember these events are happening at the same place now so $x_1' = x_2' \equiv x'$
   - Also γ always bigger than 1, so time interval between two events is longer for an observer in S than in S'
   - Time runs more slowly in a moving frame.

6. Cosmic Ray muons

   - Cosmic rays (mostly protons) hit upper atmosphere gas molecules and produce showers of particles
   - Most decay long before reaching the surface of the earth
   - Many muons survive (1 per minute through your hand – at sea level)
   - Particle lifetimes are defined for particles at rest (why?). For the muon it is 2.2 μs
   - If they are produced at an altitude of 8000m and travel with a speed of 0.998c, should they make it to sea level?
   - Classically muons would travel (on average) a distance of:

$$(2.2 \times 10^{-6}\mathrm{s})(0.998)(3 \times 10^{8}\mathrm{m/s}) = 660\mathrm{m}$$

   - In general very few of them should survive for 8000m to reach us on the surface of the earth
   - Relativistically, the muons lifetime, from the reference frame of an observer stationary on the earth's surface, is increased by a factor γ (=16 for 0.998 c), so the could travel a distance:

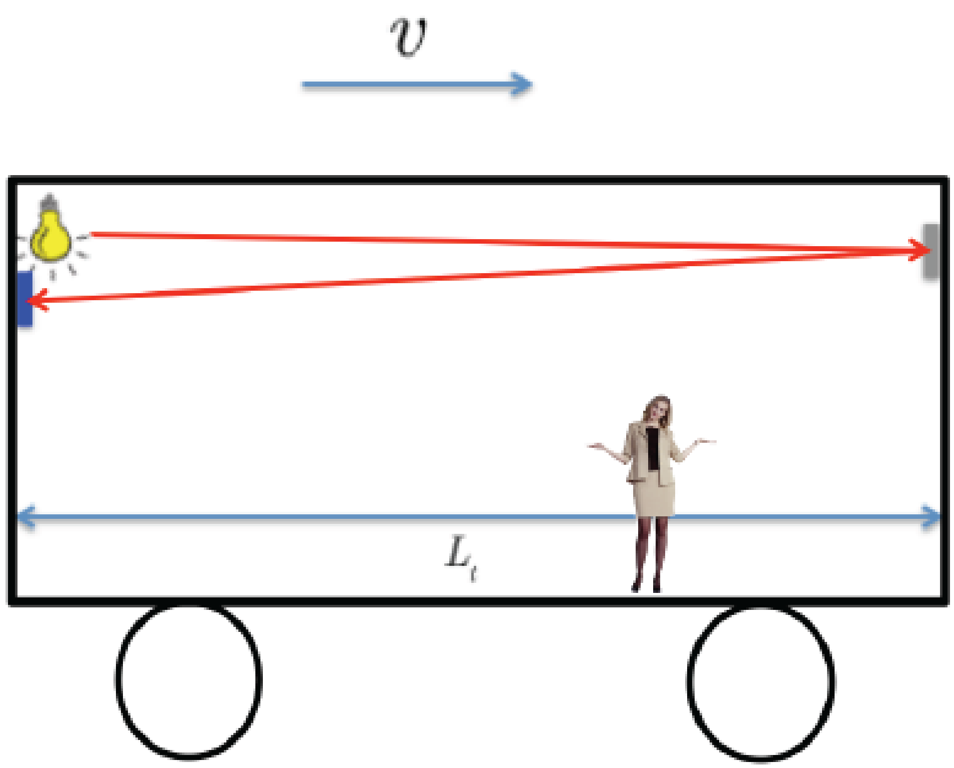
$$16(2.2 \times 10^{-6}\mathrm{s})(0.998)(3 \times 10^{8}\mathrm{m/s}) = 10,500\mathrm{m}$$

   - So many of them (more than half) could reach the earth's surface.
   - Lifetime is a 'exponential' phenomenon. 37% survive one decay length, 10% survive two and so on.

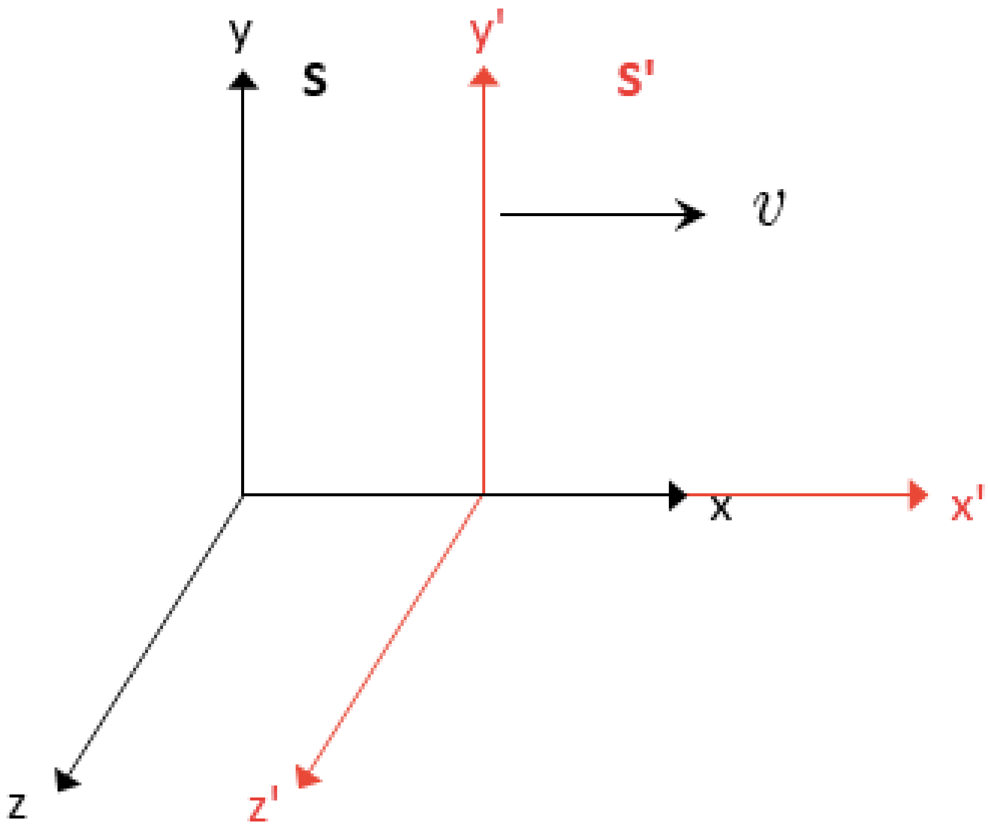
7. An alternate perspective

   - Consider this from the point of view of the muons (imagine that we are observing from their rest frame)
   - Here the muon only has a lifetime of 2.2 μs
   - But in this frame the distance to the earth is contracted by the same factor γ (16)
   - As we have seen 2.2 μs at 0.998c corresponds to a distance of 660m
   - But the distance to the earth (from the muon's perspective) is only 8000m/16 = 500m, so again, more than half of them should reach the earth before decaying.
   - The conclusion is the same in both frames ... as Einstein's second postulate says it must be.
   - Note that 0.998c corresponds to a momentum of the muons of about 2 GeV (more on these units when we get to the end of the SR section) ... which is 3000 times less energetic than the LHC protons and by far not the most energetic muons from cosmic rays can be (this is more like the mean or medium momentum muon). So even relatively modest particle momenta correspond to speeds very near the speed of light.

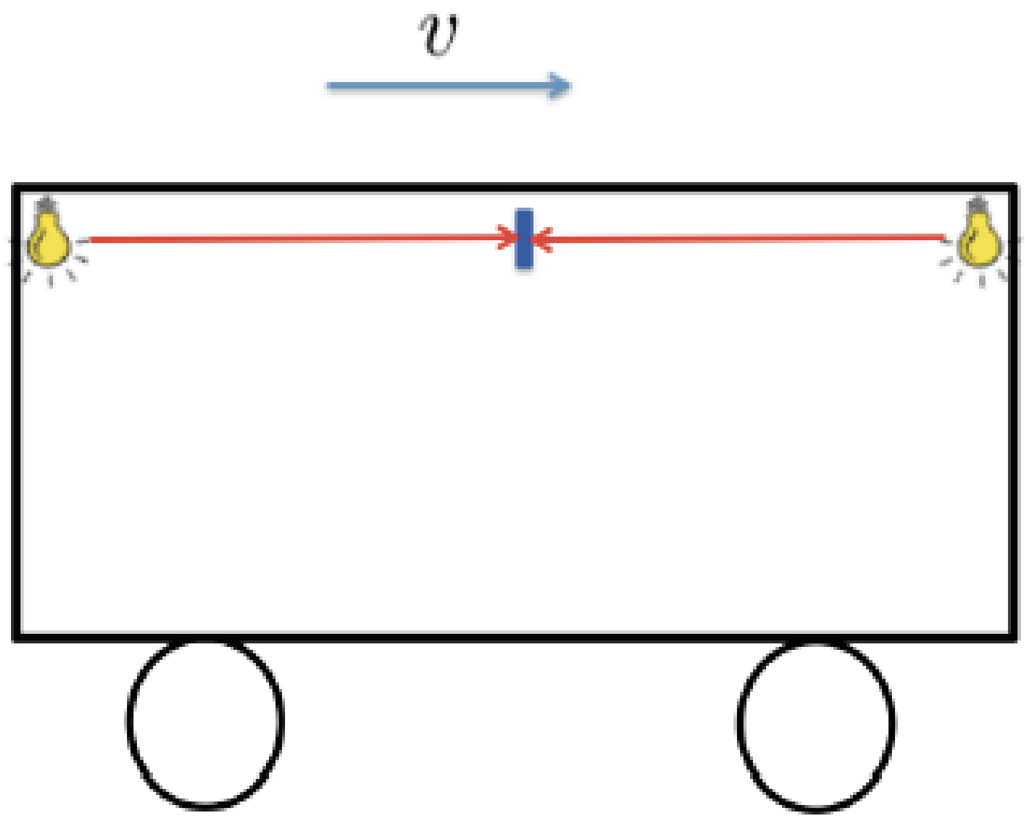
Distance Measurement for Moving Observer



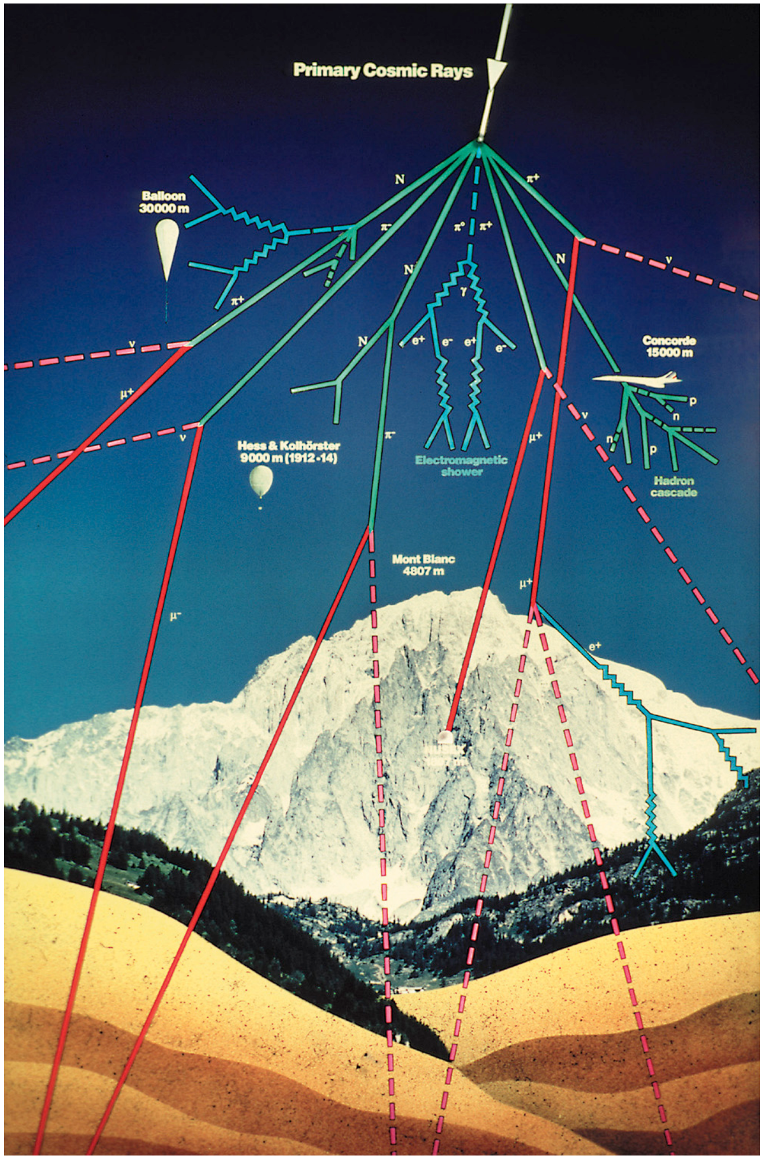
Inertial Frame Coordinates

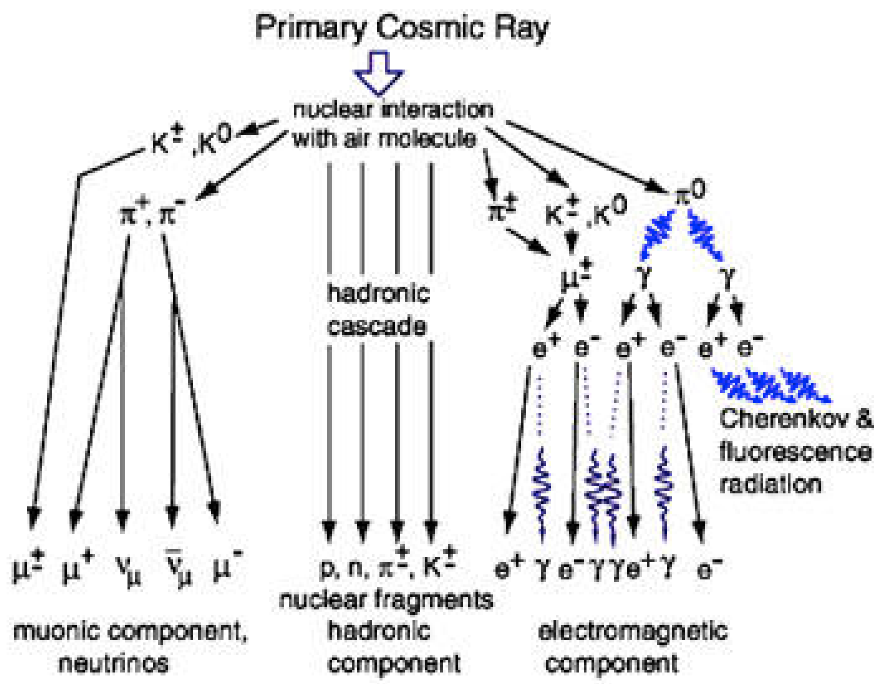


Single Detector Simultaneity



Cosmic Rays


Primary Cosmic Rays
Balloon 30000 m
Hess & Kolhörster 9000 m (1912-14)
Electromagnetic shower
Concorde 15000 m
Hadron cascade
Mont Blanc 4807 m


Primary Cosmic Ray
nuclear interaction with air molecule
$K^{\pm}, K^0$
π^+, π^-
hadronic cascade
$\pi^{\pm}$
$K^{\pm}, K^0$
π^0
$\mu^{\pm}$
γ
γ
e^+ e^- e^+ e^- e^+ e^-
Cherenkov & fluorescence radiation
$\mu^{\pm}$ μ^+ ν_μ $\bar{\nu}_\mu$ μ^-
muonic component, neutrinos
p, n, $\pi^{\pm}$, $K^{\pm}$ nuclear fragments
hadronic component
e^+ γ e^- γ γ e^+ γ e^-
electromagnetic component